Final #1

Mark all correct answers in each of the following questions.

1. An experiment consists of several stages, at each of which several coins are tossed. At the first stage, n coins are tossed. At the second stage, the number of coins tossed is as the number of those showing a head at the first stage. At the third stage, the number of coins tossed is as the number of those showing a head at the second stage. The process is continued until at some stage none of the coins shows a head. (Remark: Alternatively, you may think of each of the original n coins as if it is tossed until it shows a tail for the first time.) Denote by X the total number of heads obtained and by S the number of stages. For example, if all but two of the coins show a tail at the first toss, and both of the coins tossed next show a tail, then $X = 2$ and $S = 2$.

   (a) The total number of coins tossed at all stages is distributed $\overline{B}(n, 1/2)$.

   (b) $E(X) = n$.

   (c) $P(S = 2) = \left(\frac{1}{2}\right)^n$.

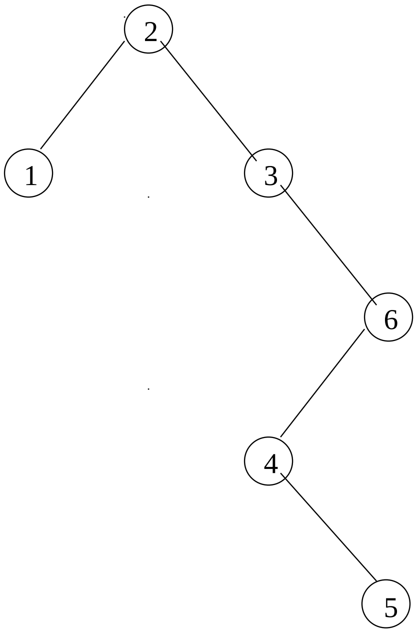
   (d) $F_S(l) = \left(1 - \frac{1}{2^l}\right)^n$ for $l = 1, 2, \ldots$.

   (e) $P(X = k|S = 2) = \binom{n}{k} \cdot \frac{1/2^k}{(3/2)^n - 1}$ for $k = 1, 2, \ldots, n$.

   (f) If $n = 2$, then $E(S) = 8/3$.

   (g) None of the above.

2. The numbers 1, 2, …, n are ordered randomly, and then taken one by one to generate a binary search tree. For example, if $n = 6$ and the initial order is 2, 3, 6, 4, 1, 5, then the following tree is obtained:



Let H be the height of the obtained tree, R the number at the root of the tree, S_{left} the sum of the numbers at the nodes in the left subtree and S_{right} the sum of the numbers at the nodes in the right subtree. For example, for the tree above we have $H = 4,\ R = 2,\ S_{\text{left}} = 1$ and $S_{\text{right}} = 18$.

(a) $P(H = n-1) = \frac{2^{n-1}}{n!}$.

(b) $V(R) = \frac{n^2}{6}$.

(c) $E(S_{\text{left}}) = \frac{n^2-1}{6}$.

(d) $E(S_{\text{left}}) = E(S_{\text{right}})$.

(e) For every $n \geq 2$, not all binary search trees (with data fields $1, 2, \ldots, n$ at the nodes) have the same probability of being obtained by this procedure.

(f) Let A be the event that the second number in the sequence is placed at the left child of the root. Then:

$$P(R = j|A) = \frac{2(j-1)}{n(n-1)} \ .$$

(g) None of the above.

3. The lifetime in years of light-bulbs produced by Light++ is distributed Exp(2), whereas for those produced by Bite & Lite it is distributed Exp(3). Each of the two companies owns half the market. Let X denote the lifetime of a random light-bulb.

   (a) A busy customer, whose main consideration in choosing a type of light-bulb is maximizing the time the light-bulb will last, prefers Bite & Lite light-bulbs.

   (b) Suppose the price of a Light++ light-bulb is 1 shekel, and that of a Bite & Lite light-bulb is b shekels. The density functions of the lifetimes of the two types of bulbs intersect at the point $\ln \frac{3}{2}$. Therefore, an economical customer, whose main consideration in choosing a type of light-bulb is minimizing his expected expenses (in the long run) will prefer Bite & Lite light-bulbs if and only if $b < \ln \frac{3}{2}$.

   (c) $X \sim \text{Exp}(2.5)$.

   (d) The expected lifetime of a random light-bulb is 5 months.

