Course Notes:

Homological Algebra

BGU, Fall 2021-22

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Available here:
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0. Introduction

comment: Start of Lecture 1, 20 Oct 2021

Homological algebra is a generalization of *linear algebra over a field*. The *vector spaces* over a field $\mathbb{K}$ are replaced here by *modules* over a *ring* A, possibly *noncommutative*.

The main feature of linear algebra over a field $\mathbb{K}$ is that *every* $\mathbb{K}$*-module is free*, namely it has a basis. Thus the only invariant of a $\mathbb{K}$-module M is its *rank* (traditionally called the *dimension*), which is the size of a basis of M.

A homomorphism $\phi : M \to N$ of $\mathbb{K}$-modules can be described by a matrix, after we choose bases for M and N.

When A is a commutative ring that is not a field, this is not true. A-modules can be very complictaed.

For example, for $A = \mathbb{Z}$, a $\mathbb{Z}$-module M is just an *abelian group*. We all know that many abelian groups are not free; indeed, every nonzero finite abelian group M is not free.

When the ring A is not commutative, things can become even more complicated.

Homological algebra provides us with strong tools to describe what A-modules look like, and what their homomorphisms look like.

Homological algebra also has methods to describe how some algebraic objects are related to other objects of the same kind.

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Let me preview some results that we will prove using tools of homological algebra, either at the end of this course or in the subsequent course on *commutative algebra*.

Many of the concepts appearing in these statements will not be familiar to you. That's all right. Eventually – later today or in the coming months – everything will be defined and proved.

comment: I probably went too far with this preview – I hope that I will have time during this course to actually teach these things...

Given a commutative ring $\mathbb{K}$ and an integer $n \geq 1$, we denote by $\mathrm{Mat}_n(\mathbb{K})$ the ring of $n \times n$ matrices with entries in $\mathbb{K}$. This is a *central* $\mathbb{K}$*-ring*. Traditionally the name is "unital associative $\mathbb{K}$-algebra".

If A and B are central $\mathbb{K}$-rings, then so is their tensor product $A \otimes_{\mathbb{K}} B$. In case A itself is commutative, then $A \otimes_{\mathbb{K}} B$ is actually a central A-ring.

We now consider the fields $\mathbb{R}$ and $\mathbb{C}$. We say that a central $\mathbb{R}$-ring A is an $\mathbb{R}$-*form of the central* $\mathbb{C}$-*ring* $\operatorname{Mat}_2(\mathbb{C})$ if there is a $\mathbb{C}$-ring isomorphism

$$C \otimes_{\mathbb{R}} A \cong \operatorname{Mat}_2(\mathbb{C}). \tag{0.1}$$

I hope everybody heard about ring of *Hamilton quaternions*. It is a NC central $\mathbb{R}$-ring. As an $\mathbb{R}$-module it free with basis $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The multiplication satisfies $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{i} \cdot \boldsymbol{j} \cdot \boldsymbol{k} = -1$.

Let A be the $\mathbb{R}$-ring $\operatorname{Mat}_2(\mathbb{R})$, and let B be the $\mathbb{R}$-ring of quaternions. Observe that both A abd B are free $\mathbb{R}$-modules of rank 4, so they are *isomorphic as* $\mathbb{R}$-*modules*. But A and B are *not isomorphic as* $\mathbb{R}$-*rings*. Indeed, B is a *division ring*, namely every nonzero $b \in B$ is invertible; whereas there are nonzero matrices $a \in A$ s.t. $a^2 = 0$.

Theorem 0.2. *Let A be the* $\mathbb{R}$-*ring* $\operatorname{Mat}_2(\mathbb{R})$*, and let B be the* $\mathbb{R}$-*ring of quaternions. Then A and B are* $\mathbb{R}$-*forms of the* $\mathbb{C}$-*ring* $\operatorname{Mat}_2(\mathbb{C})$. *Furthermore, every* $\mathbb{R}$-*form of* $\operatorname{Mat}_2(\mathbb{C})$ *is isomorphic to A or to B.*

This classification theorem relies on *group cohomology*. Specifically we will need an analysis of the cohomologies

$$\mathrm{H}^1\big(G, \mathrm{PGL}_2(\mathbb{C})\big) \quad \text{and} \quad \mathrm{H}^2\big(G, \mathrm{GL}_1(\mathbb{C})\big). \tag{0.3}$$

Here G is the Galois group of the field extension $\mathbb{R} \to \mathbb{C}$. Note that the group $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^{\times}$ is abelian, but $\mathrm{PGL}_2(\mathbb{C})$ is not abelian.

This material belongs to topic 10 in the syllabus, and I hope we will reach it.

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comment: The next discussion – from here until formula (0.9) – was not done in class. Please read it, but not too carefully, since it is just a vague description of things to come.

In the course "Commutative Algebra" we will prove the next theorem.

Theorem 0.4. *Let A be a noetherian commutative ring, and let M be a finitely generated A-module. The following three conditions are equivalent*:

(i) *The A-module M is a projective.*
(ii) *For every maximal ideal $\mathfrak{m} \subseteq A$, the $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is free.*
(iii) *The A-module M is flat.*

This theorem belongs to topic 9.

For Theorem 0.4 we shall require the following tools from homological algebra. Given a ring A, left A-modules M, N and an integer $q \geq 0$, there is the *q-th Ext group*

$$\operatorname{Ext}^q_A(M, N). \tag{0.5}$$

It is an abelian group, and it depends *functorially* on M and N.

For $q = 0$ there is a *functorial isomorphism*

$$\operatorname{Ext}^0_A(M, N) \cong \operatorname{Hom}_A(M, N). \tag{0.6}$$

The other tool needed for the proof is this: given a right A-module L and an integer $q \geq 0$, there is an abelian group

$$\operatorname{Tor}^A_q(L, N), \tag{0.7}$$

called the *q-th Tor group*. It depends functorially on L and N.

For $q = 0$ there is a functorial isomorphism

$$\operatorname{Tor}^A_0(L, N) \cong L \otimes_A N. \tag{0.8}$$

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There is a connection between group cohomology and Ext.

Given a group G we can form its *group ring* $A := \mathbb{Z}[G]$.

If M is an abelian group with an action of G on it, then M is a left A-module.

For every $q \in \mathbb{N}$ there is a functorial isomorphism of abelian groups

$$\mathrm{H}^q(G, M) \cong \operatorname{Ext}^q_A(\mathbb{Z}, M). \tag{0.9}$$

Here $\mathbb{Z}$ is given the trivial G action.

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To end the introduction, let me say that $\operatorname{Ext}^q_A(-,-)$ is the q-th right derived functor of $\operatorname{Hom}_A(-,-)$; and $\operatorname{Tor}^A_q(-,-)$ is the q-th left derived functor of $(-) \otimes_A (-)$. These are the derived functors appearing in topic 9.

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Here is a the syllabus for this course. It is tentative: I will change material, and the order of presentation, as I go along. Most of the material is in these older course notes: [Yek1], [Yek2], [Yek4] and [Yek5],

(1) **Review of prior material**. On rings, ideals and modules (including non-commutative rings).
(2) **Categories and functors**. Emphasis on linear categories and functors. (This topic will be introduced gradually, as we go along.)
(3) **Universal constructions**. Free modules, products, direct sums, polynomial rings.
(4) **Tensor products**. Definition, construction and properties.
(5) **Exactness**. Exact sequences and functors.
(6) **Special modules**. Projective, injective and flat modules.
(7) **Complexes of modules**. Operations on complexes, homotopies, the long exact cohomology sequence.
(8) **Resolutions**. Projective, flat and injective resolutions.

(9) **Left and right derived functors**. Applications to commutative algebra.
(10) **Further applications of derived functors**. Classification problems, extensions.
(11) **Morita Theory**. Equivalences of module categories and invertible bimodules.

Some of the material might move to the subsequent course "Commutative Algebra".

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Here are a few words on **administration**.

(1) Read the handout.
(2) You are required to *register* only if you want to get credit for the course.
(3) I expect all the students (registered or not) to *attend all lectures*. In case you must be absent, please send me an *email in advance*.
(4) The *homework* will be assigned as material labeled “exercise” during the lecture (and in the notes). This is often complementary material. You should do it all and submit to me in writing each week. I will usually just indicate in my list who submitted the homework (but sometimes, randomly, I will look at it).
(5) If you want help with homework, or to discuss some other math, you can send me an email. A zoom meeting can also be arranged (by email).

1. Review of Rings and Ideals

Most of this review material should be familiar to you, so I will go over it quickly.

A *ring* is a mathematical structure $(A, 0, 1, +, \cdot)$ consisting of:

- A set A.
- Distinguished elements $0, 1 \in A$, called *zero* and *one* (or the *unit*).
- Binary oprations $+$ and $\cdot$, called addition and multiplication.

The axioms are:

- ▹ The sytem $(A, 0, +)$ is an abelian group. (This is called the *additive group* of A.)
- ▹ Multiplication is associative.
- ▹ Multiplication is distributive on both sides w.r.t. addition.
- ▹ The element 1 is neutral for multiplication.

We usually say that A is a ring, leaving the rest of the structure implicit.

A ring A is called *the zero ring* if $A = \{0\}$.

Exercise 1.1. Show that a ring A is the zero ring iff $1 = 0$ in A.

The ring A is called *commutative* if

$$b \cdot a = a \cdot b \ \text{ for all } \ a, b \in A. \tag{1.2}$$

When we say that a ring A is *noncommutative* (NC), we mean that it is not necessarily commutative. This is a bit confusing.

We will often encounter noncommutative rings, such as the ring of matrices $A = \mathrm{Mat}_n(\mathbb{K})$ where $\mathbb{K}$ is a field and $n \geq 2$. (We already saw the case $n = 2$.)

Exercise 1.3. Let $\mathbb{K}$ be a nonzero commutative ring, and let $A := \mathrm{Mat}_n(\mathbb{K})$ for some integer $n \geq 1$. Show that A is commutative iff $n = 1$.

Remark 1.4. If any of the statements in this review is not clear to you, then try to prove it (or ask some other student, or look it in one of the basic references such as [Art] or [Jac]).

On the other hand, if an exercise seems too easy for you, and you are sure of the answer, then you don't have to solve it – just write "this is too easy for me" as your solution. It is your responsibility to know the answer!

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Let A be a ring. An element $a \in A$ is called *invertible* if there exists an element $b \in A$ satisfying $a \cdot b = b \cdot a = 1$. If such b exists, then it is unique, it is called the *inverse* of a, and is denoted by a^{-1}.

The set of invertible elements of A is denoted by $A^{\times}$. The system $(A^{\times}, 1, \cdot)$ is a group, called the *multiplicative group of A*.

A commutative ring A is called a *field* if it is a nonzero ring, and every nonzero element $a \in A$ is invertible. In other words, A is a field iff $A^{\times} = A - \{0\}$, the set of nonzero elements of A.

The NC analogue of a field is called a *division ring*. The same conditions, except that A is not necessarily commutative. The easiest (and historically first) division ring that's not commutative if the ring of quaternions, which we saw earlier.

Exercise 1.5. Try to prove that the ring B of quaternions is in fact a division ring. If it is hard then look it up in one of the reference. (I don't know a proof. If you find a nice proof, tell it to us in class.)

Exercise 1.6.

(1) Find the group $\mathbb{Z}^{\times}$.
(2) Let $\mathbb{K}$ be a nonzero commutative ring, and let $\mathbb{K}[t]$ be the polynomial ring in one variable over $\mathbb{K}$. Find the group $\mathbb{K}[t]^{\times}$.
(3) Let $A := \mathbb{Z}[\boldsymbol{i}] \subseteq \mathbb{C}$, the subring of $\mathbb{C}$ generated by $\boldsymbol{i} = \sqrt{-1}$. It is called the ring of *Gauss integers*. Find the group $A^{\times}$. (Hint: consider $|a|$.)
(4) Let $\mathbb{K}$ be a field, and let $A := \mathrm{Mat}_n(\mathbb{K})$ for some $n \geq 1$. Find the group $A^{\times}$.
(5) Let $\mathbb{K}$ be a nonzero commutative ring, and let $A := \mathrm{Mat}_n(\mathbb{K})$ for some $n \geq 1$. Find the group $A^{\times}$. (Hint: Cramer's formula.)

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A *subring* B of a ring A is a subset $B \subseteq A$ s.t.

- $0, 1 \in B$.
- $a, b \in B \Rightarrow a + b \in B$.
- $a, b \in B \Rightarrow a \cdot b \in B$.

Thus $(B, 0, 1, +, \cdot)$ is itself a ring.

Let A be a ring. A *left ideal* I in A is a subset $I \subseteq A$ s.t.

- $0 \in I$.
- $a, b \in I \Rightarrow a + b \in I$.
- $a \in A$ and $b \in I \Rightarrow a \cdot b \in I$.

Note that $(I, 0, +)$ is a subgroup of the additive group $(A, 0, +)$.

A *right ideal* of A is defined likewise, except that the last condition is $b \cdot a \in I$.

A *two-sided ideal* of A is a subset $I \subseteq A$ that's both a left and a right ideal.

Of course when A is a commutative ring, all three types of ideals are the same.

comment: End of Lecture 1

comment: Start of Lecture 2, 27 Oct 2021

We continue with the review, a bit faster now.

Given a ring A and subsets $S, T \subseteq A$, we write

$$S + T := \{a + b \mid a \in S, b \in T\} \tag{1.7}$$

and

$$S \cdot T := \Big\{ \sum_{k=1,\dots,n} a_k \cdot b_k \ \Big|\ n \geq 0,\ a_k \in S,\ b_k \in T \Big\}. \tag{1.8}$$

Using this notation, a subset $I \subseteq A$ is a left ideal iff it satisfies $A \cdot I = I$. Likewise I is a right ideal iff $I \cdot A = I$, and I is a two-sided ideal iff $A \cdot I = I \cdot A = I$.

If $I, J \subseteq A$ are left ideals, then so is $I + J$. The same for right ideals and two-sided ideals.

Here is another piece of useful notation.

Definition 1.9. By a *collection* of elements of a set S, indexed by some set X, we mean a function

$$f : X \to S.$$

We usually denote this collection by

$$\boldsymbol{s} = \{s_x\}_{x \in X}$$

where $s_x := f(x) \in S$.

Warning: do not confuse the collection $\{s_x\}_{x \in X}$ with the subset $f(X) \subseteq S$.

Example 1.10. If the indexing set $X = \mathbb{N}$, then the collection $\boldsymbol{s} = \{s_x\}_{x \in X}$ is just a sequence of elements of S, i.e. $\boldsymbol{s} = (s_0, s_1, \dots)$. And if $X = \{1, \dots, n\}$ then $\boldsymbol{s} = (s_1, \dots, s_n)$, an n-tuple.

Exercise 1.11. Given a collection $\{I_x\}_{x \in X}$ of left ideals of A, indexed by a (possibly infinite) set X, try define the set $\sum_{x \in X} I_x \subseteq A$, generalizing formula (1.7). Prove that $\sum_{x \in X} I_x$ is a left ideal of A.

