Course Notes:

Homological Algebra

BGU, Spring 2019-20

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Available here:
https://www.math.bgu.ac.il/~amyekut/teaching/2019-20/homol-alg/course_page.html

Contents

comment: Start of Lecture 1, 11 March 2020. Lecture via SKYPE.

0. Introduction

This course is a continuation of the course "Commutative Algebra" from the previous semester. We will use the notes [Yek1] a lot.

Here is a revised syllabus for this course. It is tentative: I will change material, and the order of presentation, as I go along.

Course Topics:

(1) Adjoint functors, equivalences and exactness.
(2) Bimodules and noncommutative tensor products.
(3) Projective modules, invertible bimodules and Morita Theory.
(4) Injective modules, including Matlis Theory for noetherian commutative rings.
(5) Complexes of modules, homotopies and homotopy equivalences, quasi-isomorphisms.
(6) The long exact cohomology sequence.
(7) Projective, flat and injective resolutions of modules.
(8) Left and right derived functors.
(9) Applications of derived functors to commutative algebra.
(10) Further applications of derived functors and cohomology, including non-abelian cohomology.

The notes from the course "Homological Algebra" [Yek2] from two years ago cover topics 5-8 roughly. (They also include the material on categories and functors, that we already covered in [Yek1].)

1. Adjoint Functors and Equivalences

The material we start with is very abstract.

Categories and functors were introduced in Sections 4, 7 and 9 of [Yek1].

The product category $\mathsf{C} \times \mathsf{D}$ and the opposite category C^{op} were also introduced in [Yek1].

Recall that we are ignoring set theoretical issues (see [Yek1, Remark 4.3] or [Yek3, Section 1.1]).

We shall use the expression *morphism of functors* instead of *natural transformation*.

comment: (200317) next def corrected

Definition 1.1 (Adjoints)**.** Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors between categories.

An *adjunction between F and G* is a bijection of sets

$$\alpha_{D,C} : \mathrm{Hom}_{\mathsf{D}}\big(D, F(C)\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{C}}\big(G(D), C\big),$$

which is functorial in $D \in \mathsf{D}$ and $C \in \mathsf{C}$.

Namely the collection of isomorphisms

$$\alpha := \{\alpha_{D,C}\}_{(D,C) \in \mathsf{D} \times \mathsf{C}}$$

is an isomorphism

$$\alpha : \mathrm{Hom}_{\mathsf{D}}\big(-, F(-)\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{C}}\big(G(-), -\big)$$

of functors

$$\mathsf{D}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}\,.$$

Given such an adjunction, we say that *G is a left adjoint of F*, and that *F is a right adjoint of G*.

Remark 1.2. Adjoint functors were invented by D. Kan in 1958.

The name is borrowed from functional analysis. Let V and W be Hilbert spaces with inner products $\langle -, - \rangle_V$ and $\langle -, - \rangle_W$. Continuous linear operators $F : V \to W$ and $G : W \to V$ are called adjoint to each other if

$$\langle G(v), w \rangle_W = \langle v, F(w) \rangle_V$$

for all $v \in V$ and $w \in W$. The analogy is clear – but, since the inner products are symmetric, one just talks about $G = F^*$ being the adjoint of F.

Let's recall *equivalences of categories* from [Yek1, Definition 13.28]. A functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence if there is a functor $G : \mathsf{D} \to \mathsf{C}$ that is a quasi-inverse of F. In detail, there is an isomorphism of functors

$$\eta : G \circ F \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{C}} \tag{1.3}$$

from C to itself, and an isomorphism of functors

$$\zeta : F \circ G \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{D}} \tag{1.4}$$

from D to itself.

Theorem 1.5. *Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors. Assume F is an equivalence. TFAE*:

(i) *G is a quasi-inverse of F.*
(ii) *G is a left adjoint of F.*
(iii) *G is a right adjoint of F.*

A direct proof is very messy; see Remark 1.9 regarding an elegant (but not easy nor quick) proof.

I don't think we will need this theorem. In case it will be needed (for Morita Theory), I will give a proof of this theorem later.

The same goes for the next theorem:

Theorem 1.6. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between categories, and let $G, G' : \mathsf{D} \to \mathsf{C}$ be left adjoints of F. Then there is a unique isomorphism of functors $G \xrightarrow{\simeq} G'$ respecting the adjunction with F.*

Likewise for right adjoints.

Here is a variant of Theorem 1.6 that's easier:

Proposition 1.7. *If $F : \mathsf{C} \to \mathsf{D}$ is an equivalence, and if G and G' are quasi-inverses of F, then there is a unique isomorphism of functors $G \xrightarrow{\simeq} G'$ that respects the isomorphisms* (1.3) *and* (1.4).