   (e) The distribution of X is memoryless.

   (f) $V(X) = \frac{3}{16}$.

   (g) None of the above.

4. In days with bad weather, the number of students entering the university bookstore per minute is distributed $P(1)$, while that of students entering the cafeteria is distributed $P(2)$, the two numbers being independent. In days with good weather, these numbers are distributed $P(2)$ and $P(3)$, respectively (and again they are independent). Assume that the weather is good half the days and bad half the days. Let X be the number of students entering the bookstore within some minute in a randomly chosen day, and Y the number of students entering the cafeteria during the same minute.

   (a) The random variables X and Y are independent, and in particular uncorrelated.

   (b) $P_{X,Y}(k, l) > P_{X,Y}(l, k)$ for any integers $l > k \geq 0$.

   (c) $X < Y$ on the whole sample space (except, perhaps, for a set of probability 0).

(d) $Y - X \sim P(1)$.

(e) $V(Y) = \frac{11}{4}$.

(f) $0 < \rho(X, Y) < 1$.

(g) None of the above.

5. A point (X, Y) is selected uniformly in the triangle T whose vertices are at the points $(0, 0)$, $(1, 0)$ and $(1, 1)$.

   (a) The random variables X and Y have distinct distributions, but $E(X) = E(Y)$.

   (b) The distribution function $F_{X,Y}$ is linear on the triangle T. Thus, for example, since $F_{X,Y}(0, 0) = 0$ and $F_{X,Y}(1, 1) = 1$, we have $F_{X,Y}(0.5, 0.5) = 0.5$.

   (c) $V(X) = \frac{1}{18}$.

   (d) There exist positive constants c_1 and c_2 such that the random variables c_1X and c_2Y are independent. However, X and Y themselves are dependent.

   (e) With probability 1 we have $X > Y$. Consequently, $E(X) > E(Y)$ and $V(X) > V(Y)$.

   (f) The random variable X/Y is of infinite expectation.

   (g) None of the above.

Solutions

1. Denote by Y the total number of coins tossed. We may view Y as counting the number of times a coin is tossed until it shows a tail n times. Therefore $Y \sim \overline{B}(n, 1/2)$, and in particular $E(Y) = \frac{n}{1/2} = 2n$. As $X = Y - n$, we have $E(X) = n$.

   The probability of each coin to show a tail at least once during the first l tosses is $1 - \frac{1}{2^l}$, and therefore the probability of all to show a tail within that period is $\left(1 - \frac{1}{2^l}\right)^n$. In other words

$$F_S(l) = \left(1 - \frac{1}{2^l}\right)^n,$$

and in particular

$$P(S = 2) = F_S(2) - F_S(1) = \left(\frac{3}{4}\right)^n - \left(\frac{1}{2}\right)^n .$$

Now

$$P(X = k|S = 2) = \frac{P(X = k, S = 2)}{P(S = 2)} .$$

The event $\{X = k, S = 2\}$ occurs if exactly k of the n coins show a head at the first toss, and all those k coins show a tail next time. Hence

$$P(X = k, S = 2) = \binom{n}{k} \frac{1}{2^n} \cdot \frac{1}{2^k} ,$$

which yields

$$P(X = k|S = 2) = \frac{\binom{n}{k}\frac{1}{2^n} \cdot \frac{1}{2^k}}{\left(\frac{3}{4}\right)^n - \left(\frac{1}{2}\right)^n} = \binom{n}{k} \cdot \frac{1/2^k}{(3/2)^n - 1} .$$

For $n = 2$ we calculate $E(S)$ as follows:

$$E(S) = \sum_{l=1}^{\infty} l \left(\left(1 - \frac{1}{2^l}\right)^2 - \left(1 - \frac{1}{2^{l-1}}\right)^2 \right) = 2 \sum_{l=1}^{\infty} l \cdot \frac{1}{2^l} - \sum_{l=1}^{\infty} l \cdot \frac{3}{4^l} .$$

We recognize the first sum on the right hand side as giving the expectation of a $G(1/2)$-distributed random variable, and the second sum as giving the expectation of a $G(3/4)$-distributed random variable. Consequently:

$$E(S) = 2 \cdot 2 - \frac{4}{3} = \frac{8}{3} .$$

Thus, (a), (b), (d), (e) and (f) are true.