Likewise for collections of right ideals and two-sided ideals.

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Given an element $a \in A$, the *principal left ideal generated by a* is

$$A \cdot a := A \cdot \{a\}$$

in terms of formula (1.8). Thus

$$A \cdot a = \{b \cdot a \mid b \in A\}.$$

Given a subset $S \subseteq A$, possibly infinite, the *left ideal of A generated by S* is $A \cdot S \subseteq A$.

Note that

$$A \cdot S = \sum_{a \in S} A \cdot a,$$

i.e. it is the sum of the corresponding principal left ideals.

Similarly we can talk about the *right ideal of A generated by S*, which is $S \cdot A \subseteq S$.

The *two-sided ideal of A generated by S* is

$$A \cdot S \cdot A := (A \cdot S) \cdot A = A \cdot (S \cdot A) \subseteq A.$$

Definition 1.12. Let A be a ring. The *opposite ring of A* is the ring A^{op} define as follows: The underlying abelian group of A^{op}, and its unit element, are the same as those of A.

The multiplication $\cdot^{\mathrm{op}}$ of A^{op} is reversed:

$$a \cdot^{\mathrm{op}} b := b \cdot a$$

for all $a, b \in A$.

Exercise 1.13. Let A be a ring.

(1) Verify that A^{op} is indeed a ring.
(2) Show that A is commutative iff $A = A^{\mathrm{op}}$.
(3) Let I be a right ideal of A. Prove that the abelian subgroup $I^{\mathrm{op}} := I \subseteq A^{\mathrm{op}}$ is a left ideal of the ring A^{op}.

Exercise 1.14. Let $\mathbb{K}$ be a nonzero commutative ring, let $r \geq 1$, and let $A := \mathrm{M}_r(\mathbb{K})$. For a matrix $a \in A$ we denote its transpose by a^{t}. Show that $a \mapsto a^{\mathrm{t}}$ is a ring isomorphism $A \xrightarrow{\simeq} A^{\mathrm{op}}$.

Warning: there are NC rings A for which A is not isomorphic to A^{op}.

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Given a two-sided ideal $I \subseteq A$, the quotient abelian group

$$\bar{A} := A/I$$

has a ring structure:

- ▹ The unit and zero elements are $1_{\bar{A}} := 1_A + I$ and $0_{\bar{A}} := 0_A + I$.
- ▹ Addition is
$$\bar{a} + \bar{b} := \overline{a + b}$$
where $a, b \in A$, and $\bar{a} := a + I$ etc.
- ▹ Multiplication is
$$\bar{a} \cdot \bar{b} := \overline{a \cdot b}.$$

We call $\bar{A}$ the *quotient ring of A modulo I*.

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Let A and B be rings. A *ring homomorphism*

$$f : A \to B$$

is a function f satisfying:

- $f(0_A) = 0_B$.
- $f(1_A) = 1_B$.
- $f(a_1 + a_2) = f(a_1) + f(a_2)$ for all $a_1, a_2 \in A$.
- $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2)$ for all $a_1, a_2 \in A$..

Note that a ring homomorphism $f : A \to B$ induces group homomorphisms

$$f : (A, 0_A, +) \to (B, 0_B, +)$$

and

$$f : (A^{\times}, 1_A, \cdot) \to (B^{\times}, 1_B, \cdot).$$

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Let $f : A \to B$ be a ring homomorphism. The *kernel* of f is the set

$$\operatorname{Ker}(f) := \{a \in A \mid f(a) = 0_B\} \subseteq A.$$

It is a two-sided ideal of A.

The *image* of f is the set

$$\operatorname{Im}(f) := f(A) \subseteq B.$$

It is a subring of B.

The ring homomorphism f induces a *ring isomorphism*

$$\bar{f} : A/\operatorname{Ker}(f) \xrightarrow{\simeq} \operatorname{Im}(f). \tag{1.15}$$

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Here is some terminology regarding functions. A function $f : X \to Y$ between sets is called:

- *injective* if it is "one-to-one", i.e. $\big(f(x_1) = f(x_2)\big) \Rightarrow (x_1 = x_2)$. Notation: $\rightarrowtail$.
- *surjective* if it is "onto", i.e. $f(X) = Y$. Notation: $\twoheadrightarrow$.
- *bijective* if it is both injective and surjective. Notation: $\xrightarrow{\simeq}$.

We know that a function $f : X \to Y$ is bijective iff it has an inverse, namely a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Here id_X is the identity automorphism of X, etc. In this case we write $g := f^{-1}$.

We sometimes use the notation f^{-1} for the *preimage*. For a subset $Y' \subseteq Y$ its preimage is

$$f^{-1}(Y') := \{x \in X \mid f(x) \in Y'\}.$$

Example 1.16.

(1) If B is a subring of A, then the inclusion $f : B \to A$ is an injective ring homomorphism.

(2) If I is a two-sided ideal of A, and $B := A/I$ is the quotient ring, then the function $\pi : A \to B$, $\pi(a) := a + I$, is called the *canonical projection*, and it is a surjective ring homomorphism.

(3) Let $h : A \to B$ be a ring isomorphism (i.e. a ring homomorphism that is bijective as a function). Then the inverse function $h^{-1} : B \to A$ is also a ring isomorphism.

Exercise 1.17. Let $f : A \to B$ be a surjective ring homomorphism with $I := \operatorname{Ker}(f)$.

(1) Show that the rule $J \mapsto f^{-1}(J)$ give a bijection of sets

$$\{\text{left ideals of } B\} \xrightarrow{\simeq} \{\text{left ideals of } A \text{ that contain } I\}.$$

This bijection preserves inclusions of left ideals.

(2) The same for right ideals and two-sided ideals.

Exercise 1.18. Given a ring A, show that there is a unique ring homomorphism $\mathbb{Z} \to A$.

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Definition 1.19. A ring A is called a *simple ring* if it is nonzero, and the only two-sided ideals in it are 0 and A.

Example 1.20. A commutative ring A is a field iff it is a simple ring. This is false for NC rings, as we shall see in Exer 1.22 below.

Exercise 1.21. Let $f : A \to B$ be a ring homomorphism. Assume A is a simple ring and B is nonzero. Show that f is injective.

Exercise 1.22. Let $\mathbb{K}$ be a field, and let $A := \operatorname{Mat}_n(\mathbb{K})$, the ring of $n \times n$ matrices for some $n \geq 1$. Prove that A is a simple ring.

Definition 1.23. Let A be a ring. The *center of* A is the subset

$$\operatorname{Cent}(A) := \{a \in A \mid a \cdot b = b \cdot a \text{ for all } b \in B\}.$$

The center of A is a commutative subring of A. Of course A is commutative iff $A = \operatorname{Cent}(A)$.

Definition 1.24. A ring homomorphism is called *central* if $f(\operatorname{Cent}(A)) \subseteq \operatorname{Cent}(B)$.

Exercise 1.25.

(1) Find a central ring homomorphism $f : A \to B$ between rings that are both not commutative.
(2) Find a ring homomorphism $f : A \to B$ such that A is commutative but f is not central. (Hint: look for a NC ring B and a commutative subring $A \subseteq B$ that's not in the center of B.)

2. Review of Modules

comment: (211103 AY) This next portion was deleted from my copy of the notes. I hope that I have restored it correctly...

Let A be a ring. A *left A-module* is an abelian group $(M, 0, +)$, together with a function

$$A \times M \to M, \quad (a, m) \mapsto a \cdot m$$

called *multiplication*, satisfying these axioms:

- ▹ Associativity:
$$a \cdot (b \cdot m) = (a \cdot b) \cdot m$$
for all $a, b \in A$ and $m \in M$.
- ▹ Distributivity on both sides:
$$a \cdot (m + n) = (a \cdot m) + (a \cdot n)$$
and
$$(a + b) \cdot m = (a \cdot m) + (b \cdot m)$$
for all $a, b \in A$ and $m, n \in M$.
- ▹ Neutrality of one: $1 \cdot m = m$ for all $m \in M$.

Note that in these formulas the symbols $\cdot$ and $+$ refer both to the operations of A and to the operations of M.

A *submodule* N of an A-module M is an abelian subgroup $N \subseteq M$ that is closed under multiplication by elements of A. In other words, if $A \cdot N = N$.

Example 2.1. Let A be a ring. The left ideals of A are precisely the submodules of A, when A is viewed as a left module over itself.

Right modules are defined similarly. The operation is

$$M \times A \to M, \quad (m, a) \mapsto m \cdot a,$$

and the axioms are modified accordingly.

Let A and B be rings. An *A-B-bimodule* is an abelian group M, equipped with a left A-module structure and a right B-module structure, such that

$$a \cdot (m \cdot b) = (a \cdot m) \cdot b$$

for all $a \in A$, $m \in M$ and $b \in B$.

Convention 2.2. In this course all modules are by default left modules.

Exercise 2.3. Let M be a right A-module. Define a left multiplication $*$ of A^{op} on M as follows:

$$a * m := m \cdot a$$

for $a \in A^{\mathrm{op}}$ and $m \in M$, where $\cdot$ is the original right multiplication of A on M. Prove that $*$ makes M into a left A^{op}-module.

This exercise says that we don't really need to worry about right modules; left modules are sufficient (if we allow changing rings).

When we talk about categories later, we will be able to say that "the category of right A-modules is isomorphic to the category of left A^{op}-modules".

If $f : A \to B$ is a ring homomorphism and M is a left B-module, then M becomes a left A-module by this formula:

$$a \cdot m := f(a) \cdot m$$

for $a \in A$ and $m \in M$. This operation is called *restriction of scalars*.

Exercise 2.4. Let M be an A-B-module. Try to find a ring C, with ring homomorphisms $A \to C$ and $B^{\mathrm{op}} \to C$, and with a left C-module structure on M, such that the original left A-module structure and right B-module structure can be recovered from this left C-module structure on M. (We will solve this later; don't think about it too hard now.)

Exercise 2.5. Show that a $\mathbb{Z}$-module is the same as an abelian group.

Example 2.6. If $\mathbb{K}$ is a field, then a $\mathbb{K}$-module is what is traditionally called a vector space.

In this course we won't say "vector space", unless it is in a geometric context. Almost always there won't be one.

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Suppose M and N are A-modules (remember Convention 2.2). A *homomorphism of A-modules* is a function

$$\phi : M \to N$$

such that

$$\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$$

and

$$\phi(a \cdot m) = a \cdot \phi(m)$$

for all $a \in A$ and $m_1, m_2 \in M$.

If $\psi : N \to P$ is another A-module homomorphism, then so is the composition

$$\psi \circ \phi : M \to P.$$

Definition 2.7. Let M and N be A-modules. The set of A-module homomorphisms $\phi : M \to N$ is denoted by $\mathrm{Hom}_A(M, N)$.

Proposition 2.8. *Let M and N be A-modules.*

(1) *The set* $\mathrm{Hom}_A(M, N)$ *is an abelian group, with these operation*:
- *The zero element is the constant function* $0 : M \to N$, $0(m) := 0_N$.
- *Addition is*

$$(\phi_1 + \phi_2)(m) := \phi_1(m) + \phi_2(m).$$

(2) *In case A is a commutative ring, the abelian group* $\mathrm{Hom}_A(M, N)$ *is an A-module, with multiplication*

$$(a \cdot \phi)(m) := a \cdot \phi(m) = \phi(a \cdot m).$$

Exercise 2.9. Prove this proposition.

Definition 2.10. Let M be an A-module. The set of A-module homomorphisms $\phi : M \to M$ is denoted by $\mathrm{End}_A(M)$.

By Proposition 2.8 the set $\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M)$ is an abelian group. But in fact more is true:

Proposition 2.11. *Let M be an A-module.*

(1) *The abelian group* $\mathrm{End}_A(M)$ *is a ring, with unit element* id_M, *and with multiplication*

$$\phi \cdot \psi := \phi \circ \psi.$$

(2) *If A is a commutative ring, then the function* $f : A \to \mathrm{End}_A(M)$, $f(a) := a \cdot \mathrm{id}_M$, *is a central ring homomorphism.*

Proposition 2.12. *Let A be a ring, and let M be an abelian group.*

(1) *Suppose* $f : A \to \mathrm{End}_{\mathbb{Z}}(M)$ *is a ring homomorphism. Then M acquires a left A-module structure with multiplication* $a \cdot m := f(a)(m)$ *for* $a \in A$ *and* $m \in M$.

(2) *Conversely, given a left A-module M, the function* $f : A \to \mathrm{End}_{\mathbb{Z}}(M)$, $f(a)(m) := a \cdot m$, *is a ring homomorphism.*

Exercise 2.13. Prove Proposition 2.12 .

comment: End of Lecture 2

comment: Start of Lecture 3, 3 Nov 2021

We continue with a review of modules. Recall that a ring A is NC, i.e. not necessarily commutative, and an A-module is a left A-module, by default.

Example 2.14. Let A be a nonzero ring, let , and let $M := A^r$, the set of r-tuples of elements of A. The module operations are coordinatewise:

$$(a_1, \cdots, a_r) + (b_1, \cdots, b_r) := (a_1 + b_1, \ldots, a_r + b_r)$$

and

$$b \cdot (a_1, \cdots, a_r) := (b \cdot a_1, \cdots, b \cdot a_r)$$

for $r \geq 0$ and $a_i, b_i, b \in A$.

This is called the *free left A-module of rank r*.

Exercise 2.15. Let A be a ring and r be a positive integer. Prove that there is a canonical ring isomorphism

$$\operatorname{End}_A(M) \cong \operatorname{Mat}_r(A^{\mathrm{op}}).$$

(Hint: write elements of $M = A^r$ as rows, and then matrices in $\operatorname{Mat}_r(A)$ act on them by matrix multiplication from the right.)

Suppose $f : A \to B$ is a ring homomorphism and N is a (left) B-module. Then N becomes an A-module by the formula

$$a \cdot m := f(a) \cdot m \tag{2.16}$$

for $a \in A$ and $m \in M$. This operation is called *restriction*.

Later we will introduce another operation called *induction*, that takes an A-module M and produces a B-module N.

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Given a ring A and an A-module M, an *A-submodule* of M is a subset $N \subseteq M$ s.t. $0 \in N$, and N is closed under addition and multiplication by elements of A.

comment: (211103 AY) Since in class today the next fact was not familiar, it is now an exercise.

Exercise 2.17. Given a ring A and an A-submodule $N \subseteq M$, show that the quotient abelian group $\bar{M} := M/N$ has a unique A-module structure such that the canonical projection $\pi : M \twoheadrightarrow \bar{M}$ is an A-module homomorphism. In other words,

$$a \cdot (m + N) := (a \cdot m) + N$$

for $a \in A$ and $m \in M$.

We call $\bar{M}$ the *quotient module of M modulo N*.

Exercise 2.18. Let A be a ring, $I \subseteq A$ a left ideal, and M a left A-module.

