Midterm #1

Mark all correct answers in each of the following questions.
Σ denotes an arbitrary alphabet unless otherwise specified.

1. (a) For any two words $w_1, w_2 \in \Sigma^*$ we have $(w_1w_2)^2 = w_1^2w_2^2$.

   (b) The equality $(w_1w_2)^2 = w_1^2w_2^2$ holds for infinitely many pairs of words w_1, w_2. There exist alphabets Σ for which the inequality $(w_1w_2)^2 \neq w_1^2w_2^2$ also holds for infinitely many pairs of words w_1, w_2.

   (c) $(w_1w_2)^2 = w_1^2w_2^2$ if and only if $w_1w_2 = w_2w_1$.

   (d) $(w_1w_2)^2 = w_1^2w_2^2$ if and only if $\{w_1\}^*\{w_2\}^* = \{w_1w_2\}^*$.

   (e) None of the above.

   We have $(w_1w_2)^2 = w_1w_2w_1w_2$. Hence the equality $(w_1w_2)^2 = w_1^2w_2^2$ is equivalent to $w_1w_2w_1w_2 = w_1w_1w_2w_2$. In this equality, the first $|w_1|$ letters of both words coincide, and so do the last $|w_2|$ letters. Hence the equality is equivalent to $w_1w_2 = w_2w_1$. Taking all pairs of words w_1, w_2 with $w_1 = w_2$, we have infinitely many pairs for which the equality in question holds. If Σ consists of two or more letters, say $\Sigma = \{a, b\}$, then $(w_1w_2)^2 \neq w_1^2w_2^2$ for $w_1 = a$ and $w_2 = b^n$ with any $n \geq 1$, so that the inequality holds for infinitely many pairs of words. We have $\{w_1\}^*\{w_2\}^* \neq \{w_1w_2\}^*$ for many pairs of commuting words, say $w_1 = w_2$ – any non-empty word.

   Thus, only (b) and (c) are true.

2. (a) The equality $LL_1 = LL_2$ for any languages $L, L_1, L_2 \subseteq \Sigma^*$ implies $L_1 = L_2$.

(b) There exist infinitely many languages $L \subseteq \Sigma^*$ for which the equality $LL_1 = LL_2$ implies $L_1 = L_2$, and there exist infinitely many languages $L \subseteq \Sigma^*$ for which this implication is false.

(c) The claim in part (a) is true if Σ consists of a single letter, but is false if it consists of two or more letters.

(d) The equality $LL_1 = LL_2$ does not necessarily imply $L_1 = L_2$, but it necessarily implies $L_1 L = L_2 L$.

(e) None of the above.

If L consists of a single word, then $LL_1 = LL_2$ obviously implies $L_1 = L_2$. On the other hand, if $\{\varepsilon, a\} \subseteq L \subseteq \{a\}^*$, then for any language L' satisfying $\{a^2\}^* \subseteq L' \subseteq \{a\}^*$ we have $LL' = \{a\}^*$. If L, L_1 and L_2 correspond to the regular expressions a^*, b^* and a^*b^*, respectively, then $LL_1 = LL_2$ yet $L_1 L \neq L_2 L$.

Thus, only (b) is true.

3. Let $\mathcal{L}_0$ be the minimal collection of languages over Σ which includes the language $\emptyset$ and the languages $\{\sigma\}$ for $\sigma \in \Sigma$, and is closed under union, intersection and concatenation. $\mathcal{L}_1$ is similarly defined, with the condition that it is also closed under complementation. $\mathcal{L}_{\text{Reg}}$ denotes the collection of regular languages.

   (a) $\mathcal{L}_0$ is the collection of all finite languages, $\mathcal{L}_1$ is the collection of all languages L such that either L or $\Sigma^* - L$ is finite, and $\mathcal{L}_0 \subsetneqq \mathcal{L}_1 \subsetneqq \mathcal{L}_{\text{Reg}}$.

   (b) $\mathcal{L}_0$ is the collection of all finite languages, and $\mathcal{L}_0 = \mathcal{L}_1 \subsetneqq \mathcal{L}_{\text{Reg}}$.

   (c) $\mathcal{L}_0$ contains properly the collection of all finite languages, and $\mathcal{L}_0 = \mathcal{L}_1 \subsetneqq \mathcal{L}_{\text{Reg}}$.