2. For the tree to be of maximal length $n-1$, it is necessary and sufficient that each added node will increase its height. In other words, every new element has to make all the way to the current unique leaf of the tree before being inserted as a left or a right child of this leaf. Thus, either all elements are larger than the root or all are smaller than it, which means that the root is either 1 or n. Similarly, at each stage (up

to the second last stage) the number adjoined to the tree is either the maximum of the remaining numbers or their minimum. It follows that the event $H = n-1$ consists of 2^{n-1} initial orderings of the numbers $1,\ 2, \ldots,\ n$, so that $P(H = n-1) = \frac{2^{n-1}}{n!}$.

Each of the numbers $1,\ 2, \ldots,\ n$ has the same probability $\frac{1}{n}$ of being the first in the initial ordering, and therefore at the root, which implies that $R \sim U[1, n]$. In particular $V(R) = \frac{n^2-1}{12}$.

We have $S_{\text{left}} = 1 + 2 + \ldots + (R-1) = \frac{R(R-1)}{2}$. Consequently:

$$\begin{aligned} E\left(S_{\text{left}}\right) &= E\left(\frac{R(R-1)}{2}\right) = \frac{1}{2}E(R^2 - R) \\ &= \frac{1}{2} \cdot \left(V(R) + E(R)^2 - E(R)\right) \\ &= \frac{1}{2} \cdot \left(\frac{n^2-1}{12} + \left(\frac{n+1}{2}\right)^2 - \frac{n+1}{2}\right) = \frac{n^2-1}{6} . \end{aligned}$$

Clearly, $S_{\text{right}} = \frac{n(n+1)}{2} - R - S_{\text{left}}$, which implies:

$$E\left(S_{\text{right}}\right) = \frac{n(n+1)}{2} - E(R) - E\left(S_{\text{left}}\right) = \frac{n^2-1}{3} .$$

Similarly to the discussion at the beginning of the answer, we can see that any binary tree of length $n-1$ is obtained from a unique initial ordering of the numbers. Hence each such tree is obtained with probability $\frac{1}{n!}$. However, taking any ordering starting with $2,\ 1,\ 3$, and changing it so that it starts with $2,\ 3,\ 1$ (leaving the order of all other numbers intact), we obviously obtain the same tree from both orderings, so that the probability of that tree is at least $\frac{2}{n!}$. (An alternative to the second part of the argument is to notice that the number of binary trees on n nodes, which is $\frac{1}{n+1}\binom{2n}{n}$, is much smaller than $n!$, so some trees must be obtained with a probability exceeding $\frac{1}{n!}$.) Hence for $n \geq 3$ not all trees are obtained with the same probability. However, for $n = 2$ the two possible trees are obtained with a probability $1/2$ each.

The second number in the sequence is placed at the left child of the root if and only if it is smaller than the first number in the sequence.

By symmetry, this gives $P(A) = \frac{1}{2}$. Hence:

$$P(R=j|A) = \frac{P((R=j) \bigcap A)}{P(A)} = \frac{\frac{1}{n} \cdot \frac{j-1}{n-1}}{\frac{1}{2}} = \frac{2(j-1)}{n(n-1)} \ .$$

Thus, only (a), (c) and (f) are true.

3. The expectation of an Exp(θ)-distributed random variable is $\frac{1}{\theta}$, so that the expected lifelength of Light++ light-bulbs is $\frac{1}{2}$ a year, or 6 months, while that of Bite & Lite light-bulbs is only $\frac{1}{3}$ of a year, or 4 months. (Moreover, the distribution function of the lifelength of Light++ bulbs is smaller than that of the lifelength of Bite & Lite bulbs throughout the positive axis, so that the first distribution assumes generally larger values than the second.) Hence, if the main consideration is the frequency of replacing bulbs, Light++ is the better firm.

   Buying Light++ bulbs, one consumes on the average 2 bulbs a year, for which he pays 2 shekels. Similarly, with Bite & Lite bulbs, the average expense per year is $3b$ shekels. Hence, from the point of view of expenses, one should prefer Bite & Lite bulbs if and only if $3b < 2$, namely $b < \frac{2}{3}$.