(1) Show that the subset $I \cdot M \subseteq M$ (see (1.8) is a left A-submodule of M.

By Exer 2.17 the abelian group $\bar{M} := M/(I \cdot M)$ is a left A-module, and the canonical projection $\pi_M : M \twoheadrightarrow \bar{M}$ is an A-module homomorphism.

(2) If I is a two-sided ideal we have a ring $\bar{A} := A/I$ with a surjective ring homomorphism $\pi_A : A \twoheadrightarrow \bar{A}$. Prove that the abelian group $\bar{M} = M/(I \cdot M)$ from item (1) has a unique structure of left $\bar{A}$-module, such that every $a \in A$ there is equality

$$\pi_A(a) \cdot \pi_M(m) = \pi_M(a \cdot m)$$

for all $a \in A$ and $m \in M$.

comment: (211103 AY) The things I said today about group actions, in analogy with the previous exercise, were mostly wrong, so please delete from your mental memory (good thing it was not recorded).

Given an A-module homomorphism $\phi : M \to N$, the kernel $\operatorname{Ker}(\phi)$ is a submodule of M, the image $\operatorname{Im}(\phi)$ is a submodule of N, and there is an induced A-module isomorphism

$$M/\operatorname{Ker}(\phi) \xrightarrow{\simeq} \operatorname{Im}(\phi). \tag{2.19}$$

Definition 2.20. Let A be a ring, let M be a left A-module, and let $S \subseteq M$ be a subset. The *A-submodule of M generated by S* is the A-submodule

$$A \cdot S \subseteq M.$$

We are using the notation (1.8).

It is easy to see that $N := A \cdot S$ is a submodule on M using the "telescopic product criterion":

$$A \cdot N = A \cdot (A \cdot S) = (A \cdot A) \cdot S = A \cdot S = N$$

inside M.

Definition 2.21. Let A be a ring and let M be an A-module. We say that M is a *finitely generated A-module* if M is generated as an A-module by some finite subset S.

Very soon we will have a better way to state these generation conditions, using collections instead of subsets.

comment: (211103 AY) The material below was discussed briefly in class.

Everybody seems to have learned Thm 3.2 already. I had a proof typed anyhow, so just read it.

Then please read Thm 3.3 and try to prove it. This is Exer 3.4 and there are hints.

Next week I will define free modules precisely, and will prove Thm 3.7 in class.

3. Universal Constructions for Commutative Rings

Here is the current standing assumption:

Convention 3.1. In this section all rings are *commutative* by default.

This convention will make life much easier. Later we will generalize what can be generalized to NC rings.

Let A be a nonzero ring and M an A-module.

A collection $\boldsymbol{m} = \{m_x\}_{x \in X}$ of elements is called a *basis* of M is it *generates* M and it is *linearly independent*. I will give a precise definition next week, but when A is a field it is the definition you already know.

An A-module M is called *free* if it has a basis.

Theorem 3.2. *If* $\mathbb{K}$ *is a field, then every* $\mathbb{K}$*-module is free.*

Proof. Let M be a $\mathbb{K}$-module. In this proof I and J are sets (instead of X and Y like we had before). We say that a function $\sigma : J \to M$ is linearly independent, or generating, or a basis, if the collection $\boldsymbol{m} = \{m_j\}_{j \in J}$, $m_j := \sigma(j)$, has this property.

Choose a set I with cardinality greated than that of M. Let S be the set of pairs (J, σ), where $J \subseteq I$, and $\sigma : J \to M$ is a linearly independent function. The set S is partially ordered by this relation: $(J, \sigma) \leq (J', \sigma')$ if $J \subseteq J'$ and $\sigma'|_J = \sigma$. The set S is nonempty, because it contains $\sigma : \varnothing \to M$.

Every chain $S' \subseteq S$ has a supremum (J, σ) in S: we take

$$J := \bigcup\nolimits_{(J', \sigma') \in S'} J'$$

and

$$\sigma := \bigcup\nolimits_{(J', \sigma') \in S'} \sigma'.$$

The function $\sigma : J \to M$ is linearly independent, since linear dependence is checked of finite subsets of J.

The assumptions of Zorn's Lemma are satisfied. Therefore S has a maximal element (J, σ). This must be a generating function. Otherwise, there is an element $m^+ \in M$ that's not is the $\mathbb{K}$-submodule $M_\sigma \subseteq M$ generated by σ. Now the cardinality of J satisfies $|J| \leq |M_\sigma| \leq |M| < |I|$. Choose some element $j^+ \in I - J$. Let $J^+ := J \cup \{j^+\}$, and let $\sigma^+ : J^+ \to M$ be the unique function that extends σ and has $\sigma^+(j^+) := m^+$. The usual linear algebra calculation shows that the function σ^+ is linearly independent, so $(J^+, \sigma^+) \in S$. But $(J, \sigma) < (J^+, \sigma^+)$, and this contradicts the maximality of (J, σ). □

In the linear algebra course you learned that when $\mathbb{K}$ is a field, and M is a finitely generated $\mathbb{K}$-module, then any two bases of M have the same cardinality. This cardinality was called the "dimension of M".

It turns out to be true also for infinitely generated $\mathbb{K}$-modules:

Theorem 3.3. *Let $\mathbb{K}$ be a field, let M be a $\mathbb{K}$-module, and let $\boldsymbol{m} = \{m_x\}_{x \in X}$ and $\boldsymbol{n} = \{n_y\}_{y \in Y}$ be collections of elements of M. Assume that $\boldsymbol{m}$ is linearly independent, and that $\boldsymbol{n}$ is generating. Then $|X| \leq |Y|$.*

Exercise 3.4. Prove Thm 3.3. Here are some hints:

(1) Show, by usual linear algebra, that it is enough to consider the case when both X and Y are infinite.

(2) Let $\operatorname{Fin}(Y)$ be the set of finite subsets of Y. Define a function $f : X \to \operatorname{Fin}(Y)$ by letting $f(x) := Y'$, where $Y' \subseteq Y$ is the smallest finite subset such that m_x is in the linear span of the collection $\{n_y\}_{y \in Y'}$.

(3) Show that $|\operatorname{Fin}(Y)| = |Y|$, and that the fibers of f are finite sets. (The fiber of a function $g : X \to Z$ over a point $z \in Z$ is the subset $g^{-1}(z) \subseteq X$.) Conclude that $|X| \leq |Y|$.

An immediate consequence is:

Corollary 3.5. *Let $\mathbb{K}$ be a field, let M be a $\mathbb{K}$-module, and let $\boldsymbol{m} = \{m_x\}_{x \in X}$ and $\boldsymbol{n} = \{n_y\}_{y \in Y}$ be two bases of M. Then $|X| = |Y|$.*

This corollary justifies:

Definition 3.6. Let $\mathbb{K}$ be a field, let M be a $\mathbb{K}$-module, and let $\boldsymbol{m} = \{m_x\}_{x \in X}$ be a basis of M. The *rank* of M, denoted by $\operatorname{rank}_{\mathbb{K}}(M)$, is defined to be the cardinality of X.

Here is a rather surprising generalization of Corollary 3.5.

Theorem 3.7. *Let A be a nonzero ring, let M be a free A-module, and let $\boldsymbol{m} = \{m_i\}_{i \in I}$ and $\boldsymbol{n} = \{n_j\}_{j \in J}$ be two bases of M. Then $|I| = |J|$.*

Recall that our rings are commutative. This result is false for NC rings!

comment: (211103 AY) The proof will be done next week, by reduction to the case of a field.

comment: End of Lecture 3

comment: Start of Lecture 4, 10 Nov 2021

comment: (211111 AY) 2nd revision of notes. Added missing "start of lecture 3". Small improvements.

comment: (211110 AY) We started the lecture by going over Exer 2.15, with matrix multiplication over a NC ring.

We continue with Convention 3.1, namely all rings are commutative by default.

This convention will make life much easier. Later we will generalize what can be generalized to NC rings.

Definition 3.8. Fix a ring A.

(1) An A-*ring* is a ring B equipped with a ring homomorphism $\phi_B : A \to B$, called the structural homomorphism.

(2) Suppose C is another A-ring, with structural homomorphism $\phi_C : A \to C$. An A-*ring homomorphism* $\psi : B \to C$ is a ring homomorphism ψ s.t. $\psi \circ \phi_B = \phi_C$.

In a commutative diagram:

$$\begin{tikzcd} A \arrow[d, "\phi_B"'] \arrow[dr, "\phi_C"] & \\ B \arrow[r, "\psi"] & C \end{tikzcd}$$

Let M be an abelian group and X a set. We already talked about collections $\boldsymbol{m} = \{m_x\}_{x \in X}$ of elements of M indexed by the set X. These are just functions $\phi : X \to M$, and the translation $\boldsymbol{m} \mapsto \phi$ is $\phi(x) = m_x$.

Let us denote by $\mathrm{F}(X, M)$ the set of all such functions.

Suppose A is a ring and M is an A-module. Then $\mathrm{F}(X, A)$ is a ring, and $\mathrm{F}(X, M)$ is an $\mathrm{F}(X, A)$-module. The operations are in the target, e.g. the multiplication of $\phi \in \mathrm{F}(X, A)$ and $\psi \in \mathrm{F}(X, M)$ is

$$(\phi \cdot \psi)(x) := \phi(x) \cdot \psi(x) \in \mathrm{F}(X, M). \tag{3.9}$$

Another way of writing (3.9), in terms of collections, is this:

$$\boldsymbol{a} \cdot \boldsymbol{m} = \{a_x \cdot m_x\}_{x \in X} \tag{3.10}$$

for $\boldsymbol{a} = \{a_x\}_{x \in X} \in \mathrm{F}(X, A)$ and $\boldsymbol{m} = \{m_x\}_{x \in X} \in \mathrm{F}(X, M)$.

There is a ring homomorphism $A \to \mathrm{F}(X, A)$, sending $a \in A$ to the constant function $\phi_a \in \mathrm{F}(X, A)$, $\phi_a(x) = a$.

In this way, using restriction, $\mathrm{F}(X, M)$ becomes an A-module. Explicitly

$$a \cdot \boldsymbol{m} = \{a \cdot m_x\}_{x \in X}. \tag{3.11}$$

Definition 3.12. Let M be an abelian group and X a set.

(1) The *support* of a function $\phi : X \to M$ is the set
$$\mathrm{Supp}(\phi) := \{x \in X \mid \phi(x) \neq 0\} \subseteq X.$$

(2) A function $\phi \in \mathrm{F}(X, M)$ is called *finitely supported* if $\mathrm{Supp}(\phi)$ is a finite set.

(3) The set of finitely supported functions $\phi : X \to M$ is denoted by $\mathrm{F}_{\mathrm{fin}}(X, M)$.

Let A be a ring and M an A-module.

If either $\boldsymbol{a} \in \mathrm{F}(X, A)$ or $\boldsymbol{m} \in \mathrm{F}(X, M)$ is finitely supported, then $\boldsymbol{a} \cdot \boldsymbol{m}$ is finitely supported. Indeed,

$$\mathrm{Supp}(\boldsymbol{a} \cdot \boldsymbol{m}) \subseteq \mathrm{Supp}(\boldsymbol{a}) \cap \mathrm{Supp}(\boldsymbol{m}).$$

If $\boldsymbol{m}, \boldsymbol{n} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ then $\boldsymbol{m} + \boldsymbol{n} \in \mathrm{F}_{\mathrm{fin}}(X, A)$, because

$$\mathrm{Supp}(\boldsymbol{m} + \boldsymbol{n}) \subseteq \mathrm{Supp}(\boldsymbol{m}) \cup \mathrm{Supp}(\boldsymbol{n}).$$

We see that $\mathrm{F}_{\mathrm{fin}}(X, M)$ is an $\mathrm{F}(X, A)$-submodule of $\mathrm{F}(X, M)$.

Therefore $\mathrm{F}_{\mathrm{fin}}(X, M)$ is also an A-submodule of $\mathrm{F}(X, M)$.

comment: (211110 AY) This as far as we got in class today.

comment: (211110 AY) The revision: I changed my mind and included all the material that was in the prelim notes, but in a different order. There are no extra compulsory exercises, but there are many optional ones.

Definition 3.13. Let M be an abelian group and X a set. Given $\boldsymbol{m} = \phi \in \mathrm{F}_{\mathrm{fin}}(X, M)$, its *sum* is the element

$$\sum_X \boldsymbol{m} = \sum_X \phi := \sum_{x \in \mathrm{Supp}(\phi)} \phi(x) = \sum_{x \in \mathrm{Supp}(\boldsymbol{m})} \boldsymbol{m}_x \in M.$$

This makes sense: the sum above is finite.

Moreover, if $X_0 \subseteq X$ is any finite subset containing $\mathrm{Supp}(\boldsymbol{m})$, then

$$\sum_X \boldsymbol{m} = \sum_{x \in X_0} m_x. \tag{3.14}$$

It is easy to see that the function

$$\textstyle\sum_X : \mathrm{F}_{\mathrm{fin}}(X, M) \to M$$

is A-linear.

I am not sure how useful the next notation is. Let's try it.

Definition 3.15. Let $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ and $\boldsymbol{m} \in \mathrm{F}(X, M)$. We define

$$\langle \boldsymbol{a} | \boldsymbol{m} \rangle := \sum_X \boldsymbol{a} \cdot \boldsymbol{m} = \sum_{x \in \mathrm{Supp}(\boldsymbol{a} \cdot \boldsymbol{m})} a_x \cdot m_x \in M.$$

Those who learned physics may recognize the $\langle \mathrm{bra} | \mathrm{ket} \rangle$ notation of Dirac.

Using these new concepts, we can give an elegant and practical way of defining generation of modules, compare Definition 2.20. We can also talk about linear independence of collections.

Definition 3.16. Let A be a nonzero ring, let M be an A-module, and let $\boldsymbol{m} = \{m_x\}_{x \in X}$ be a collection of elements of M indexed by a set X.

(1) We say that the collection $\boldsymbol{m}$ *generates* M as an A-module if for every element $m \in M$ there *exists* some collection $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ such that $m = \langle \boldsymbol{a} | \boldsymbol{m} \rangle$.

(2) We say that the collection $\boldsymbol{m}$ is *linearly independent* over A if the only collection $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ satisfying $\langle \boldsymbol{a} | \boldsymbol{m} \rangle = 0$ is $\boldsymbol{a} = 0$.

(3) The collection $\boldsymbol{m}$ is called a *basis of* M as an A-module if it generates M and it is linearly independent.

Definition 3.17. Let A be a nonzero ring and let M be an A-module. The module M is called a *free A-module* if it has a basis.

The next three results are very similar to things you have learned in linear algebra. So the exercises might be very easy...

