Quiz Section 4 - Best approximation and orthogonal projection

Theorem:

Let V be an inner-product space, $W \subset V$ a subspace,. Then $u^* \in W$ is the best approximation of $u \in V$ within W if and only if $(u - u^*) \perp W$, that is $\langle u - u^*, w \rangle = 0 \ \forall w \in W$.

Definition:

Let V be an inner-product space, $W \subset V$ a subspace, and $u \in V$. Then a vector $u^* \in W$ satisfying $(u - u^*) \perp W$ is called the *orthogonal projection* of u into W.

Theorem:

Let V be an inner-product space, and $W \subset V$ a subspace of dimension n, spanned by the orthonormal system $\{e_1, \ldots, e_n\}$. Then $\forall u \in V$ the orthogonal projection of u into W is given by:

$$u^* = \sum_{i=1}^{n} \langle u, e_i \rangle e_i.$$

Proof:

We will show that u^* is indeed the orthogonal projection, that is, $\langle u - u^*, w \rangle = 0 \ \forall w \in W$.

Let $w \in W$, $W =$span$\{e_1, \ldots, e_n\}$. Then there exist $\alpha_1, \ldots \alpha_n \in \mathbb{C}$ such that $w = \sum_{i=1}^{n} \alpha_i e_i$.

$$\begin{aligned}
\langle u-u^{*},w\rangle &= \left\langle u-\sum_{i=1}^{n}\langle u,e_i\rangle e_i,\sum_{j=1}^{n}\alpha_j e_j\right\rangle \\
&= \sum_{j=1}^{n}\overline{\alpha_j}\left\langle u-\sum_{i=1}^{n}\langle u,e_i\rangle e_i,e_j\right\rangle \\
&= \sum_{j=1}^{n}\overline{\alpha_j}\left(\langle u,e_j\rangle-\left\langle\sum_{i=1}^{n}\langle u,e_i\rangle e_i,e_j\right\rangle\right) \\
&= \sum_{j=1}^{n}\overline{\alpha_j}\left(\langle u,e_j\rangle-\sum_{i=1}^{n}\langle u,e_i\rangle\cdot\langle e_i,e_j\rangle\right) \\
&\underset{*}{=} \sum_{j=1}^{n}\overline{\alpha_j}\left(\langle u,e_j\rangle-\langle u,e_j\rangle\right) \\
&= \sum_{j=1}^{n}\overline{\alpha_j}\cdot 0 \\
&= 0.
\end{aligned}$$

The transition marked with * is justified by the fact that $\{e_1,\ldots e_n\}$ is an orthonormal system, namely:

$$\langle e_i,e_j\rangle=\begin{cases}1, & i=j.\\ 0, & i\neq j.\end{cases}$$

This proves the theorem.

Exercise 1 (From a quiz in 2008)

Let V be the set of continuous functions from $\mathbb{R}$ to $\mathbb{C}$ with the property:

$$\int_{\mathbb{R}}|f(x)|^2 e^{-x^2}dx<\infty$$

1. Prove that V is a vector space (with the standard operations of addition and multiplication by a scalar), and that V is an inner-product space with the inner product:

$$\langle f,g\rangle=\int_{\mathbb{R}}f(x)\,\overline{g(x)}e^{-x^2}dx$$

2. Prove that all polynomials belong to V.
3. Find the orthogonal projection of x^3 onto the subspace of polynomials of degree ≤ 2. (Hint: you may use the fact that $\int_{\mathbb{R}}e^{-x^2}=\sqrt{\pi}$)