   Now:
$$P(X > x) = \frac{1}{2} \cdot e^{-2x} + \frac{1}{2} \cdot e^{-3x} \qquad x \geq 0.$$
   Hence
$$F_X(x) = 1 - \frac{1}{2} \cdot e^{-2x} - \frac{1}{2} \cdot e^{-3x}$$
   and
$$f_X(x) = e^{-2x} + \frac{3}{2} \cdot e^{-3x}$$
   throughout the positive x-axis. Hence
$$E(X) = \int_0^\infty x \left(e^{-2x} + \frac{1}{2} \cdot e^{-3x} \right) dx = \frac{5}{12} \ ,$$
   so that the expected lifetime of a random light-bulb is $\frac{5}{12}$ years, or 5 months. Similarly
$$E(X^2) = \int_0^\infty x^2 \left(e^{-2x} + \frac{1}{2} \cdot e^{-3x} \right) dx = \frac{13}{36} \ ,$$

and therefore

$$V(X) = \frac{13}{36} - \frac{25}{144} = \frac{3}{16}\,.$$

Since for $s, t \geq 0$ we have

$$P(X \geq s+t) = \frac{1}{2} \cdot e^{-2(s+t)} + \frac{1}{2} \cdot e^{-3(s+t)},$$

while

$$P(X \geq s)P(X \geq t) = \left(\frac{1}{2} \cdot e^{-2s} + \frac{1}{2} \cdot e^{-3s}\right) \cdot \left(\frac{1}{2} \cdot e^{-2t} + \frac{1}{2} \cdot e^{-3t}\right),$$

the distribution of X is not memoryless.

Thus, only (d) and (f) are true.

4. Denoting by A the event that the chosen day is with bad weather, we obtain

$$\begin{aligned} P_{X,Y}(k,l) &= P(A) \cdot P(X = k, Y = l \,|\, A) \\ &\quad + P(\overline{A}) \cdot P(X = k, Y = l \,|\, \overline{A}) \\ &= \frac{1}{2} \cdot \frac{1^k}{k!} e^{-1} \cdot \frac{2^l}{l!} e^{-2} + \frac{1}{2} \cdot \frac{2^k}{k!} e^{-2} \cdot \frac{3^l}{l!} e^{-3}. \end{aligned}$$

Thus:

$$P_{X,Y}(l,k) = \frac{1}{2} \cdot \frac{1^l}{l!} e^{-1} \cdot \frac{2^k}{k!} e^{-2} + \frac{1}{2} \cdot \frac{2^l}{l!} e^{-2} \cdot \frac{3^k}{k!} e^{-3}.$$

If $l > k \geq 0$, then the two terms on the right hand side of this equality are smaller than those of the preceding equality (by factors of 2^{l-k} and $(3/2)^{l-k}$, respectively). Hence $P_{X,Y}(k,l) > P_{X,Y}(l,k)$ in this case.

Obviously

$$P_X(k) = \sum_{l=0}^{\infty} P_{X,Y}(k,l) = \frac{1}{2} \cdot \frac{1^k}{k!} e^{-1} + \frac{1}{2} \cdot \frac{2^k}{k!} e^{-2}, \qquad k = 0, 1, \dots\ ,$$

and:

$$P_Y(l) = \sum_{k=0}^{\infty} P_{X,Y}(k,l) = \frac{1}{2} \cdot \frac{2^l}{l!} e^{-2} + \frac{1}{2} \cdot \frac{3^l}{l!} e^{-3}, \qquad l = 0, 1, \dots\ .$$

In calculating $E(X)$ and $E(X^2)$ we shall use the results regarding the analogous series for a Poissonian distribution:

$$E(X) = \sum_{k=0}^{\infty} k \left(\frac{1}{2} \cdot \frac{1^k}{k!} e^{-1} + \frac{1}{2} \cdot \frac{2^k}{k!} e^{-2} \right) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = \frac{3}{2} \,.$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \left(\frac{1}{2} \cdot \frac{1^k}{k!} e^{-1} + \frac{1}{2} \cdot \frac{2^k}{k!} e^{-2} \right) = \frac{1}{2} \cdot (1^2+1) + \frac{1}{2} \cdot (2^2+2) = 4.$$

Therefore:

$$V(X) = 4 - \left(\frac{3}{2} \right)^2 = \frac{7}{4} \,.$$

Analogously

$$E(Y) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 3 = \frac{5}{2} \,,$$

$$E(Y^2) = \frac{1}{2} \cdot (2^2 + 2) + \frac{1}{2} \cdot (3^2 + 3) = 9,$$

and

$$V(Y) = 9 - \left(\frac{5}{2} \right)^2 = \frac{11}{4} \,.$$

Since (X, Y) assumes with positive probability values (k, l) with $k \geq l$, the inequality $X < Y$ is satisfied only on part of the sample space (although it is quite conceivable that the probability for this event exceeds $\frac{1}{2}$). The same implies that $Y - X$ assumes with positive probability any integer value, and in particular this random variable is not Poisson distributed.