Proposition 3.18. *Let $\phi : M \xrightarrow{\simeq} N$ be a homomorphism of A-modules. Assume ϕ is injective (resp. surjective, resp. bijective). Let $\boldsymbol{m} = \{m_x\}_{x \in X} \in \mathrm{F}(X, M)$. Define $n_x := \phi(m_x) \in N$ and $\boldsymbol{n} := \{n_x\}_{x \in X} \in \mathrm{F}(X, N)$. Assume $\boldsymbol{m}$ is linearly independent (resp. generating, resp. a basis) in M. Show that $\boldsymbol{n}$ is linearly independent (resp. generating, resp. a basis) in N.*

Exercise 3.19. Prove Prop 3.18.

Proposition 3.20. *Let A be a nonzero ring, let M be an A-module, and let $\boldsymbol{m} \in \mathrm{F}(X, M)$. The following are equivalent:*

(i) *$\boldsymbol{m}$ is a basis of M (Definition 3.16(3)).*

(ii) *For every element $m \in M$ there is a unique element $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ such that $m = \langle \boldsymbol{a} | \boldsymbol{m} \rangle$.*

Exercise 3.21. Prove Prop 3.20.

The collection $\boldsymbol{a}$ in condition (ii) of Prop 3.20 is called the *coordinates* of m w.r.t. the basis $\boldsymbol{m}$.

Next we have a *universal characterization* of free modules.

Theorem 3.22. *Let A be a nonzero ring, let M be an A-module, and let $\boldsymbol{m} = \{m_x\}_{x \in X} \in \mathrm{F}(X, M)$. The following are equivalent*:

(i) *The collection $\boldsymbol{m}$ is a basis of M.*

(ii) *Let N be an A-module, and let $\boldsymbol{n} = \{n_x\}_{x \in X} \in \mathrm{F}(X, N)$. Then there is a unique A-module homomorphism $\phi : M \to N$ such that $\phi(m_x) = n_x$ for all $x \in X$.*

Exercise 3.23. Prove Thm 3.22.

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Example 3.24. The protoypical free A-module is $M := \mathrm{F}_{\mathrm{fin}}(X, A)$, where X is some set.

As a basis of M we take the collection $\boldsymbol{\delta} := \{\delta_x\}_{x \in X}$, where for every x the *delta function* $\delta_x : X \to A$ is defined by

$$\delta_x(y) := \begin{cases} 1 & \text{if } \ y = x \\ 0 & \text{if } \ y \neq x. \end{cases}$$

Let's prove that $\boldsymbol{\delta}$ is a basis. We need to prove that given an arbitrary element

$$\boldsymbol{m} = \{m_x\}_{x \in X} \in M = \mathrm{F}_{\mathrm{fin}}(X, A)$$

there is a unique element $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ such that

$$\langle \boldsymbol{a} \,|\, \boldsymbol{\delta} \rangle = \boldsymbol{m}.$$

We will actually prove that this unique $\boldsymbol{a}$ is $\boldsymbol{a} = \boldsymbol{m}$.

We go about it indirectly (it is a bit tricky). Take an arbitrary element

$$\boldsymbol{a} = \{a_x\}_{x \in X} \in \mathrm{F}_{\mathrm{fin}}(X, A).$$

Then the function

$$\langle \boldsymbol{a} \,|\, \boldsymbol{\delta} \rangle \in M = \mathrm{F}_{\mathrm{fin}}(X, A)$$

applied to some $y \in X$ is

$$\begin{aligned} \langle \boldsymbol{a} \,|\, \boldsymbol{\delta} \rangle (y) &= \Big(\sum_X \boldsymbol{a} \cdot \boldsymbol{\delta} \Big)(y) = \Big(\sum_{x \in \mathrm{Supp}(\boldsymbol{a} \cdot \boldsymbol{\delta})} a_x \cdot \delta_x \Big)(y) \\ &= \sum_{x \in \mathrm{Supp}(\boldsymbol{a} \cdot \boldsymbol{\delta})} a_x \cdot \delta_x(y) = \sum_{x \in \mathrm{Supp}(\boldsymbol{a}) \cup \{y\}} a_x \cdot \delta_x(y) = a_y = \boldsymbol{a}(y). \end{aligned} \tag{3.25}$$

Here we use that $\mathrm{Supp}(\boldsymbol{a} \cdot \boldsymbol{\delta}) \subseteq \mathrm{Supp}(\boldsymbol{a}) \cup \{y\})$ and the latter is a finite set.

We see that

$$\langle \boldsymbol{a} \,|\, \boldsymbol{\delta} \rangle = \boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A) = M.$$

Therefore, as claimed, given $\boldsymbol{m} \in M$, the unique element $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ such that $\langle \boldsymbol{a} \,|\, \boldsymbol{\delta} \rangle = \boldsymbol{m}$ is $\boldsymbol{a} = \boldsymbol{m}$.

Let M be an A-module and $\boldsymbol{m} = \{m_x\}_{x \in X} \in \mathrm{F}(X, M)$. The A-linear homomorphism

$$\langle - \,|\, \boldsymbol{m} \rangle : \mathrm{F}_{\mathrm{fin}}(X, A) \to M, \quad \boldsymbol{a} \mapsto \langle \boldsymbol{a} \,|\, \boldsymbol{m} \rangle \tag{3.26}$$

sends $\delta_x \mapsto m_x$. This was essentially done in (3.25).

Proposition 3.27. *Let A be a nonzero ring, let M be an A-module, and let $\boldsymbol{m} \in \mathrm{F}(X, M)$. The following are equivalent*:

(i) *$\boldsymbol{m}$ is a basis of M.*

(ii) *The homomorphism $\langle - \,|\, \boldsymbol{m} \rangle$ in formula* (3.26) *is bijective.*

Exercise 3.28 (Optional)**.** Prove this proposition.

Remark 3.29. Later we will see that $X \mapsto \mathrm{F}_{\mathrm{fin}}(X, A)$ is a *functor* from the category Set to the category $\mathsf{Mod}(A)$ of A-modules, and moreover it is a *left adjoint* to the *forgetful functor* $\mathsf{Mod}(A) \to \mathsf{Set}$.

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We now repeat Thm 3.7 and prove it.

Theorem 3.30. *Let A be a nonzero ring, let M be a free A-module, and let $\boldsymbol{m} = \{m_x\}_{x \in X}$ and $\boldsymbol{n} = \{n_y\}_{y \in Y}$ be two bases of M. Then $|X| = |Y|$.*

Proof. Since A is a nonzero ring, it has some maximal ideal $\mathfrak{m} \subsetneq A$. (The proof relies on the axiom of choice.)

The residue field is $\bar{A} := A/\mathfrak{m}$, the quotient $\bar{A}$-module is $\bar{M} := M/(\mathfrak{m} \cdot M)$, and the canonical projections are $\pi_A : A \to \bar{A}$ and $\pi_M : M \to \bar{M}$.

Write $\bar{m}_x := \pi_M(m_x)$ and $\bar{\boldsymbol{m}} := \{\bar{m}_x\}_{x \in X}$. We will prove that $\bar{\boldsymbol{m}}$ is a basis of the $\bar{A}$-module $\bar{M}$,

It is clear that the collection $\bar{\boldsymbol{m}}$ generates $\bar{M}$ as an $\bar{A}$-module. See Proposition 3.18.

We claim that $\bar{\boldsymbol{m}}$ is also linearly independent in $\bar{M}$ over $\bar{A}$.

Indeed, suppose $\langle \bar{\boldsymbol{a}} \,|\, \bar{\boldsymbol{m}} \rangle = 0$ for some collection of coefficients $\bar{\boldsymbol{a}} = \{\bar{a}_x\}_{x \in X} \in \mathrm{F}_{\mathrm{fin}}(X, \bar{A})$.

For every x s.t. $\bar{a}_x \neq 0$ choose some lifting $a_x \in A$, i.e. $\pi_A(a_x) = \bar{a}_x$. For every x s.t. $\bar{a}_x = 0$ take $a_x := 0 \in A$. We get a collection $\boldsymbol{a} = \{a_x\}_{x \in X} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ s.t. $\pi_M(\boldsymbol{a}) = \bar{\boldsymbol{a}}$ and $\mathrm{Supp}(\boldsymbol{a}) = \mathrm{Supp}(\bar{\boldsymbol{a}})$.

Let's calculate:

$$\pi_M(\langle \boldsymbol{a} \,|\, \boldsymbol{m} \rangle) = \pi_M\Big(\sum_{x \in \mathrm{Supp}(\bar{\boldsymbol{a}})} a_x \cdot m_x \Big) = \sum_{x \in \mathrm{Supp}(\bar{\boldsymbol{a}})} \bar{a}_x \cdot \bar{m}_x = \langle \bar{\boldsymbol{a}} \,|\, \bar{\boldsymbol{m}} \rangle = 0.$$

We see that

$$\langle \boldsymbol{a} \,|\, \boldsymbol{m} \rangle \in \operatorname{Ker}(\pi_M) = \mathfrak{m} \cdot M.$$

By the definition of $\mathfrak{m} \cdot M$ we know that $\langle \boldsymbol{a} \,|\, \boldsymbol{m} \rangle = \langle \boldsymbol{b} \,|\, \boldsymbol{m} \rangle$ for some collection of coefficients $\boldsymbol{b} \in \mathrm{F}_{\mathrm{fin}}(X, \mathfrak{m})$. Since $\boldsymbol{m}$ is a basis of M, according to Proposition 3.20 we must have $\boldsymbol{a} = \boldsymbol{b}$. This implies that

$$\bar{a}_x = \pi_A(a_x) = \pi_A(b_x) = 0$$

for all $x \in X$. We see that $\bar{\boldsymbol{a}} = 0$. The conclusion is that the collection $\bar{\boldsymbol{m}}$ is linearly independent over $\bar{A}$, as claimed.

At this point we know that $\bar{\boldsymbol{m}} = \{\bar{m}_x\}_{x \in X}$ is a basis of the $\bar{A}$-module $\bar{M}$.

By the same token, the collection $\bar{\boldsymbol{n}} = \{\bar{n}_y\}_{y \in Y}$, where $\bar{n}_y := \pi_M(n_y)$, is also a basis of $\bar{M}$.

Corollary 3.5 tells us that $|X| = |Y|$. □

This theorem allows us to make the next definition.

Definition 3.31. Let A be a nonzero ring, let M be a free A-module, and let $\boldsymbol{m} = \{m_x\}_{x \in X}$ be a basis of M. The *rank* of M, denoted by $\operatorname{rank}_A(M)$, is defined to be the cardinality of X.

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Here are a few optional exercises for your general math education (we won't need these later).

Exercise 3.32 (Optional)**.** We fix some nonzero commutative ring A. Let M and N be free A-modules with bases $\boldsymbol{m} = \{m_x\}_{x \in X}$ and $\boldsymbol{n} = \{n_y\}_{y \in Y}$.

(1) Given an A-linear homomorphism $\phi : M \to N$, try to express ϕ as a $Y \times X$ matrix $\boldsymbol{a}$, namely as a collection

$$\boldsymbol{a} = \{a_{y,x}\}_{(y,x) \in Y \times X} \in \mathrm{F}(Y \times X, A)$$

with suitable finiteness conditions. (Hint: viewing M and N as columns of sizes X and Y, the matrices $\boldsymbol{a}$ have finitely supported columns.)

(2) Now assume that $M = N$ and $\boldsymbol{m} = \boldsymbol{n}$. Show that the $X \times X$ matrices with the finiteness condition above is a NC ring under matrix multiplication. It is the ring $\operatorname{End}_A(M)$.

Exercise 3.33 (Optional)**.** We fix some nonzero ring A.

(1) Let $f : X \to Y$ be a function between sets. Show that there is a unique A-module homomorphism

$$\textstyle\sum_f : \mathrm{F}_{\mathrm{fin}}(X, A) \to \mathrm{F}_{\mathrm{fin}}(Y, A)$$

such that $\sum_f (\delta_x) = \delta_{f(x)}$ for every $x \in X$. (Hint: Thm 3.22.)

(2) Try to understand why the notation $\sum_f$ is used. Hint: show that for every $\phi \in \mathrm{F}_{\mathrm{fin}}(X, A)$ there is equality

$$\Big(\sum\nolimits_f \phi\Big)(y) = \sum_{x \in f^{-1}(y)} \phi(x).$$

Remark 3.34. Later we will prove that $\mathrm{F}_{\mathrm{fin}}(-, A)$ is a functor $\mathsf{Set} \to \mathsf{Mod}(A)$. The homomorphism $\sum_f$ will play a role there.

Exercise 3.35 (Optional)**.** We fix some nonzero ring A.

Define the free A-module

$$B := \mathrm{F}_{\mathrm{fin}}(\mathbb{N}, A).$$

We know that the collection $\boldsymbol{\delta} = \{\delta_i\}_{i \in \mathbb{N}}$ is a basis of B.

(1) Show that B has an A-ring structure with multiplication $\delta_i \cdot \delta_j = \delta_{i+j}$ and unit $1_B = \delta_0$.

(2) Show that B is isomorphic to the polynomial ring $A[t]$, by $\delta_i \mapsto t^i$.

(3) Try to express the polynomial ring $A[t_1, \ldots, t_n]$, in the variables $t_1, \ldots, t_n$, is a similar way.

The next proposition describes the universal property of the polynomial ring. I think we all know it.

Proposition 3.36. *Let A be a nonzero ring, and let $A[\boldsymbol{t}] = A[t_1, \ldots, t_n]$ be the polynomial ring over A in n variables, for some $n \in \mathbb{N}$. Suppose B is an A-ring, and we are given a sequence $\boldsymbol{b} = (b_1, \ldots, b_n)$ of elements of B. Then there is a unique A-ring homomorphism $\phi : A[\boldsymbol{t}] \to B$ s.t. $\phi(t_i) = b_i$.*

Another way to understand the homomorphism ϕ above is as *substitution*: for a polynomial $p(\boldsymbol{t}) \in A[\boldsymbol{t}]$, the element $p(\boldsymbol{b}) := \phi(p(\boldsymbol{t})) \in B$ can be viewed a the result of the substitution $t_i \mapsto b_i$ in $p(\boldsymbol{t})$.

Definition 3.37. In the situation of Proposition 3.36, the subring $\mathrm{Im}(f) \subseteq B$ is denoted by $A[\boldsymbol{b}] = A[b_1, \ldots, b_n]$. It is called the A-subring of B generated by the sequence $\boldsymbol{b} = (b_1, \ldots, b_n)$.

The next exercise shows how to generalize polynomial rings.

Exercise 3.38 (Optional)**.** We fix some nonzero ring A.

Given sets X, Y and elements $\phi \in \mathrm{F}_{\mathrm{fin}}(X, A)$ and $\psi \in \mathrm{F}_{\mathrm{fin}}(Y, A)$, define the function

$$\phi \boxtimes \psi : X \times Y \to A, \quad (\phi \boxtimes \psi)(x, y) := \phi(x) \cdot \psi(y).$$

Show that

$$\phi \boxtimes \psi \in \mathrm{F}_{\mathrm{fin}}(X \times Y, A).$$

Exercise 3.39 (Optional)**.** Let G be a monoid, namely a unital semigroup. The unit of G is e, and the multiplication of G is

$$\mathrm{m} : G \times G \to G, \quad \mathrm{m}(g_1, g_2) = g_1 \cdot g_2$$

and the unit element is e.