Exercise 1.8. Prove this Proposition. (Hint: Study [Yek1, Example 13.29] and the results near it.)

Remark 1.9. The nice proofs of Theorems 1.5 and 1.6 require the *Yoneda Lemma*.

This is something I want to avoid doing in class (it is extremely abstract and confusing).

You can talk to the postdcos about this material – they will be glad to explain.

Here are a few useful examples of adjoints.

But first two conventions.

Convention 1.10. We fix a nonzero commutative base ring $\mathbb{K}$ (e.g. $\mathbb{Z}$ or a field). All rings are assumed by default to be central $\mathbb{K}$-rings, and all ring homomorphisms are assumed by default to be $\mathbb{K}$-ring homomorphisms. (If $\mathbb{K} = \mathbb{Z}$ then this is automatically satisfied.)

More generally, all linear categories are assumed by default to be $\mathbb{K}$-linear categories, and all linear functors between them as assumed by default to be $\mathbb{K}$-linear.

The expression $\otimes$ means $\otimes_{\mathbb{K}}$.

The base ring $\mathbb{K}$ will usually remain implicit.

Warning: unlike the previous course, here rings are not assume to be commutative!

Notation 1.11. The category of central $\mathbb{K}$-rings is $\mathsf{Rng} /_{\mathrm{c}} \mathbb{K}$.

The full subcategory of commutative $\mathbb{K}$-rings is $\mathsf{Rng}_{\mathrm{c}} / \mathbb{K}$.

Another caution: A homomorphism $f : A \to B$ in $\mathsf{Rng} /_{\mathrm{c}} \mathbb{K}$ need not itself be a central homomorphism. E.g. the inclusion of $A := \mathbb{K} \times \mathbb{K}$ into $B := \mathrm{Mat}_2(\mathbb{K})$, as the diagonal matrices, is not central.

On the other hand, a homomorphism $\mathbb{K} \to A$, when A is a commutative ring, is automatically central. This is why the notation $\mathsf{Rng}_{\mathrm{c}} /_{\mathrm{c}} \mathbb{K}$ is redundant.

Convention 1.12. For a ring A we write $\mathsf{M}(A) := \mathsf{Mod}\, A$, the category of left A-modules.

The the category of right A-modules is $\mathsf{M}(A^{\mathrm{op}})$.

These are $\mathbb{K}$-linear categories.

Example 1.13. Let A be a nonzero ring. We have the forgetful functor

$$F : \mathsf{M}(A) \to \mathsf{Set}\,.$$

There is the free module functor

$$G : \mathsf{Set} \to \mathsf{M}(A), \quad G(S) := A^{\oplus S} = A^{(S)} = \mathrm{F}_{\mathrm{fin}}(S, A) = \bigoplus_{s \in S} A.$$

As basis of $\mathrm{F}_{\mathrm{fin}}(S, A)$ we take the collection of delta functions $\{\delta_s\}_{s \in S}$.

We know that the free module has a universal property: Given an A-module N and a function of sets $f : S \to N$, there is a unique A-module homomorphism

$$G(f) : G(S) \to N, \quad G(f)(\delta_s) = f(s).$$

We can interpret this as an adjunction: G is a left adjoint of F, and F is a right adjoint of G.

To be precise, given a set S and a module M we define the bijection

$$\alpha_{S,M} : \mathrm{Hom}_{\mathsf{Set}}\big(S, F(M)\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{M}(A)}\big(G(S), M\big)$$

to be

$$\alpha_{S,M}(f)(\delta_s) := f(s) \in M$$

for

$$f \in \mathrm{Hom}_{\mathsf{Set}}\big(S, F(M)\big)$$

and $s \in S$.

Something needs to be verified – see exercise.

Exercise 1.14. Prove that

$$\alpha := \{\alpha_{S,M}\}_{(S,M) \in \mathsf{Set} \times \mathsf{M}(A)}$$

is a functor

$$\alpha : \mathsf{Set}^{\mathrm{op}} \times \mathsf{M}(A) \to \mathsf{Set}\,.$$

Exercise 1.15. Let $\mathsf{Set}_{\mathrm{fin}}$ be the category of finite sets and let $\mathsf{Rng}_{\mathsf{c}}/\mathbb{K}$ be the category of commutative $\mathbb{K}$-rings.

We have the forgetful functor

$$F : \mathsf{Rng}_{\mathsf{c}}/\mathbb{K} \to \mathsf{Set},$$

and the polynomial ring functor

$$G : \mathsf{Set} \to \mathsf{Rng}_{\mathsf{c}}/\mathbb{K}, \quad G(S) = \mathbb{K}[S],$$

where $\mathbb{K}[S]$ is the ring of polymomials of the finite set of variables S.