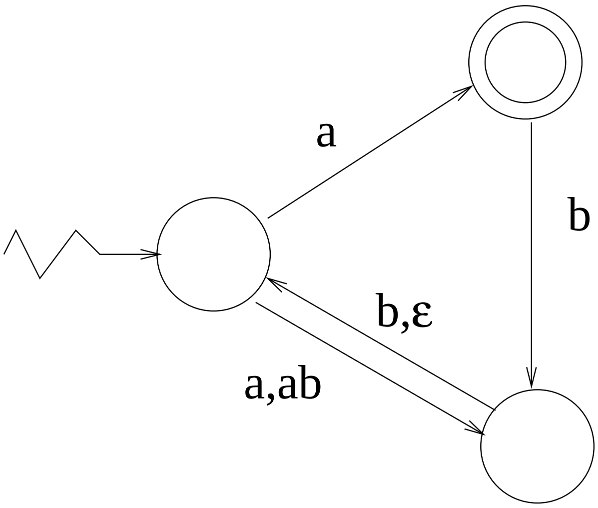
   (d) $\mathcal{L}_0, \mathcal{L}_1$ are both finite collections of languages, and $\mathcal{L}_0 \subsetneqq \mathcal{L}_1$.

   (e) None of the above.

The closure properties imply that every language consisting of a single non-empty word belongs to $\mathcal{L}_0$. Hence every finite language consisting of non-empty words belongs to $\mathcal{L}_0$. The minimality ensures that $\mathcal{L}_0$ is in fact exactly this collection of languages. Similarly, $\mathcal{L}_1$ consists of all these languages and their complements.

Thus, only (e) is true.

4. Let M be the following automaton:



Denote $L = L\left((a \cup ab)(\phi^* \cup b)\right)$. Then:

(a) $L(M) \subsetneqq L^* a\,(bL^*a)^*$.

(b) $L(M) = L^* a\,(bL^*a)^*$.

(c) $L(M) \supsetneqq L^* a\,(bL^*a)^*$.

(d) $L(M) \supsetneqq \{a\}\{(ba)^n : n \geq 0\}$.

(e) None of the above.

Denote the initial state by q_1, the accepting state by q_2 and the third state by q_3. A word in $L(M)$ is a word which may lead from q_1 to q_2. Such a word is formed by concatenating any number of words, each of which may lead from q_1 to itself without passing through q_1 in the middle, and then a word which may lead from q_1 to q_2 without passing through q_1 in the middle. Now a simple check shows that a word may lead from q_1 to itself without passing through q_1 in the middle if and only if it belongs to L. Hence $L(M) = L^*a$, and in particular $L(M) \subseteq L^*a\,(bL^*a)^*$ and $L(M) \supsetneqq \{a\}\{(ba)^n : n \geq 0\}$.

Since $ab \in L$, for any $k \geq 1$ we have:

$$L^*a\,(bL^*a)^k = (L^*ab)^k\,a \subseteq (L^*)^k\,a = L^*a\,.$$

Thus, only (b) and (d) are true.

5. Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA, and let $L = L(M)$.

   (a) If $A = \{s\}$, then $L^* = L$.

   (b) If $L^* = L$, then $A = \{s\}$.

   (c) If $A = Q$, then $L = \Sigma^*$.

   (d) If $L = \Sigma^*$, then $A = Q$.

   (e) None of the above.

If $A = \{s\}$, then L consists of those words leading from s to itself. Concatenating such words, we obtain again a word leading from s to itself, and therefore any word in L^* is accepted by M. If $A = Q$, then every word leads to an accepting state, and hence $L = \Sigma^*$. The converses are false if there are states in Q to which it is impossible to get by starting at s.

Thus, only (a) and (c) are true.

6. (a) If the two languages L_1 and $L_1 \bigcup L_2$ are accepted by finite automata, then the language L_2 is accepted by a finite automaton as well.

   (b) If the two languages L_1 and $L_1 \bigcup L_2$ are accepted by finite automata and $L_1 \bigcap L_2 = \emptyset$, then L_2 is accepted by a finite automaton as well.

   (c) If the two languages L_1 and $L_1 L_2$ are accepted by finite automata, then the language L_2 is accepted by a finite automaton as well.

(d) If the two languages L_1 and L_1L_2 are accepted by finite automata and $L_1 \bigcap L_2 = \emptyset$, then L_2 is accepted by a finite automaton as well.

(e) None of the above.

Under the assumptions of (b) we have

$$L_2 = \left(L_1 \bigcup L_2\right) \bigcap (\Sigma^* - L_1) \ ,$$

which proves that L_2 is accepted by a finite automaton. If the assumption $L_1 \bigcap L_2 = \emptyset$ is dropped, then, taking $L_1 = \Sigma^*$, we have $L_1 \bigcup L_2 = \Sigma^*$ for arbitrary L_2. Similarly, if $L_1 = \emptyset$, then $L_1L_2 = \emptyset$ for any L_2.

Thus, only (b) is true.