Define the free A-module

$$B := \mathrm{F}_{\mathrm{fin}}(G, A).$$

(1) Define a multiplication

$$B \times B \to B, \quad (\phi, \psi) \mapsto \sum\nolimits_{\mathrm{m}} \phi \boxtimes \psi$$

on B.

Prove that B is a noncommutative central A-ring with this multiplication. The unit element is $\delta_e \in B$. (Hint: Show that $\delta_g \cdot \delta_h = \delta_{g \cdot h}$.)

The ring B is denoted by $A[G]$. When G is a group, the ring $A[G]$ is called the group ring of G with coefficients in A.

(2) Prove that $A[G]$ is commutative iff G is abelian.

(3) Try to understand the multiplication of the ring B as a *convolution*, in the sense of functional analysis.

(4) When $G = \mathbb{N}$ with addition, compare to Exer 3.35.

comment: (211111 AY) End of Lecture 4

comment: Start of Lecture 5, 17 Nov 2021

4. Categories

A *category* is a very efficient way to talk about "mathematical objects of the same type".

I will give the definition, and then many examples.

Definition 4.1. A *category* C is a mathematical system consisting of these ingredients:

- A set $\operatorname{Ob}(\mathsf{C})$, called the set of *objects* of C.
- For every pair $C_0, C_1 \in \operatorname{Ob}(\mathsf{C})$, there is a set $\operatorname{Hom}_{\mathsf{C}}(C_0, C_1)$, called the set of *morphisms* from C_0 to C_1.
- For every triple $C_0, C_1, C_2 \in \operatorname{Ob}(\mathsf{C})$, there is a function
$$\operatorname{Hom}_{\mathsf{C}}(C_1, C_2) \times \operatorname{Hom}_{\mathsf{C}}(C_0, C_1) \to \operatorname{Hom}_{\mathsf{C}}(C_0, C_2)\,, \quad (f_2, f_1) \mapsto f_2 \circ f_1$$
called *composition*.
- For every $C \in \operatorname{Ob}(\mathsf{C})$, there is a morphism
$$\mathrm{id}_C \in \operatorname{Hom}_{\mathsf{C}}(C, C)\,,$$
called the *identity morphism*.

These are the axioms:

- ▹ Composition is associative: given objects $C_0, C_1, C_2, C_3 \in \operatorname{Ob}(\mathsf{C})$ and morphisms $f_i \in \operatorname{Hom}_{\mathsf{C}}(C_{i-1}, C_i)$, there is equality
$$f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1$$
in $\operatorname{Hom}_{\mathsf{C}}(C_0, C_3)$.
- ▹ The identity morphisms are neutral for composition: for every $f \in \operatorname{Hom}_{\mathsf{C}}(C_0, C_1)$ there is equality
$$f \circ \mathrm{id}_{C_0} = f = \mathrm{id}_{C_1} \circ f$$
in $\operatorname{Hom}_{\mathsf{C}}(C_0, C_1)$.

We usually write $f : C \to D$ to mean that $f \in \operatorname{Hom}_{\mathsf{C}}(C, D)$.

Example 4.2. The category Set has all sets as its objects. The morphisms $f : X \to Y$ between $X, Y \in \operatorname{Ob}(\mathsf{Set})$ are the functions. The identity morphisms are the identity functions, and composition is the usual one.

Remark 4.3. The example above brings us to a variant of *Russell's Paradox*: is the set $X := \mathrm{Ob}(\mathsf{Set})$ an element of itself? There are two main ways to avoid this set-theoretical difficulty. Both work by installing a size hierarchy among sets.

(1) Modify Definition 4.1, by declaring that $\mathrm{Ob}(\mathsf{C})$ is a *class*. This is called the *von Neumann – Bernays – Gödel set theory*.

(2) Choose a *Grothendieck universe* U, which is some big set. Elements of U are called small sets. Declare that $\mathrm{Ob}(\mathsf{C}) \subseteq \mathsf{U}$ and $\mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \in \mathsf{U}$.

We will take the second way. Thus, for us (implicitly) Set is the category of small sets inside a given (implicit) universe.

As usual with foundations, these issues are not really important or interesting. It is the algebraic properties of categories (and functors, etc.) that are interesting and useful.

Students who want to read more about these matters can look in [Mac2, Chapter I].

Example 4.4. The category Top has all topological spaces as its objects. The morphisms $f : X \to Y$ between $X, Y \in \mathrm{Ob}(\mathsf{Top})$ are the continuous functions. The identity morphisms are the identity functions, and composition is the usual one.

Example 4.5. Fix a ring A. The category $\mathsf{Mod}(A)$ of left A-modules has all left A-modules as its objects.

Regarding size: as in Remark 4.3, the ring A and the modules M have small underlying sets.

The morphisms $\phi : M \to N$ between $M, N \in \mathrm{Ob}(\mathsf{Mod}(A))$ are the A-module homomorphisms. The identity morphisms are the identity homomorphisms, and composition is the usual one.

Example 4.6. Let $\phi : M \to N$ be a morphism in $\mathsf{Mod}(A)$.

Consider this diagram

$$
\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[d, two heads, "\pi"'] & N \\
M/\mathrm{Ker}(\phi) \arrow[r, "\bar{\phi}", "\cong"'] & \mathrm{Im}(\phi) \arrow[u, hook, "\epsilon"']
\end{tikzcd}
\tag{4.7}
$$

in the category $\mathsf{Mod}(A)$.

It consists of four objects and four morphisms.

This diagram is *commutative*. This means that the two morphisms we get from M to N, namely ϕ and $\epsilon \circ \bar{\phi} \circ \pi$, are equal.

Example 4.8. The category Rng of rings has all rings as its objects. The morphisms $f : A \to B$ between $A, B \in \mathrm{Ob}(\mathsf{Rng})$ are the ring homomorphisms. The identity morphisms are the identity functions, and composition is the usual one.

Example 4.9. Let $\mathbb{K}$ be a commutative ring.

Recall that a central $\mathbb{K}$-ring is a ring A together with a homomorphism $f_A : \mathbb{K} \to A$ such that $f_A(\mathbb{K}) \subseteq \operatorname{Cent}(A)$.

Suppose (B, f_B) is another central $\mathbb{K}$-ring. A morphism of central $\mathbb{K}$-rings $g : A \to B$ is a ring homomorphism g such that $f_B = g \circ f_A$.

In a commutative diagram:

$$\begin{tikzcd} \mathbb{K} \arrow[d, "f_A"'] \arrow[dr, "f_B"] & \\ A \arrow[r, "g"] & B \end{tikzcd}$$

The category of central $\mathbb{K}$-rings is denoted by $\mathsf{Rng}/_{\mathrm{c}}\, \mathbb{K}$.

Here are some typical central $\mathbb{K}$-rings: the polynomial ring $\mathbb{K}[t]$, the matrix ring $\mathrm{M}_r(\mathbb{K})$, the group ring $\mathbb{K}[G]$ for some group G, the NC polynomial ring $\mathbb{K}\langle t_1, t_2\rangle$ in two variables.

Definition 4.10. Let C be a category. A *subcategory* D of C consists of a subset $\operatorname{Ob}(\mathsf{D}) \subseteq \operatorname{Ob}(\mathsf{C})$, and for every $D_0, D_1 \in \operatorname{Ob}(\mathsf{D})$ a subset

$$\operatorname{Hom}_{\mathsf{D}}(D_0, D_1) \subseteq \operatorname{Hom}_{\mathsf{C}}(D_0, D_1),$$

such that:

- ▹ Closure under composition: given objects $D_0, D_1, D_2 \in \operatorname{Ob}(\mathsf{D})$ and morphisms $f_i \in \operatorname{Hom}_{\mathsf{D}}(D_{i-1}, D_i)$, the composed morphism $f_2 \circ f_1$ belongs to $\operatorname{Hom}_{\mathsf{D}}(D_0, D_2)$.
- ▹ The identity morphism id_D belongs to $\operatorname{Hom}_{\mathsf{D}}(D, D)$.

Definition 4.11. Let C be a category. A subcategory $\mathsf{D} \subseteq \mathsf{C}$ is called a *full subcategory* if for every $D_0, D_1 \in \operatorname{Ob}(\mathsf{D})$ there is equality

$$\operatorname{Hom}_{\mathsf{D}}(D_0, D_1) = \operatorname{Hom}_{\mathsf{C}}(D_0, D_1).$$

Example 4.12. The category of groups is Grp.

The category of abelian groups is Ab.

Ab is a full subcategory of Grp.

Example 4.13. Let C be a category. Given a subset $S \subseteq \operatorname{Ob}(\mathsf{C})$, there is the full subcategory on S.

Definition 4.14. Let C be a category. A morphism $f : C \to D$ is called an *isomorphism* if there is a morphism $g : D \to C$ s.t. $g \circ f = \mathrm{id}_C$ and $f \circ g = \mathrm{id}_D$.

Definition 4.15. A category G is called a *groupoid* if all the morphism in it are isomorphisms.

Exercise 4.16. Let C be a category.

(1) If $f : C \to D$ is an isomorphism in C, then its inverse g is unique, and it too is an isomorphism.

(2) Show that the composition of isomorphisms is an isomorphism.

(3) Conclude the there is a subcategory $\mathsf{C}^\times$ of C, with all the objects, but the morphisms in $\mathsf{C}^\times$ are the isomorphisms in C. The category $\mathsf{C}^\times$ is a groupoid, sometimes called the gauge groupoid of C.

Example 4.17. Let G be a monoid (unital semigroup) with unit e.

Consider the category G that has a single object, say x, and

$$\mathrm{Hom}_{\mathsf{G}}(x, x) := G.$$

The composition and the unit are those of the monoid G.

Observation: there is no difference between a single-object category and a monoid.

Likewise for a group and a groupoid.

comment: (211117 AY) End of live lecture. Below are some exercises and examples I added. Please solve them.

◊ ◊ ◊

Exercise 4.18. This is about the abstract notion of *product*. Let C be a category, and let $\{C_i\}_{i \in I}$ be a collection of objects of C. A *product* of this collection is an object C of C, and a collection $\{p_i\}_{i \in I}$ of morphisms $p_i : C \to C_i$, with this universal property: for every object $D \in \mathsf{C}$ with a collection $\{f_i\}_{i \in I}$ of morphisms $f_i : D \to C_i$, there is a unique morphism $f : D \to C$ such that $f_i = p_i \circ f$. See commutative diagram (4.19).

(4.19)
$$\begin{tikzcd} D \arrow[d, dashed, "f"'] \arrow[dr, dashed, "f_i"] & \\ C \arrow[r, "p_i"] & C_i \end{tikzcd}$$

(1) Prove that a product $\big(C, \{p_i\}_{i \in I}\big)$, if it exists, is unique up to a unique isomorphism. (You need to make this statement precise!)

(2) Show that products exist in these categories: Set, Ab, $\mathsf{Mod}(A)$, where A is some ring. Write the explicit formula in each case.

(3) Let $\mathsf{Ab}_{\mathrm{fin}}$ be the category of finite abelian groups. (A full subcategory of Ab.) Find a collection $\{M_i\}_{i \in I}$ in $\mathsf{Ab}_{\mathrm{fin}}$ that does not have a product in this category. (Hint: the indexing set I is infinite. Even then the proof could be a bit tricky.)

Exercise 4.20. This is about the dual notion, the abstract notion of *coproduct*. Let C be a category, and let $\{C_i\}_{i\in I}$ be a collection of objects of C. A *coproduct* of this collection is an object C of C, and a collection $\{e_i\}_{i\in I}$ of morphisms $e_i : C_i \to C$, with this universal property: for every object $D \in \mathsf{C}$ with a collection $\{g_i\}_{i\in I}$ of morphisms $g_i : C_i \to D$, there is a unique morphism $g : C \to D$ such that $g_i = g \circ e_i$.

(4.21)
$$\begin{tikzcd} C_i \arrow[r, "e_i"] \arrow[dr, dashed, "g_i"'] & C \arrow[d, dashed, "g"] \\ & D \end{tikzcd}$$

(1) Prove that a coproduct $(C, \{e_i\}_{i\in I})$, if it exists, is unique up to a unique isomorphism. (Again you need to make this statement precise!)

(2) Show that products exist in these categories: Set, Ab, $\mathsf{Mod}(A)$. Write the explicit formula in each case. (Warning: do not copy from the prior exercise. The coproducts look different for these various categories.)

(3) Let $\mathsf{Ab}_{\mathrm{fin}}$ be the category of finite abelian groups. Find a collection $\{M_i\}_{i\in I}$ in $\mathsf{Ab}_{\mathrm{fin}}$ that does not have a coproduct in this category. (Hint: modify the answer from the previous exercise.)

Exercise 4.22. This is about *fibered products*. We will consider only the important case of the fiber product of two objects over a third one.

In a category C we are given objects C_1, C_2, C and morphisms $f_1 : C_1 \to C$ and $f_2 : C_2 \to C$.

A fibered product of C_1 and C_2 over C is an object D, with morphisms $p_1 : D \to C_1$ and $p_2 : D \to C_2$, such that these conditions hold:

(a) There is equality
$$f_1 \circ p_1 = f_2 \circ p_2$$
of morphisms $D \to C$.

$$\begin{tikzcd} & D \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ C_1 \arrow[dr, "f_1"'] & & C_2 \arrow[dl, "f_2"] \\ & C & \end{tikzcd}$$

(b) The triple (D, p_1, p_2) is universal for property (a). Namely given a triple (D', p'_1, p'_2) of morphisms $p'_i : D' \to C_i$ such that
$$f_1 \circ p'_1 = f_2 \circ p'_2,$$

there is a unique morphism $g : D' \to D$ such that

$$p'_i = p_i \circ g$$

for $i = 1, 2$.

$$\begin{tikzcd}
 & D' \arrow[d, dashed, "g"] \arrow[ddl, dashed, bend right, "p'_1"'] \arrow[ddr, dashed, bend left, "p'_2"] & \\
 & D \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
C_1 \arrow[dr, "f_1"'] & & C_2 \arrow[dl, "f_2"] \\
 & C &
\end{tikzcd}$$

(1) Prove that a fibered product (D, p_1, p_2), if it exists, is unique up to a unique isomorphism. (Again you need to make this statement precise!)

(2) Show that fibered products exist in these categories: Set, Ab, $\mathsf{Mod}(A)$. Write the explicit formula in each case.

Exercise 4.23. Let C be a category.

(1) Define the *fibered coproduct* in C. (Hint: start with the fibered product from Exer 4.22, and then make the suitable change in the direction of arrows, like in the passage from product to coproduct.)

(2) Prove that a fibered coproduct (D, p_1, p_2), if it exists, is unique up to a unique isomorphism.