Prove that G is a left adjoint of F.

Exercise 1.16. Let $f : A \to B$ be a homomorphism between commutative rings.

We have seen the restriction functor

$$\mathrm{Rest}_f : \mathsf{M}(B) \to \mathsf{M}(A),$$

which is just a forgetful functor.

We have also seen the induction functor

$$\mathrm{Ind}_f : \mathsf{M}(A) \to \mathsf{M}(B), \quad M \mapsto B \otimes_A M.$$

Prove that Ind_f is a left adjoint of Rest_f.

We end this section with an example. It will be made more general later, and that general *Hom-tensor adjunction* is a very important fact.

Example 1.17. Let A be a commutative ring. Given $L, M, N \in \mathsf{M}(A)$ there is an isomorphism

$$\mathrm{adj}_{L,M,N} : \mathrm{Hom}_A(L \otimes_A M, N) \xrightarrow{\simeq} \mathrm{Hom}_A(L, \mathrm{Hom}_A(M, N)) \tag{1.18}$$

in $\mathsf{M}(A)$, called Hom-tensor adjunction.

It is functorial in L, M, N – see Exercise 1.19.

The formula is this: given

$$\phi : L \otimes_A M \to N$$

we define $\mathrm{adj}_{L,M,N}(\phi)$ to be

$$\mathrm{adj}_{L,M,N}(\phi)(l)(m) := \phi(l \otimes M) \in N.$$

Conversely, given

$$\psi \in \mathrm{Hom}_A(L, \mathrm{Hom}_A(M, N))$$

we have a homomorphism

$$\chi : L \otimes_A M \to N$$

with formula

$$\chi(l \otimes m) := \psi(l)(m) \in N.$$

The homomorphism $\psi \mapsto \chi$ is the inverse of $\mathrm{adj}_{L,M,N}$.

Exercise 1.19. Prove that

$$\mathrm{adj} : \mathsf{M}(A)^{\mathrm{op}} \times \mathsf{M}(A)^{\mathrm{op}} \times \mathsf{M}(A) \to \mathsf{M}(A)$$

is a morphism of functors.

comment: Start of Lecture 2, 18 March 2020. Lecture via ZOOM.

It turns out that we shall need Theorems 1.5 and 1.6 for Morita theory.

Most likely I will prove these theorems using the Yoneda Lemma. This will be done next week.

I have a feeling that Exer 1.8, i.e. proving Proposition 1.7, was too difficult. But this will be taken care of when we prove Theorem 1.6.

Let me discuss [Yek1, Thm 13.15], whose proof was also an exercise, but I doubt many were able to solve it. Therefore I will give the proof now.

Consider a functor $F : \mathsf{C} \to \mathsf{D}$. For every pair of objects $C_0, C_1 \in \mathsf{C}$ there is a function of sets

$$F_{C_0,C_1} : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}(F(C_0), F(C_1)).$$

The functor F is called *full* (resp. *faithful*, resp. *fully faithful*) if the functions F_{C_0,C_1} are all surjective (resp. injective, resp. bijective).

The functor F is said to be *essentially surjective on objects* if for every object $D \in \mathsf{D}$ there exists some object $C \in \mathsf{C}$ and an isomorphism $F(C) \xrightarrow{\simeq} D$ in D.

Theorem 1.20. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. TFAE:*

(i) *F is an equivalence.*
(ii) *F is fully faithful and essentially surjective on objects.*

Proof. The proof is in a few steps.

Step 1. Assume F is an equivalence. In this step we shall prove that F is essentially surjective on objects.

Let $G : \mathsf{D} \to \mathsf{C}$ be a quasi-inverse of F, equipped with isomorphisms of functors

$$\eta : G \circ F \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{C}}$$

and

$$\zeta : F \circ G \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{D}};$$

see formula (1.3) and (1.4).

Given an object $D \in \mathsf{D}$, let $C := G(D) \in \mathsf{C}$. We then have an isomorphism $\zeta_D : F(C) \xrightarrow{\simeq} D$ in D.

This establishes that F is essentially surjective on objects.

Step 2. Again assume F is an equivalence, with G, η and ζ as in step 1. Here we prove that F is faithful.

Take a pair of objects $C_1, C_2 \in \mathsf{C}$. Consider the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{C}}(C_1, C_2) & \xrightarrow{F_{C_1,C_2}} & \mathrm{Hom}_{\mathsf{D}}\big(F(C_1), F(C_2)\big) \\
& \searrow^{\simeq}_{\mathrm{Hom}(\eta_{C_1}, \eta_{C_2}^{-1})} & \downarrow G_{F(C_1),F(C_2)} \\
& & \mathrm{Hom}_{\mathsf{C}}\big((G \circ F)(C_1), (G \circ F)(C_2)\big)
\end{array} \tag{1.21}
$$

in Set. We claim it is commutative.