A large value of X indicates that the chosen day is one with good weather, so that Y also should assume a relatively large value. Hence $\rho(X, Y) > 0$. On the other hand, the value of Y is not determined by the value of X, and in particular the two random variables are not linear functions of each other. Hence $\rho(X, Y) < 1$. (In fact, one can compute the correlation coefficient. We easily find that $E(XY) = 4$, conclude that $\mathrm{Cov}(X, Y) = \frac{1}{4}$, and finally $\rho(X, Y) = \frac{1}{\sqrt{77}}$.)

Thus, only (b, (e) and (f) are true.

5. Since the total area of T is $1/2$, the probability for (X, Y) to assume a value in any subset A of T is twice the area of A. This gives easily:

$$F_{X,Y}(0.5, 0.5) = P(X \leq 0.5, Y \leq 0.5) = 2 \cdot \frac{0.5^2}{2} = 0.25.$$

Also

$$F_X(x) = 2 \cdot \frac{x^2}{2} = x^2, \qquad 0 \leq x \leq 1,$$

and

$$F_Y(y) = 2 \cdot y \cdot \frac{1 + (1-y)}{2} = 2y - y^2, \qquad 0 \leq y \leq 1.$$

(Clearly, both F_X and F_Y vanish on $(-\infty, 0)$ and assume the value 1 on $(1, \infty)$. A similar comment applies to the density functions f_X and f_Y, to be defined momentarily.) Hence

$$f_X(x) = 2x, \qquad 0 \leq x \leq 1,$$

and

$$f_Y(y) = 2 - 2y, \qquad 0 \leq y \leq 1.$$

Now calculate the expectation and variance of X and Y:

$$E(X) = \int_0^1 x \cdot 2x \, dx = \frac{2}{3} \, .$$

$$E(X^2) = \int_0^1 x^2 \cdot 2x \, dx = \frac{2}{4} = \frac{1}{2} \, .$$

$$V(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18} \, .$$

$$E(Y) = \int_0^1 y \cdot (2 - 2y) \, dy = \frac{1}{3} \, .$$

$$E(Y^2) = \int_0^1 y^2 \cdot (2 - 2y) \, dy = \frac{1}{6} \, .$$

$$V(Y) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18} \, .$$

(Some of the calculations could be avoided by noting that the transformation $(x, y) \mapsto (1-x, 1-y)$ takes T to a symmetric triangle, thereby switching the roles of X and Y. Therefore, X and $1-Y$ have the same distribution, and in particular $E(Y) = 1 - E(X)$ and $V(Y) = V(X)$.)

X and Y are dependent, as we can see, for example, from the two equalities

$$P(X \in [0, 1/2], Y \in [1/2, 1]) = 0$$

and

$$P(X \in [0, 1/2]) \cdot P(Y \in [1/2, 1]) = \frac{1}{16} .$$

Since functions of independent variables are also independent, if $c_1 X$ and $c_2 Y$ were independent for some c_1 and c_2, then X and Y themselves would be independent.

To deal with the random variable $T = X/Y$, first find its distribution function:

$$\begin{aligned} F_T(t) &= P\left(\frac{X}{Y} \le t\right) = P\left(Y \ge \frac{X}{t}\right) \\ &= 2 \cdot \frac{1 \cdot \left(1 - \frac{1}{t}\right)}{2} = 1 - \frac{1}{t}, \qquad 1 \le t < \infty. \end{aligned}$$

Hence

$$f_T(t) = \begin{cases} 1/t^2, & t \ge 1, \\ 0, & \text{otherwise}, \end{cases}$$

so that

$$E(T) = \int_1^\infty t \cdot \frac{1}{t^2}\, dt = \int_1^\infty \frac{1}{t}\, dt = \infty.$$

Thus, only (c) and (f) are true.