(3) Show that fibered coproducts exist in these categories: Set, Ab, $\mathsf{Mod}(A)$. Write the explicit formula in each case.

comment: (211117 AY) End of Lecture 5.

comment: Start of Lecture 6, 24 Nov 2021

Let's start with the homework. I checked it, and nobody solved Exer 3.23 (proving Thm 3.22) correctly. The problem was with the implication (ii) $\Rightarrow$ (i).

Here is a proof.

Proof of Thm 3.22.

(i) $\Rightarrow$ (ii): This will be brief, because most got it right. Take an element $m \in M$. Let $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ be its coordinate collection, namely the unique collection satisfying $\langle \boldsymbol{a} | \boldsymbol{m} \rangle = m$. Define $\phi(m) := \langle \boldsymbol{a} | \boldsymbol{n} \rangle \in N$.

One needs to prove that this function is A-linear, and that it is the unique A-linear function s.t. $\phi(m_x) = n_x$. This is quite easy.

(ii) $\Rightarrow$ (i): Here I will use Example 3.22 and Prop 3.18.

Take the module $N := \mathrm{F}_{\mathrm{fin}}(X, A)$. We know that N is a free A-module, with basis the collection of elements $\boldsymbol{n} := \boldsymbol{\delta} = \{\delta_x\}_{x \in X}$. This is Example 3.22.

Let $\phi : M \to N$ be a homomorphism such that $\phi(m_x) = \delta_x$ for all $x \in X$. This ϕ exists by assumption. (It is unique, but we don't care.)

Because $\boldsymbol{\delta}$ generates N, the homomorphism ϕ is surjective.

We know that $\boldsymbol{\delta}$ is a basis of N, so using the first part of the proof (the implication (i) $\Rightarrow$ (ii)) there is a homomorphism $\psi : N \to M$ s.t. $\psi(\delta_x) = m_x$ for all $x \in X$.

Consider the composed homomorphism $\psi \circ \phi : M \to M$. It satisfies

$$(\psi \circ \phi)(m_x) = \psi(\delta_x) = m_x$$

for all $x \in X$.

The identity homomorphism id_M also satisfies

$$\mathrm{id}_M(m_x) = m_x$$

for all $x \in X$.

The uniqueness in condition (ii) says that

$$\psi \circ \phi = \mathrm{id}_M .$$

We conclude that ϕ is injective. Thus

$$\phi : M \to N$$

is an isomorphism. By Prop 3.18 it follows that $\boldsymbol{m}$ is a basis on M. □

Remark 4.24. Can anybody find a more direct proof of Thm 3.22, not using the free module $\mathrm{F}_{\mathrm{fin}}(X, A)$?

comment: (211125 AY) I gave such a proof in class

◊ ◊ ◊

Today we will talk about *bilinearity*. This will go in two directions: *linear categories* and *tensor products*.

Definition 4.25. Let A be a commutative ring and let M, N, P be A-modules. An *A-bilinear function*

$$\beta : M \times N \to P$$

is a function satisfying these conditions:

- additivity in the first argument: $\beta(m_1 + m_2, n) = \beta(m_1, n) + \beta(m_2, n)$
- additivity in the second argument: $\beta(m, n_1 + n_2) = \beta(m, n_1) + \beta(m, n_2)$
- multiplication by elements of A : $\beta(a \cdot m, n) = \beta(m, a \cdot n) = a \cdot \beta(m, n)$

for all $m_1, m_2, m \in M$, $n_1, n_2, n \in N$ and $a \in A$.

Example 4.26. Let $\mathbb{K}$ be a commutative ring and $B := \mathrm{M}_r(\mathbb{K})$, the NC ring of $r \times r$ matrices over $\mathbb{K}$.

The matrix multiplication

$$B \times B \to B$$

is $\mathbb{K}$-bilinear.

More generally, if B is a NC central $\mathbb{K}$-ring, then multiplication is a $\mathbb{K}$-bilinear function $B \times B \to B$.

Definition 4.27. Fix a nonzero commutative ring $\mathbb{K}$.

A $\mathbb{K}$-*linear category* is a category M such that for every pair of objects M_0, M_1 the set $\mathrm{Hom}_{\mathsf{M}}(M_0, M_1)$ has a $\mathbb{K}$-module structure.

The condition is that for objects $M_0, M_1, M_2 \in \mathsf{M}$ the composition function

$$\mathrm{Hom}_{\mathsf{M}}(M_1, M_2) \times \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{M}}(M_0, M_2)\,,\ (m_2, m_1) \mapsto m_2 \circ m_1$$

is $\mathbb{K}$-bilinear.

The next exercises present some important examples.

Exercise 4.28. Let $\mathbb{K}$ be a commutative ring and B a NC central $\mathbb{K}$-ring.

Show that the category $\mathsf{Mod}(B)$ of left B-modules is $\mathbb{K}$-linear.

Exercise 4.29. Let $\mathbb{K}$ be a commutative ring and M be a $\mathbb{K}$-linear category.

Take an object $M \in \mathsf{M}$. Prove that

$$\mathrm{End}_{\mathsf{M}}(M) := \mathrm{Hom}_{\mathsf{M}}(M, M)$$

is a central $\mathbb{K}$-ring. Here addition is from the $\mathbb{K}$-module structure on $\mathrm{Hom}_{\mathsf{M}}(M, M)$, and multiplication is composition. The ring homomorphism $\mathbb{K} \to \mathrm{End}_{\mathsf{M}}(M)$ is $\lambda \mapsto \lambda \cdot \mathrm{id}_M$.

Example 4.30. This is like Exa 4.17. Let B be a NC central $\mathbb{K}$-ring.

Consider the category B that has a single object, say x, and

$$\mathrm{Hom}_{\mathsf{B}}(x, x) := B.$$

The $\mathbb{K}$-module structure, the composition and the unit are those of the ring B.

Then B is a $\mathbb{K}$-linear category.

Observation, from the previous exercise and example: there is no difference between a single-object $\mathbb{K}$-linear category and a NC central $\mathbb{K}$-ring.

Remark 4.31. What about ring homomorphisms? What is their categorical generalization? These are the $\mathbb{K}$-*linear functors*, and we will talk about them later.

5. Tensor Products of Modules

Let's fix a commutative ring A for this whole section.

We shall use the terms A-linear function and A-module homomorphism (between A-modules) interchangeably.

If $\beta : M \times N \to P$ is an A-bilinear function and $\psi : P \to Q$ is an A-linear function, then

$$\psi \circ \beta : M \times N \to Q$$

is A-bilinear function.

Definition 5.1. Let M and N be A-modules.

A *tensor product* of M and N is a pair (P, β), consisting of an A-module P and an A-bilinear function

$$\beta : M \times N \to P,$$

having this universal property:

(T) Given any pair (Q, γ), consisting of an A-module Q and an A-bilinear function

$$\gamma : M \times N \to Q,$$

there exists a unique A-module homomorphism $\phi : P \to Q$ such that $\gamma = \phi \circ \beta$.

In a commutative diagram:

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[dr, "\gamma"'] & P \arrow[d, dashed, "\phi"] \\ & Q \end{tikzcd}$$

Proposition 5.2 (Uniqueness)**.** *Suppose (P, β) and (P', β') are both tensor products of the A-modules M and N. Then there is a unique isomorphism of A-modules $\phi : P \xrightarrow{\simeq} P'$ such that $\beta' = \phi \circ \beta$.*

Exercise 5.3. Prove this proposition.

For existence we are going to make a very "big" construction. Later we will see that often the tensor product is not very big.

Theorem 5.4 (Existence)**.** *Let M and N be A-modules. Their tensor product (P, β) exists.*

Proof. Let $I := M \times N$, seen as an indexing set, and let $F := \mathrm{F}_{\mathrm{fin}}(I, A)$, the free A-module with basis $\boldsymbol{\delta} = \{\delta_i\}_{i \in I}$.

Inside F we have these special elements:

$$\begin{gathered} \delta_{m_1+m_2,n} - (\delta_{m_1,n} + \delta_{m_2,n}) \\ \delta_{m,n_1+n_2} - (\delta_{m,n_1} + \delta_{m,n_2}) \\ \delta_{a \cdot m,n} - a \cdot \delta_{m,n} \\ \delta_{m,a \cdot n} - a \cdot \delta_{m,n} \end{gathered} \tag{5.5}$$

for all $m_1, m_2, m \in M$, $n_1, n_2, n \in N$ and $a \in A$. Let K be the submodule of F generated by all these special elements. Define the A-module

$$P := F/K$$

and the function

$$\beta : M \times N \to P, \quad \beta(m, n) := \delta_{m,n} + K.$$

The function β is A-bilinear, because all the conditions in Definition 4.25 are satisfied, by construction.

In a commutative diagram:

$$\begin{tikzcd} M \times N \arrow[r, "(m,n) \mapsto \delta_{m,n}"] \arrow[rr, bend left, "\beta"] & F \arrow[r, "\pi"] & P \end{tikzcd}$$

Note that π is surjective.

It remains to prove that the pair (P, β) has the universal property (T) from Definition 5.1.

Suppose (Q, γ) is some pair as in condition (T). So

$$\gamma : M \times N \to Q$$

is an A-bilinear function.

By the universal property of the free module F and its basis $\boldsymbol{\delta}$ (see Theorem 3.22) there is a unique A-module homomorphism $\tilde{\phi} : F \to Q$ such that

$$\tilde{\phi}(\delta_{m,n}) = \gamma(m, n)$$

for all $(m, n) \in I = M \times N$.

In a commutative diagram:

$$\begin{tikzcd}
M \times N \arrow[r, "(m,n)\mapsto\delta_{m,n}"] \arrow[rr, bend left, "\beta"] \arrow[dr, "\gamma"'] & F \arrow[r, "\pi"] \arrow[d, "\tilde{\phi}"] & P \arrow[dl, "\phi"] \\
& Q &
\end{tikzcd}$$

Now γ is A-bilinear, and $\tilde{\phi}$ is A-linear; and therefore $\tilde{\phi}$ sends all the special elements in (5.5) to $0 \in Q$. It follows that there is a unique A-module homomorphism $\phi : P \to Q$ such that

$$\phi(\beta(m, n)) = \gamma(m, n).$$

$\square$

> **comment**: (211125 AY) To here in live class. Please read the material below. There are 4 exercises to solve (incl Exer 5.9)

Once uniqueness and existence are settled, we can introduce the next notation.

Notation 5.6. Given A-modules M and N, with tensor product (P, β), we write

$$M \otimes_A N := P$$

and

$$m \otimes n := \beta(m, n) \in M \otimes_A N$$

for $(m, n) \in M \times N$.

The element $m \otimes n$ is called a *pure tensor*.

The tensor product is a very confusing object. The proof of the theorem might give the impression that it is very big – but this is usually false; see Proposition 5.8 below.

Another confusing feature of the tensor product is that often when we need to work with it, we have to use its universal defintion (and not the construction above).

Still the contstruction tells us this:

Proposition 5.7. *The A-module $M \otimes_A N$ is generated by the pure tensors.*

Proof. The free module F is generated by the elements $\delta_{m,n}$, and π is surjective. Hence P is generated by the elements $m \otimes n = \pi(\delta_{m,n})$. □

We will see later that often there are elements of $M \otimes_A N$ that are not pure tensors; but the proof won't be so easy.

Proposition 5.8. *Suppose M and N are generated by the collections of elements $\{m_i\}_{i \in I}$ and $\{n_j\}_{j \in J}$. Then the A-module $M \otimes_A N$ is generated by the collection of elements $\{m_i \otimes n_j\}_{(i,j) \in I \times J}$.*

Exercise 5.9. Prove Proposition 5.8. (Hint: use Prop 5.7.)

comment: (211125 AY) End lecture 6

comment: (211130 AY) Some inter-lecture material

comment: (211128) Solution of Exercise 4.18(3)

Some of the students answered this correctly, but not all. Here is a good solution, and then a related example.

Solution of Exercise 4.18(3). We want to produce a collection of finite abelian groups that does not have a product in the category $\mathsf{Ab}_{\mathrm{fin}}$ of finite abelian groups.

We take the indexing set $I := \{1, 2, 3, \ldots\}$ and collection $\{M_i\}_{i \in I}$ of finite abelian groups $M_i := \mathbb{Z}/(i)$. So M_i is cyclic of size i. Suppose, for the sake of contradiction, that there exists a product $\big(M, \{p_i\}_{i \in I}\big)$ of the collection $\{M_i\}_{i \in I}$ in $\mathsf{Ab}_{\mathrm{fin}}$. Take $j := |M| + 1$. Consider the test data $\big(N, \{f_i\}_{i \in I}\big)$ where $N := M_j$, $f_j : N \to M_j$ is the identity, and $f_i := 0$ for all $i \neq j$. By the universal property there is a homomorphism $f : N \to M$ s.t. $f_i = p_i \circ f$ for all i. But for $i = j$ we get $f_j = p_j \circ f = \mathrm{id}_N$. This implies that $f : N \to M$ is an injective homomorphism, and hence $|M| \geq |N| = j$. This is a contradiction.

The mistake made by some students is this: they wrote down an infinite collection of finite abelian groups, showed that the product in the category Ab is infinite. But this does not imply that there does not exist (some other) product in the full subcategory $\mathsf{Ab}_{\mathrm{fin}}$ – since the quantifiers are different (in $\mathsf{Ab}_{\mathrm{fin}}$ there are less conditions to check).

In Example 5.10 I show that coproducts depend on the category. I wan't able to find something like this for products (but see Remark 5.11).

Example 5.10. Here we look at the category Grp and its full subcategory Ab. We will show that coproducts are different in these categories. Take $G_1 := G_2 := \mathbb{Z}$. The coproduct in Ab is

$$G_1 \coprod G_2 = \mathbb{Z}^2,$$

the free abelian group of rank 2. The embeddings are $e_1(n) = (n, 0)$ and $e_2(n) = (0, n)$.

But the coproduct in Grp is

$$G_1 \coprod G_2 = G_1 * G_2 = (\text{free nonabelian group on two generators}).$$

For those who learner algebraic topology, this is the fundamental group $\boldsymbol{\pi}_1(X)$ of the topological space gotten by joining two circles at one point (like the numeral 8).

Remark 5.11. When we talk about opposite categories, $\mathsf{Ab}^{\mathrm{op}}$ is a full subcategory of $\mathsf{Grp}^{\mathrm{op}}$, and the products in $\mathsf{Grp}^{\mathrm{op}}$ are coproducts in Grp. In this crooked way Example 5.10 can be transformed into an example about products.

comment: (211129 AY) The next def is relevant to exer 4.29

The next definition was inside Example 4.9. It is the NC variant of Definition 3.8, in which all rings were commutative.

Definition 5.12. Let $\mathbb{K}$ be a commutative ring.