To see this, take a morphism $\gamma : C_1 \to C_2$ in C.

The image of γ by going right and then down in diagram (1.21) is $(G \circ F)(\gamma)$. Note that I am omitting the subscripts from $G \circ F$.

Because η is an isomorphism of functors, the diagram

$$
\begin{array}{ccc}
(G \circ F)(C_1) & \xrightarrow{(G \circ F)(\gamma)} & (G \circ F)(C_2) \\
\eta_{C_1} \downarrow \simeq & & \simeq \downarrow \eta_{C_2} \\
C_1 & \xrightarrow{\gamma} & C_2
\end{array} \tag{1.22}
$$

is commutative.

The commutativity of diagram (1.22) says that

$$(G \circ F)(\gamma) = \mathrm{Hom}(\eta_{C_1}, \eta_{C_2}^{-1})(\gamma).$$

So indeed diagram (1.21) is commutative.

The commutativity of diagram (1.21) implies that the function F_{C_1,C_2} is injective.

Thus F is a faithful functor.

Step 3. Again assume F is an equivalence, with G, η and ζ as in step 1. Here we prove that F is full.

Doing step 2, but with the roles of F and G reversed, we see that G is faithful. Thus for every $D_1, D_2 \in \mathsf{D}$ the function

$$G : \mathrm{Hom}_{\mathsf{D}}(D_1, D_2) \to \mathrm{Hom}_{\mathsf{D}}(G(D_1), G(D_2))$$

is injective.

Taking $D_i := F(C_i)$, it follows that the function $G_{F(C_1),F(C_2)}$ in diagram (1.21) is injective.

But

$$G_{F(C_1),F(C_2)} \circ F_{C_1,C_2} = \mathrm{Hom}(\eta_{C_1}, \eta_{C_2}^{-1}),$$

and this is a bijection. We see that F_{C_1,C_2} is surjective.

Thus F is a full functor.

Step 4. Now we assume that F is fully faithful and essentially surjective on objects, and we construct a quasi-inverse G.

For every object $D \in \mathsf{D}$ choose an object $C \in \mathsf{C}$ with an isomorphism $\zeta_D : F(C) \xrightarrow{\simeq} D$ in D.

Define the function

$$G : \mathrm{Ob}(\mathsf{D}) \to \mathrm{Ob}(\mathsf{C})$$

by letting $G(D)$ be the object C chosen above. Thus we get an isomorphism

$$\zeta_D : (F \circ G)(D) \xrightarrow{\simeq} D$$

in D.

Step 5. Continuing from step 4, for a morphism $\psi : D_1 \to D_2$ in D define the morphism

$$G(\psi) : G(D_1) \to G(D_2)$$

as follows.

Let $\tilde{\psi}$ be the unique morphism in D for which the diagram

$$\begin{array}{ccc} F(G(D_1)) & \xrightarrow{\tilde{\psi}} & F(G(D_2)) \\ {\scriptstyle \zeta_{D_1}} \downarrow {\scriptstyle \simeq} & & {\scriptstyle \simeq} \downarrow {\scriptstyle \zeta_{D_2}} \\ D_1 & \xrightarrow{\psi} & D_2 \end{array} \tag{1.23}$$

is commutative.

Next let $G(\psi)$ be the unique morphism that goes to $\tilde{\psi}$ under the bijection

$$F : \mathrm{Hom}_{\mathsf{C}}\big(G(D_1), G(D_2)\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}}\big(F(G(D_1)), F(G(D_2))\big).$$

Then G is a functor $\mathsf{D} \to \mathsf{C}$. (Exercise.)

By construction,

$$\zeta = \{\zeta_D\}_{D \in \mathsf{D}} : F \circ G \to \mathrm{Id}_{\mathsf{D}}$$

is an isomorphism of functors. (Exercise.)

Step 6. Finally we define the isomorphism

$$\eta_C : G(F(C)) \xrightarrow{\simeq} C,$$

for $C \in \mathsf{C}$, as follows.

From step 4 we already have an isomorphism

$$\zeta_{F(C)} : F(G(F(C))) \xrightarrow{\simeq} F(C)$$

in D.

Since F is fully faithful, there is a unique morphism η_C that goes to $\zeta_{F(C)}$ under the bijection

$$F : \mathrm{Hom}_{\mathsf{C}}\big(G(F(C), C\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}}\big(F(G(F(C))), F(C)\big).$$

Moreover, this η_C is an isomorphism, since $\zeta_{F(C)}$ is an isomorphism. (Exercise.)