A *central* $\mathbb{K}$*-ring* is a NC ring A, together with a ring homomorphism $f_A : \mathbb{K} \to A$ such that $f_A(\mathbb{K}) \subseteq \operatorname{Cent}(A)$. The homomorphism f_A is called the structural homomorphism of A.

Suppose (B, f_B) is another central $\mathbb{K}$-ring. A *homomorphism of central* $\mathbb{K}$*-rings* $g : A \to B$ is a ring homomorphism g such that $f_B = g \circ f_A$. Note that there is no centrality condition on the homomorphism g.

In a commutative diagram:

$$\begin{tikzcd}
\mathbb{K} \arrow[d, "f_A"'] \arrow[dr, "f_B"] & \\
A \arrow[r, "g"] & B
\end{tikzcd}$$

The category of central $\mathbb{K}$-rings is denoted by $\mathsf{Rng}/_{\mathrm{c}}\, \mathbb{K}$.

comment: Start of Lecture 7, 1 Dec 2021

Today we are going to talk more about tensor products.

As before, the ring A is assumed to be commutative.

An immediate corollary of Proposition 5.8 is:

Corollary 5.13. *If M and N are finitely generated A-modules, then so is $M \otimes_A N$.*

As promised, the tensor product is not "much bigger" than the original modules.

Sometimes the tensor product is very small:

Example 5.14. Take $A := \mathbb{Z}$, $M := \mathbb{Z}/(3)$ and $N := \mathbb{Z}/(8)$.

I claim that $M \otimes_A N = 0$.

To see this, I will show that every pure tensor $m \otimes n$ is zero. (It is enough to prove that $1_M \otimes 1_N = 0$, by Prop 5.8.)

The element $9 \in A$ has the property that $9 \cdot m = 0$ in M and $9 \cdot n = 1 \cdot n = n$ in N. Hence

$$m \otimes n = m \otimes (9 \cdot n) = (9 \cdot m) \otimes n = 0 \otimes n = 0 \cdot (1 \otimes n) = 0.$$

Exercise 5.15. Find the condition on the positive numbers $r, s \in \mathbb{Z}$ such that the $\mathbb{Z}$-modules $M := \mathbb{Z}/(r)$ and $N := \mathbb{Z}/(s)$ will satisfy $M \otimes_{\mathbb{Z}} N = 0$. (Hint: the next exercise.)

Exercise 5.16. Let $\mathfrak{a}, \mathfrak{b} \subseteq A$ be ideals. Then

$$(A/\mathfrak{a}) \otimes_A (A/\mathfrak{b}) \cong A/(\mathfrak{a} + \mathfrak{b})$$

as A-modules.

Theorem 5.17. *Suppose M and N are free A-modules, with bases $\{m_i\}_{i \in I}$ and $\{n_j\}_{j \in J}$ respectively. Then the A-module $M \otimes_A N$ is free with basis $\{m_i \otimes n_j\}_{(i,j) \in I \times J}$.*

Proof. We are going to use Thm 3.22, the universal characterization of a basis. Let P be an A-module, and let $\{p_{i,j}\}_{(i,j) \in I \times J}$ be a collection of elements of P. (We often refer to data like $\big(P, \{p_{i,j}\}_{(i,j) \in I \times J}\big)$ as "test data".) We are going to prove that there is a unique A-linear homomorphism

$$\phi : M \otimes_A N \to P$$

satisfying

$$\phi(m_i \otimes n_j) = p_{i,j}.$$

We first define a function

$$\beta : M \times N \to P$$

as follows. for a pair $(m, n) \in M \times N$ let $\boldsymbol{a} = \{a_i\}_{i \in I} \in \mathrm{F}_{\mathrm{fin}}(I, A)$ and $\boldsymbol{b} = \{b_j\}_{j \in J} \in \mathrm{F}_{\mathrm{fin}}(J, A)$ be the coefficient sequences, namely

$$m = \langle \boldsymbol{a} | \boldsymbol{m} \rangle = \sum_{i \in I} a_i \cdot m_i \in M$$

and $n = \langle \boldsymbol{b} | \boldsymbol{n} \rangle$. We define

$$\beta(m, n) := \sum_{i,j} a_i \cdot b_j \cdot p_{i,j} \in P.$$

This β is an A-bilinear function. We will just prove one of the conditions (the others are very similar to prove). Suppose $m' \in M$ is some other element, with coefficient $\boldsymbol{a}' \in \mathrm{F}_{\mathrm{fin}}(I, A)$. Then the coefficient of $m + m'$ is $\boldsymbol{a} + \boldsymbol{a}'$. We get

$$\begin{aligned} \beta(m + m', n) &= \sum_{i,j} (a_i + a'_i) \cdot b_j \cdot p_{i,j} \\ &= \Big(\sum_{i,j} a_i \cdot b_j \cdot p_{i,j} \Big) + \Big(\sum_{i,j} a'_i \cdot b_j \cdot p_{i,j} \Big) = \beta(m, n) + \beta(m', n). \end{aligned}$$

The universal property (T) says that there is an A-linear homomorphism

$$\phi : M \otimes_A N \to P$$

satisfying

$$\phi(m \otimes n) = \beta(m, n).$$

Taking $m = m_i$ and $n = n_j$ the coefficients are $\boldsymbol{a} = \delta_i$ and $\boldsymbol{b} = \delta_j$, so

$$\phi(m_i \otimes n_j) = \beta(m_i, n_j) = p_{i,j}$$

as required.

This ϕ is unique because its values on the generators $m_i \otimes n_j$ of $M \otimes_A N$ are prescribed. □

Proposition 5.18. *Let $\phi : M \to M'$ and $\psi : N \to N'$ be A-module homomorphisms. There is a unique A-module homomorphisms*

$$\phi \otimes \psi : M \otimes_A N \to M' \otimes_A N'$$

such that

$$(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)$$

for all $m \in M$ and $n \in N$.

Exercise 5.19. Prove this proposition. (Hint: find a suitable bilinear function $\beta : M \times N \to M' \otimes_A N'$.)

Proposition 5.20. *Suppose B is another commutative ring and $f : A \to B$ is a ring homomorphism. Let M be an A-module. Then the A-module $B \otimes_A M$ has a unique B-module structure such that*

$$b \cdot (1 \otimes m) = b \otimes m$$

for all $m \in M$ and $b \in B$.

Exercise 5.21. Prove this proposition. (Hint: first define $b \cdot w$ for every $w \in B \otimes_A M$, using a bilinear function. Then verify the axioms of a B-module.)

Even though the next example is not crucial for our course, I think it has pedagogial value.

Example 5.22. Here we exhibit tensors which are not pure. Take a field $\mathbb{K}$, and let $M := \mathbb{K}^r$ for some integer $r \geq 2$, with standard basis $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_r)$. We view M is a module of rows. Let M^* be the dual module of M, namely $M^* := \mathbb{K}^r$ but viewed as columns. It is dual because every $\mathbb{K}$-linear homomorphism (called functional sometimes) $\phi : M \to \mathbb{K}$ is $\phi(m) = m \cdot \mu$ for a unique column μ, where $m \cdot \mu$ is matrix multiplication. The basis of M^* we take is the dual basis $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r)$, where the column ϵ_i is the transpose of the row δ_i, so $\delta_i \cdot \epsilon_j$ is either 0 or 1.

The tensor product module is canonically isomorphic to the module of matrices:

$$M \otimes_{\mathbb{K}} M^* \cong \operatorname{Mat}_r(\mathbb{K}).$$

The basis element $\delta_i \otimes \epsilon_j$ goes to the matrix with 1 in the (j, i) place and 0 elsewhere.

Mini exercise: if $m = (a_1, \ldots)$ and $\mu = (b_1, \ldots)^{\mathrm{t}}$, what is the (j, i) entry of the matrix $m \otimes \mu$?

Now take some pure tensor

$$m \otimes \mu \in M \otimes_{\mathbb{K}} M^* \cong \operatorname{Mat}_r(\mathbb{K}).$$

The rows of the matrix $m \otimes \mu$ are $\mathbb{K}$-linear combinations of m. This implies that the rank of the matrix (the rank of its row module) is at most 1.

But there are matrices in $\operatorname{Mat}_r(\mathbb{K})$ of all ranks between 2 and r. The corresponding tensors in $M \otimes_{\mathbb{K}} M^*$ are not pure.

Remark 5.23. Later in the course we will prove that if A and B are (possibly NC) central $\mathbb{K}$-rings ($\mathbb{K}$ is some commutative ring), then the $\mathbb{K}$-module $A \otimes_{\mathbb{K}} B$ is also a central $\mathbb{K}$-ring, with multiplication satisfying

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 \cdot a_2) \otimes (b_1 \cdot b_2).$$

The function $A \to A \otimes_{\mathbb{K}} B$, $a \mapsto a \otimes 1_B$, is a ring homomorphism, and so is the analogous $B \to A \otimes_{\mathbb{K}} B$. Moreover, $A \otimes_{\mathbb{K}} B$ is the *coprodcut* of A and B in the category $\mathsf{Rng}/_{\mathrm{c}}\, \mathbb{K}$.

6. Functors

In Section 4 we learned about *categories*. We saw a few examples of categories:

- The category Set of sets. Its morphisms are the functions.
- The category Grp of groups. Its morphisms are the group homomorphisms.
- The category Ab of abelian groups. Its morphisms are the group homomorphisms. It is a full subcategory of Grp.

- The category $\mathsf{Mod}(A)$ of modules over a ring A. Its morphisms are the A-linear homomorphisms.
- The category Rng of NC rings. Its morphisms are the ring homomorphisms.

The categories above have large sets of objects.

But we also saw categories with very few objects. In Exa 4.30 we saw how a ring can be made into a single-object category.

Say A and B are rings, and $f : A \to B$ is a ring homomorphism. Let A and B be the corresponding single-object categories. What plays the role of f in the theory of categories?

Definition 6.1. Let C and D be categories. A *functor*

$$F : \mathsf{C} \to \mathsf{D}$$

consists of a function

$$F_{\mathrm{ob}} : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D}),$$

and for every pair of objects $C_0, C_1 \in \mathrm{Ob}(\mathsf{C})$ a function

$$F_{C_0,C_1} : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}\big(F_{\mathrm{ob}}(C_0), F_{\mathrm{ob}}(C_1)\big).$$

There are two conditions:

(i) Identities: $F_{C,C}(\mathrm{id}_C) = \mathrm{id}_{F_{\mathrm{ob}}(C)}$.

(ii) Composition: For all $C_0, C_1, C_2 \in \mathrm{Ob}(\mathsf{C})$ and $\phi_i \in \mathrm{Hom}_{\mathsf{C}}(C_{i-1}, C_i)$ there is equality

$$F_{C_1,C_2}(\phi_2) \circ F_{C_0,C_1}(\phi_1) = F_{C_0,C_2}(\phi_2 \circ \phi_1).$$

Usually we suppress the subscripts from F_{ob} and F_{C_0,C_1}.

Example 6.2. Again A and B are rings, and $f : A \to B$ is a ring homomorphism. Let A and B be the corresponding single-object categories, with $\mathrm{Ob}(\mathsf{A}) = \{x\}$ and $\mathrm{Ob}(\mathsf{B}) = \{y\}$.

We construct a functor $F : \mathsf{A} \to \mathsf{B}$ as follows.

On objects there is no choice but $F(x) := y$.

On morphisms we define

$$F : \mathrm{Hom}_{\mathsf{A}}(x, x) = A \to \mathrm{Hom}_{\mathsf{B}}(y, y) = B$$

to be f.

The axioms of a functor are satisfied:

$$F(\mathrm{id}_x) = f(1_A) = 1_B = \mathrm{id}_y \,.$$

Similarly composition is respected, due to the second axiom.

The fact that f respects addition will be discussed later (linear functors).

comment: (211201 AY) End lecture 7

comment: Start of Lecture 8, 8 Dec 2021

Since a few students did answer Exer 4.28 correclty, and since this is important for us, here is a proof.

Solution of Exercise 4.28. We are given a commutative ring $\mathbb{K}$ a NC central $\mathbb{K}$-ring B. We need to prove that the category $\mathsf{Mod}(B)$ of left B-modules is $\mathbb{K}$-linear.

Let us denote the structural ring homomorphism of B by $f_B : \mathbb{K} \to B$.

The first task is to give each Hom set in the category a $\mathbb{K}$-module structure.

Take objects $M_0, M_1 \in \mathsf{Mod}(B)$. For $\phi \in \operatorname{Hom}_B(M_0, M_1)$ and $k \in \mathbb{K}$ we define the function $k \cdot \phi : M_0 \to M_1$ to be

$$(k \cdot \phi)(m) := f_B(k) \cdot \phi(m)$$

for $m \in M_0$. We must prove that $k \cdot \phi \in \operatorname{Hom}_B(M_0, M_1)$, i.e. it is B-linear. It is very easy to show that $k \cdot \phi$ respects addition, so we'll skip this. What is interesting and important is that $k \cdot \phi$ respects the action of B, and we will prove this. For $b \in B$ and $m \in M_0$ we have

$$\begin{aligned} &(k \cdot \phi)(b \cdot m) =^{(\mathrm{i})} f_B(k) \cdot \phi(b \cdot m) =^{(\mathrm{ii})} f_B(k) \cdot b \cdot \phi(m) =^{(\mathrm{iii})} b \cdot f_B(k) \cdot \phi(m) \\ &\quad =^{(\mathrm{i})} b \cdot (k \cdot \phi)(m). \end{aligned}$$

The equalities (i) are by the definition of $k \cdot \phi$. The equality (ii) is because ϕ is B-linear. The equality (iii) is because $f_B(k)$ *is in the center of* B.

Now we have to show that the composition in $\mathsf{Mod}(B)$ is $\mathbb{K}$-bilinear. Again we will skip showing that composition is additive in the first and second arguments. As for the action of B, let $\phi_i \in \operatorname{Hom}_B(M_{i-1}, M_i)$, $m \in M_0$ and $k \in \mathbb{K}$. Then

$$\begin{aligned} ((k \cdot \phi_1) \circ \phi_0)(m) &= (k \cdot \phi_1)(\phi_0(m)) = f_B(k) \cdot \phi_1((\phi_0(m))) \\ &= f_B(k) \cdot (\phi_1 \circ \phi_0)(m) = (k \cdot (\phi_1 \circ \phi_0))(m). \end{aligned}$$

As m changes we get $(k \cdot \phi_1) \circ \phi_0 = k \cdot (\phi_1 \circ \phi_0)$. A similar calculation shows that $\phi_1 \circ (k \cdot \phi_0) = k \cdot (\phi_1 \circ \phi_0)$.

This finishes the exercise.

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Here is another exercise on linear categories.

Exercise 6.3. Let $\mathsf{M} := \mathsf{Mod}(\mathbb{Z}) = \mathsf{Ab}$. Calculate the rings $\operatorname{End}_{\mathsf{M}}(M)$ and where:

(1) $M := \mathbb{Z}^{12}$.

(2) $M := \mathbb{Z} \oplus (\mathbb{Z}/(10))$.

(3) $M := (\mathbb{Z}/(3)) \oplus (\mathbb{Z}/(10))$.

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Now to new material on functors.

Many functors are "forgetful functors". Here is the protoypical example.

Example 6.4. Let A be a ring. The forgetful functor

$$F : \mathsf{Mod}(A) \to \mathsf{Set}$$

sends an A-module M to its underlying set, and a module homomorphism ϕ to the function between the underlying sets.

Definition 6.5. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between categories. The functor F is called *full* (resp. *faithful*) if for every pair of objects $C_0, C_1 \in \mathsf{C}$ the function

$$F : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}\big(F(C_0), F(C_1)\big)$$

is surjective (resp. injective).

Exercise 6.6.

(1) Show that the forgetful functor $F : \mathsf{Mod}(A) \to \mathsf{Set}$ is faithful but not full.

(2) Find a functor $F : \mathsf{C} \to \mathsf{D}$ which is full but not faithful.

Exercise 6.7. Functors can be *composed*. The exercise is to write the precise formulas for the composition $G \circ F$ of a functor $F : \mathsf{C} \to \mathsf{D}$ with a functor $G : \mathsf{D} \to \mathsf{E}$.

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Exercise 6.8. Let A be a nonzero ring (commutative). Show that there is a unique functor

$$G : \mathsf{Set} \to \mathsf{Mod}(A)$$

such that on objects it is

$$G(X) = \mathrm{F}_{\mathrm{fin}}(X, A),$$

the module of finitely supported functions, and for a function $f : X \to Y$ it is the homomorphism

$$G(f) = \textstyle\sum_f : \mathrm{F}_{\mathrm{fin}}(X, A) \to \mathrm{F}_{\mathrm{fin}}(Y, A)$$

from Exercise 3.33. The "integration" formula in that exercise can be replaced by the universal property of a basis in Thm 3.22; the homomorphism $G(f)$ must send the basis element $\delta_x \in \mathrm{F}_{\mathrm{fin}}(X, A)$ to the element $\delta_{f(x)} \in \mathrm{F}_{\mathrm{fin}}(Y, A)$.

G is called the *free module functor*.

Later we will explore the relation between the free module functor G and the forgetful functor F from Example 6.4. The next exercise is a first step in this direction.

Exercise 6.9. Continuing with the setting of Example 6.4 and Exer 6.8:

(1) Show that for $X \in \mathsf{Set}$ and $M \in \mathsf{Mod}(A)$ there is a "canonical" bijection

$$\eta_{X,M} : \mathrm{Hom}_{\mathsf{Set}}(X, F(M)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{Mod}(A)}(G(X), M).$$

Hint: Use the universal propoerty of the free module.

(2) Given morphisms $f : X \to Y$ in Set and $\phi : N \to M$ in $\mathsf{Mod}(A)$, find "canonical" morphisms $\mathrm{Hom}_{\mathsf{Set}}(f, F(\phi))$ and $\mathrm{Hom}_{\mathsf{Mod}(A)}(G(f), \phi)$ tha make a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Set}}(X, F(M)) \arrow[r, "\eta_{X,M}"] & \mathrm{Hom}_{\mathsf{Mod}(A)}(G(X), M) \\
\mathrm{Hom}_{\mathsf{Set}}(Y, F(N)) \arrow[u, "\mathrm{Hom}_{\mathsf{Set}}(f, F(\phi))"] \arrow[r, "\eta_{Y,N}"] & \mathrm{Hom}_{\mathsf{Mod}(A)}(G(Y), N) \arrow[u, "\mathrm{Hom}_{\mathsf{Mod}(A)}(G(f), \phi)"']
\end{tikzcd}$$

in the category Set.

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Here are two examples of functors from $\mathsf{Mod}(A)$ to itself, where A is now a commutative ring.

Example 6.10. Let A be a commutative ring.

Fix $M \in \mathsf{Mod}(A)$. There is a unique functor

$$G : \mathsf{Mod}(A) \to \mathsf{Mod}(A)$$

such that

$$G(N) = M \otimes_A N$$

on objects, and

$$G(\psi : N \to N') = \mathrm{id}_M \otimes \psi$$

on morphisms.

In other words, on pure tensors

$$G(\psi)(m \otimes n) = m \otimes \psi(n) \in M \otimes_A N'.$$

This is an easy consequence of Proposition 5.18.

Definition 6.11. Given morphisms $\phi : M' \to M$ and $\psi : N \to N'$ in $\mathsf{Mod}(A)$, let

$$\mathrm{Hom}_A(\phi, \psi) : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N')$$

be

$$\mathrm{Hom}_A(\phi, \psi)(\mu) := \psi \circ \mu \circ \phi.$$

Example 6.12. Fix $M \in \mathsf{Mod}(A)$. There is a unique functor

$$F : \mathsf{Mod}(A) \to \mathsf{Mod}(A)$$

such that

$$F(N) = \mathrm{Hom}_A(M, N)$$

on objects, and

$$F(\psi : N \to N') = \mathrm{Hom}_A(\mathrm{id}_M, \psi)$$

on morphisms.

Example 6.13. Suppose we try to change things around in the previous example; namely we fix $N \in \mathsf{Mod}(A)$, and we try to define a functor

$$H : \mathsf{Mod}(A) \to \mathsf{Mod}(A)$$

such that

$$H(M) = \mathrm{Hom}_A(M, N)$$

on objects.

The only formula that seems to make sense on morphisms is

$$H(\phi : M \to M') = \mathrm{Hom}_A(\phi, \mathrm{id}_M).$$

But this is a homomorphism

$$H(M') \to H(M).$$

This is in the wrong direction! What to do?

The answer is: a new definition.

Definition 6.14. Let C and D be categories. A *contravariant functor*

$$F : \mathsf{C} \to \mathsf{D}$$

consists of a function

$$F : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D}),$$

and for every pair of objects $C_0, C_1 \in \mathrm{Ob}(\mathsf{C})$ a function

$$F : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}\big(F(C_1), F(C_0)\big).$$

There are two conditions:

(i) Identities: $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$.
(ii) Composition, reversed: For all $C_0, C_1, C_2 \in \mathrm{Ob}(\mathsf{C})$ and $\phi_i \in \mathrm{Hom}_{\mathsf{C}}(C_{i-1}, C_i)$ there is equality

$$F(\phi_1) \circ F(\phi_2) = F(\phi_2 \circ \phi_1)$$

in $\mathrm{Hom}_{\mathsf{D}}\big(F(C_2), F(C_0)\big)$.

An ordinary functor (Definition 6.1) is sometimes called a *covariant functor*.

Remark 6.15. Later we might talk about the *opposite category* C^{op} of a given category C. If so, then we will see that contravariant functors $\mathsf{C} \to \mathsf{D}$ are the same as (covariant) functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$.

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Definition 6.16. Let $\mathbb{K}$ be a commutative ring (the base ring) and let M and N be $\mathbb{K}$-linear categories. An $\mathbb{K}$-*linear functor* $F : \mathsf{M} \to \mathsf{N}$ is a functor such that for every pair of objects $M_0, M_1 \in \mathsf{M}$ the function

$$F : \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N}}\big(F(M_0), F(M_1)\big)$$

is $\mathbb{K}$-linear.

Example 6.17. Consider $\mathsf{M} = \mathsf{N} := \mathsf{Mod}(A)$. The functors F and G from Examples 6.10 and 6.12 are A-linear functors. The functor H from Example 6.13 is a contravariant A-linear functor.

Exercise 6.18. Let $F : \mathsf{M} \to \mathsf{N}$ be a $\mathbb{K}$-linear functor between $\mathbb{K}$-linear categories. Let $M \in \mathsf{M}$. Prove that the function

$$F : \mathrm{End}_{\mathsf{M}}(M) \to \mathrm{End}_{\mathsf{N}}\big(F(M)\big)$$

is a homomorphism of $\mathbb{K}$-rings. Cf. Exer 4.29.

comment: (211208 AY) End lecture 8

comment: Start of Lecture 9, 15 Dec 2021

comment: (211213 AY) Reminder: Please solve the exercises from lecture 8, plus Exercise 7.10.

7. Exact Linear Functors

Setup 7.1. In this section $\mathbb{K}$ is a commutative ring (the base ring), and A and B are central $\mathbb{K}$-rings.

A *sequence* in $\mathsf{Mod}(A)$ is a diagram

$$\boldsymbol{S} = \big(\cdots M_{i-1} \xrightarrow{\phi_{i-1}} M_i \xrightarrow{\phi_i} M_{i+1} \xrightarrow{\phi_{i+1}} M_{i+2} \cdots \big) \tag{7.2}$$

in $\mathsf{Mod}(A)$, finite or infinite in either direction.

Thus the M_i are A-modules, and the ϕ_i are A-module homomorphisms $M_i \to M_{i+1}$.

The sequence $\boldsymbol{S}$ in (7.2) can extend infinitely to the left, but if not, then there is a first module appearing in $\boldsymbol{S}$, say M_{i_0}. Likewise, the sequence $\boldsymbol{S}$ can extend infinitely to the right, but if not, then there is a last module appearing in $\boldsymbol{S}$, say M_{i_0}. A module M_i appearing in $\boldsymbol{S}$ that's neither first nor last is called internal.

Definition 7.3. Consider a sequence $\boldsymbol{S}$ in $\mathsf{Mod}(A)$, in the notation of (7.2).

(1) $\boldsymbol{S}$ is called *exact* at some internal module M_i if
$$\mathrm{Im}(\phi_{i-1}) = \mathrm{Ker}(\phi_i)$$
as submodules of M_i.

(2) The sequence $\boldsymbol{S}$ is called *exact* if it exact at every internal module M_k in it.

Definition 7.4. A *short exact sequence* in $\mathsf{Mod}(A)$ is an exact sequence $\boldsymbol{S}$ with 5 modules appearing in it, in which the first and last modules are 0.

Explicitly, it is an exact sequence of this shape:

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0. \tag{7.5}$$

Example 7.6. Let M be a module, $M' \subseteq M$ a submodule with inclusion homomorphism $\epsilon : M' \rightarrowtail M$, and $M'' := M/M'$ with projection $\pi : M \twoheadrightarrow M''$. Then
$$0 \to M' \xrightarrow{\epsilon} M \xrightarrow{\pi} M'' \to 0,$$
is a short exact sequence.

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Example 7.7. Let X be a differentiable real manifold (of type C^∞) of dimension n.

If you are not familiar with this concept, you may assume that $X = \mathbb{R}^n$.

For every $0 \leq p \leq n$ let $\Omega^p(X)$ be the $\mathbb{R}$-module of degree p differential forms on X. In particular, $\Omega^0(X) = \mathcal{O}(X)$, the ring of differentiable functions $X \to \mathbb{R}$.

If $X = \mathbb{R}^n$, with coordinate functions $t_i : \mathbb{R}^n \to \mathbb{R}$, then

$$\Omega^1(X) = \bigoplus_{i=1}^{n} \mathcal{O}(X) \cdot \mathrm{d}(t_i),$$

a free $\mathcal{O}(X)$-module of rank n.

And $\Omega^p(X)$ is a $\mathcal{O}(X)$-module of rank $\binom{n}{p}$, with basis the differential forms $\mathrm{d}(t_{i_1}) \wedge \cdots \wedge \mathrm{d}(t_{i_p})$ indexed by $0 \leq i_1 < \cdots i_p \leq n$.

The exterior derivative

$$\mathrm{d} : \Omega^p(X) \to \Omega^{p+1}(X)$$

satisfies $\mathrm{d} \circ \mathrm{d} = 0$.

The formula for d is thi. For a function $a \in \mathcal{O}(X)$ it is

$$\mathrm{d}(a) = \sum_i \frac{\partial f}{\partial t_i} \cdot \mathrm{d}(t_i).$$

For forms $\omega \in \Omega^p(X)$ and $\xi \in \Omega^q(X)$ the differential is

$$\mathrm{d}(\omega \wedge \xi) = \mathrm{d}(\omega) \wedge \xi + (-1)^{p+1} \omega \wedge \mathrm{d}(\xi).$$

The *de Rham complex* is the sequence

$$\Omega(X) = \big(0 \to \Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(X) \to 0\big).$$

Note that each $\Omega^p(X)$ is an $\mathcal{O}(X)$-module, but d is not $\mathcal{O}(X)$-linear, so this is a sequence in $\mathsf{Mod}(\mathbb{R})$ but not in $\mathsf{Mod}(\mathcal{O}(X))$.

A differential form ω is called *exact* if $\mathrm{d}(\omega) = 0$.

A differential form ω is called *closed* if $\omega = \mathrm{d}(\xi)$ for some ξ.

The *augmented de Rham complex* is the sequence

$$0 \to \mathbb{R} \to \Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(X) \to 0. \tag{7.8}$$

Thus the de Rham complex $\Omega(X)$ is an exact sequence iff every exact form is closed.

The *Poincaré Lemma* tells us that if X is contractible, then the augmented de Rham complex is an exact sequence.

The fact that $\mathrm{d} \circ \mathrm{d} = 0$ implies that

$$\mathrm{Im}\big(\mathrm{d} : \Omega^{p-1}(X) \to \Omega^p(X)\big) \subseteq \mathrm{Ker}\big(\mathrm{d} : \Omega^p(X) \to \Omega^{p+1}(X)\big) \subseteq \Omega^p(X).$$

The $\mathbb{R}$-module

$$\mathrm{H}^p_{\mathrm{DR}}(X) := \frac{\mathrm{Ker}\big(\mathrm{d} : \Omega^p(X) \to \Omega^{p+1}(X)\big)}{\mathrm{Im}\big(\mathrm{d} : \Omega^{p-1}(X) \to \Omega^p(X)\big)}$$

is called the *p-th de Rham cohomology* of X.

We see that the sequence (7.8) is exact iff $\mathrm{H}^0_{\mathrm{DR}}(X) = \mathbb{R}$ and $\mathrm{H}^p_{\mathrm{DR}}(X) = 0$ for all $p > 0$.

We are going to mimic this construction next.

Definition 7.9. A *complex of A-modules*, or a *complex in* $\mathsf{Mod}(A)$, is a diagram

$$M = \big(\cdots \to M^{i-1} \xrightarrow{\mathrm{d}^{i-1}} M^i \xrightarrow{\mathrm{d}^i} M^{i+1} \xrightarrow{\mathrm{d}^{i+1}} \cdots\big)$$

extending infinitely to both sides, such that $\mathrm{d}^{i+1} \circ \mathrm{d}^i = 0$.

The *i-th cohomology* of M is the A-module

$$\mathrm{H}^i(M) := \mathrm{Ker}(\mathrm{d}^i)/\mathrm{Im}(\mathrm{d}^{i-1}).$$

Exercise 7.10. Let M be a complex of A-modules. Prove that M is an exact sequence iff $\mathrm{H}^i(M) = 0$ for all i.

comment: (211208 AY) End lecture 9

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