By construction,

$$\eta = \{\eta_C\}_{C \in \mathsf{C}} : G \circ F \to \mathrm{Id}_{\mathsf{C}}$$

is an morphism of functors. (Exercise.) □

Exercise 1.24. Finish the arguments in the four places in the proof above marked "Exercise".

Let us examine a concrete situation of adjoint functors. This will be generalized later to the noncommutative setting.

Let $f : A \to B$ be a homomorphism of commutative rings. A B-module P is also an A-module by Rest_f.

Therefore it makes sense to look at the functors

$$G_P : \mathsf{M}(A) \to \mathsf{M}(B), \quad G_P(M) := P \otimes_A M. \tag{1.25}$$

and

$$F_P : \mathsf{M}(B) \to \mathsf{M}(A), \quad F_P(N) := \mathrm{Hom}_B(P, N). \tag{1.26}$$

These are both A-linear functors.

Example 1.27. If we take $P := B$, then $G_P = \mathrm{Ind}_f$ and $F_P = \mathrm{Rest}_f$.

Proposition 1.28. *Given a homomorphism $A \to B$ of commutative rings and a B-module P, the functor G_P is left adjoint to the functor F_P.*

Proof. We have to construct an A-linear isomorphism

$$\alpha_{M,N} : \mathrm{Hom}_{\mathsf{M}(A)}(M, F_P(N)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{M}(B)}(G_P(M), N) \tag{1.29}$$

that's functorial in $M \in \mathsf{M}(A)$ and $N \in \mathsf{M}(B)$.

In other words, we are looking for a "canonical" A-linear homomorphism

$$\alpha_{M,N} : \mathrm{Hom}_A\big(M, \mathrm{Hom}_B(P, N)\big) \to \mathrm{Hom}_B\big(P \otimes_A M, N\big).$$

"Canonical" will imply functorial. Also it should be bijective.

We define $\alpha_{M,N}$ as follows: for

$$\phi \in \mathrm{Hom}_A\big(M, \mathrm{Hom}_B(P, N)\big),$$

$p \in P$ and $m \in M$, we let

$$\alpha_{M,N}(\phi)(p \otimes m) := \phi(m)(p) \in N.$$

This is a well-defined A-linear homomorphism, and it is functorial in M and N.

Next we construct and A-linear homomorphism

$$\beta_{M,N} : \mathrm{Hom}_B\big(P \otimes_A M, N\big) \to \mathrm{Hom}_A\big(M, \mathrm{Hom}_B(P, N)\big).$$

Its formula is

$$\beta_{M,N}(\psi)(m)(p) := \psi(p \otimes m) \in N.$$

This is a well-defined A-linear homomorphism.

The homomorphisms $\alpha_{M,N}$ and $\beta_{M,N}$ are inverses of each other. Hence $\alpha_{M,N}$ is an isomorphism. □

Remark 1.30. Prop 1.28 will be generalized greatly next week – the rings A and B will be noncommutative central $\mathbb{K}$-rings, and there won't be a ring homomorphism $A \to B$. Instead of a module P, we will have a B-A-bimodule P. And so on. Even in this generality the functors G_P and F_P will exists, and they will be $\mathbb{K}$-linear, and adjoints to each other (with basically the same proof!).

Remark 1.31. Still in the commutative setting of Prop 1.28, assume that F_P is an equivalence. By Theorem 1.5 the functor G_P is a quasi-inverse of G_P. (And vice versa).

For the sake of simplicity assume A and B are noetherian. We will prove that $A = B$, and that the A-module P is projective of rank 1; i.e. P is a finitely generated projective A-module, and for every prime ideal $\mathfrak{p} \subseteq A$ the $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ is free of rank 1.

comment: Start of Lecture 3, 25 March 2020. Lecture via ZOOM.

For new students (new to my courses): If a certain homework exercise is too easy for you, you can answer it by “This is clear to me”. It is your responsibility to know the full answer!

2. Opposite Categories and Opposite Rings

After trying to write clean proofs of Thms 1.5 and 1.6 using the *Yoneda Lemma*, I realized this is impossible. More precisely: it would take too much time to give a nice full proof, and giving a sketchy proof will be very obscure and not useful. Interested students can discuss these matters with Mattia – he will explain gladly.

As a consequence, I won’t be able to prove the *Morita Theorems* in their full generality. So I will just state them (later on) without proofs. (Unfortunately, I do not know a good reference for these theorems. Standard ring theory books give lousy treatment of this topic. You can glance at the derived version, which is Chapter 14 of my book [Yek3].)

Instead of the full Morita Theorems, I will give a detailed proof of the “Baby Morita Theorem”, on the equivalence between the categories $\mathsf{M}(A)$ and $\mathsf{M}(B)$, where A is a commutative ring and $B := \operatorname{Mat}_n(A)$ for $n \geq 2$.

Today I will begin by explaining carefully what is the *opposite category*. This concept is needed in order to understand adjoint functors (that we already defined last week) and other constructions.

Given a category C, its opposite category C^{op} has the same objects, but its arrows and their compositions are reversed.

Let me describe this in detail.

First, define the set of objects of the new category C^{op} to be

$$\operatorname{Ob}(\mathsf{C}^{\mathrm{op}}) := \operatorname{Ob}(\mathsf{C}). \tag{2.1}$$

The identity automorphism of the set $\operatorname{Ob}(\mathsf{C})$ is now written as a bijection

$$\operatorname{Op} : \operatorname{Ob}(\mathsf{C}) \xrightarrow{\simeq} \operatorname{Ob}(\mathsf{C}^{\mathrm{op}}). \tag{2.2}$$

This means that every object $D \in \operatorname{Ob}(\mathsf{C}^{\mathrm{op}})$ can be expressed as $D = \operatorname{Op}(C)$ for a unique $C \in \operatorname{Ob}(\mathsf{C})$.

Now to morphisms. Given a pair of objects $C_0, C_1 \in \operatorname{Ob}(\mathsf{C})$, we let

$$\operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}\big(\operatorname{Op}(C_1), \operatorname{Op}(C_0)\big) := \operatorname{Hom}_{\mathsf{C}}\big(C_0, C_1\big). \tag{2.3}$$

There is a bijection of sets (the identity automorphism in disguise)

$$\operatorname{Op} : \operatorname{Hom}_{\mathsf{C}}\big(C_0, C_1\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}\big(\operatorname{Op}(C_1), \operatorname{Op}(C_0)\big). \tag{2.4}$$

This means that every morphism $\psi : D_1 \to D_0$ in C^{op} can be expressed as $\psi = \operatorname{Op}(\phi)$ for a unique morphism $\phi : C_0 \to C_1$ in C, with $D_i = \operatorname{Op}(C_i)$.

The composition $\circ^{\mathrm{op}}$ of C^{op} is as follows. Given morphisms $\psi_2 : D_2 \to D_1$ and $\psi_1 : D_1 \to D_0$ in C^{op}, let's express them as $\psi_i = \mathrm{Op}(\phi_i)$, for morphisms $\phi_i : C_{i-1} \to C_i$, with $D_i = \mathrm{Op}(C_i)$. Then the composition is

$$\psi_1 \circ^{\mathrm{op}} \psi_2 = \mathrm{Op}(\phi_1) \circ^{\mathrm{op}} \mathrm{Op}(\phi_2) := \mathrm{Op}(\phi_2 \circ \phi_1). \tag{2.5}$$

In diagrams: first the commutative diagram of the composition in C.

$$\overset{\phi_2 \circ \phi_1}{\overbrace{C_0 \xrightarrow{\ \phi_1\ } C_1 \xrightarrow{\ \phi_2\ } C_2}}$$

Now the commutative diagram of the composition in C^{op}.

$$\overset{\psi_1 \circ \psi_2 = \mathrm{Op}(\phi_2 \circ \phi_1)}{\overbrace{D_2 = \mathrm{Op}(C_2) \xrightarrow{\ \psi_2 = \mathrm{Op}(\phi_2)\ } D_1 = \mathrm{Op}(C_1) \xrightarrow{\ \psi_1 = \mathrm{Op}(\phi_1)\ } D_0 = \mathrm{Op}(C_0)}}$$

Lastly, the identity automorphism of an object $D \in \mathsf{C}^{\mathrm{op}}$ is

$$\mathrm{id}_D := \mathrm{Op}(\mathrm{id}_C) \tag{2.6}$$

where $C \in \mathrm{Ob}(\mathsf{C})$ and $D = \mathrm{Op}(C)$.

Exercise 2.7. Let C be a category.

(1) Prove that C^{op} is indeed a category, with set of objects (2.1), sets of morphisms (2.3), composition (2.5) and identities (2.6).
(2) Prove that $\mathrm{Op} : \mathsf{C} \to \mathsf{C}^{\mathrm{op}}$ is a contravariant functor.

Proposition 2.8. *Let C and D be categories. The formula $F \mapsto F \circ \mathrm{Op}$ is a bijection from the set of contravariant functors $F : \mathsf{C} \to \mathsf{D}$ to the set of* (*ordinary, or covariant*) *functors* $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$.

Exercise 2.9. Prove Prop 2.8.

Since we often have several categories under discussion, we may decorate the opposite functor of C like this:

$$\mathrm{Op}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}^{\mathrm{op}} . \tag{2.10}$$

Exercise 2.11. Let C be a category. Prove that

$$\mathrm{Op}_{\mathsf{C}^{\mathrm{op}}} \circ \mathrm{Op}_{\mathsf{C}} = \mathrm{Id}_{\mathsf{C}} .$$

(These are functors from C to itself.)

From now on we will usually deal only with (covariant) functors. Whenever we encounter a contravariant functor, we make it covariant be replacing the source category wiht its opposite. Fopr instance:

Example 2.12. Let C be a category and $C \in \mathsf{C}$. Then $\mathrm{Hom}_{\mathsf{C}}(-, C)$ is a contravariant functor $\mathsf{C} \to \mathsf{Set}$, but we prefer to see it a (covariant) functor

$$\mathrm{Hom}_{\mathsf{C}}(-, C) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set} .$$

* * *

Now to rings. Recall that according to Convention 1.10, all rings are $\mathbb{K}$-central, and $\otimes = \otimes_{\mathbb{K}}$.

Definition 2.13. Given a ring A, its *opposite ring* A^{op} has the same underlying $\mathbb{K}$-module as A, the same unit element, but the multiplication $\cdot^{\mathrm{op}}$ is the reversed:

$$\mathrm{op}(a_1) \cdot^{\mathrm{op}} \mathrm{op}(a_0) := \mathrm{op}(a_0 \cdot a_1).$$

Here $\mathrm{op} : A \to A^{\mathrm{op}}$ is the identity automorphism of the $\mathbb{K}$-module A.

The $\mathbb{K}$-linear bjection $\mathrm{op} : A \to A^{\mathrm{op}}$ is a *ring anti-automorphism*. Of course $A = A^{\mathrm{op}}$ iff A is a commutative ring.

Example 2.14. Some noncommutative rings are isomorphic to their opposites. Here are two such cases:

(1) A is a nonzero commutative ring, $n \geq 2$ and $B := \mathrm{Mat}_n(A)$. Then transposition is a ring isomorphism $B \xrightarrow{\simeq} B^{\mathrm{op}}$.
(2) Let G be a nonabelian group. Then $g \mapsto g^{-1}$ is a group isomorphism $\phi_G : G \to G^{\mathrm{op}}$. Let A be a nonzero commutative ring, and let $B := A[G]$, the *group ring*. (B is commutative iff G is abelian.) Then ϕ_G extends to a ring isomorphism $\phi_B : B \xrightarrow{\simeq} B^{\mathrm{op}}$.

Example 2.15. Given a ring A, we can make it into a single-object linear category A, say $\mathrm{Ob}(\mathsf{A}) = \{P\}$, such that $\mathrm{End}_{\mathsf{A}}(P) = A$.

Then the category corresponding to the opposite ring A^{op} is the opposite category A^{op}.

Exercise 2.16. Let A be a ring and let A be the corresponding single-object linear category. Define $\mathsf{LinFun}_{\mathbb{K}}(\mathsf{A}, \mathsf{M}(\mathbb{K}))$ to be the category whose objects are the $\mathbb{K}$-linear functors $F : \mathsf{A} \to \mathsf{M}(\mathbb{K})$, and whose morphisms are the morphisms of functors.

Prove that there is an equivalence (in fact an isomorphism) of $\mathbb{K}$-linear categories

$$\mathsf{LinFun}_{\mathbb{K}}(\mathsf{A}, \mathsf{M}(\mathbb{K})) \xrightarrow{\simeq} \mathsf{M}(A).$$

* * *

Recall that a *right A-module* is a $\mathbb{K}$-module M, equipped with an action

$$M \times A \to M, \quad (m, a) \mapsto m \cdot a$$

satisfying the obvious axioms, esp.

$$(m \cdot a_0) \cdot a_1 = m \cdot (a_0 \cdot a_1).$$

Exercise 2.17. Let M be a right A-module. Prove that the action

$$\mathrm{op}(a) \cdot m := m \cdot a$$

makes M into a left A^{op}-module.

The exercise shows that the category of right A-modules is isomorphic, as a $\mathbb{K}$-linear category, to $\mathsf{M}(A^{\mathrm{op}})$. This fact is of great importance.

Recall that if M is an object in a linear category N, then

$$\mathrm{End}_{\mathsf{N}}(M) = \mathrm{Hom}_{\mathsf{N}}(M, M)$$

is a ring. In case $\mathsf{N} = \mathsf{M}(A)$ for a ring A, then

$$\mathrm{Hom}_{\mathsf{N}}(M, N) = \mathrm{Hom}_A(M, N).$$

Exercise 2.18. Let A be a nonzero ring. Let $n \geq 1$, and let $P_1, \ldots, P_n \in \mathsf{M}(A^{\mathrm{op}})$ be distinct free A^{op}-modules of rank 1, namely $P_i \cong A$ as A^{op}-modules (i.e. as right A-modules).

(1) Prove that there is a ring isomorphism

$$\phi_i : A \xrightarrow{\simeq} \mathrm{End}_{\mathsf{M}(A^{\mathrm{op}})}(P_i).$$

(2) Prove that the isomorphism ϕ_i you found in item (1) depends on a choice of basis element for $e_i \in P_i$. Give a formula for this dependence.

(3) Let A be the full subcategory of $\mathsf{M}(A^{\mathrm{op}})$ on the set of objects $\{P_1, \ldots, P_n\}$. Show that

$$B := \prod_{1 \leq i, j \leq n} \mathrm{Hom}_{\mathsf{A}}(P_i, P_j)$$

is a ring (give formulas for the multiplication and the unit element), and find a ring isomorphism $B \xrightarrow{\simeq} \mathrm{Mat}_n(A)$.

comment: End of Lecture 3, 25 March 2020. Lecture via ZOOM.

comment: Start of Lecture 4, 1 April 2020. Lecture via ZOOM.

First let me call everybody's attention to the statement of the Yoneda Lemma in [Yek3, Section 1.7], and to Mattia's proof of it (posted on the course web page). Please ask Mattia if you want to see how the Yoneda Lemma is used to prove Theorems 1.5 and 1.6.

3. Bimodules over NC Rings and Related Operations

Recall that all rings are central $\mathbb{K}$-rings, and $\otimes = \otimes_{\mathbb{K}}$.

Since I presume most of you are not familiar with bimodules, I will define them (rather than "recalling").

Definition 3.1. Let A and B be rings. A $\mathbb{K}$-*central* A-B-*bimodule* is a $\mathbb{K}$-module M, equipped with a left A-module structure and a right B-module structure, that commute with each other. Namely

$$a \cdot (m \cdot b) = (a \cdot m) \cdot b$$

for all $a \in A$, $b \in B$ and $m \in M$.

Moreover, the given $\mathbb{K}$-module structure of M is respected by the left A-module structure and the right B-module structure. Namely for $k \in \mathbb{K}$ we have

$$f_A(k) \cdot m = m \cdot f_B(k) = k \cdot m,$$

where $f_A : \mathbb{K} \to A$ and $f_B : \mathbb{K} \to B$ are the structural homomorphisms

Convention 3.2. All bimodules are by default $\mathbb{K}$-central.

We want a more effective description of such bimodules, analogous to Exer 2.17.

Proposition 3.3. *Let A and B be central $\mathbb{K}$-rings. Then $A \otimes B = A \otimes_{\mathbb{K}} B$ is a central $\mathbb{K}$-ring, with unit element $1 \otimes 1$, and multiplication*

$$(a_0 \otimes b_0) \cdot (a_1 \otimes b_1) := (a_0 \cdot a_1) \otimes (b_0 \cdot b_1).$$

Proposition 3.4. *Let A and B be central $\mathbb{K}$-rings. Given an A-B-bimodule M, the formula*

$$(a \otimes \operatorname{op}(b)) \cdot m := a \cdot m \cdot b$$

makes M into a left $(A \otimes B^{\mathrm{op}})$-module.

In this way we get an equivalence (in fact an isomorphism) of $\mathbb{K}$-linear categories from the category of A-B-bimodules to $\mathsf{M}(A \otimes B^{\mathrm{op}})$.

Exercise 3.5. Prove Props 3.3 and 3.4.

Prop 3.4 is extremely useful. Unfortunately most ring theory books do not adhere to it, thus limiting their possibility for a deeper understanding of the theory.

Example 3.6. Even commutative rings admit noncentral bimodules, namely bimodules whose left and right actions are not the same.

Let $\gamma : A \to A$ be a nontrivial ring automorphism. (If you want to be concrete, you can either look at $\mathbb{K} = \mathbb{R}$, $A = \mathbb{C}$ and γ is conjugation; or $A = \mathbb{K}[t]$ and $\gamma(t) := t + 1$.)

The ring A is a bimodule over itself, and it is central. But now we define the γ-twist of A to be the bimodule $M = A(\gamma)$. As a left A-module it is just A. But the right action $\cdot^{\gamma}$ is twisted by γ, as follows:

$$m \cdot^{\gamma} a := m \cdot \gamma(a)$$

for $m \in M = A(\gamma)$ and $a \in A$.

I leave it to you to verify that this is a legitimate A-A-bimodule.

It is not central, since if we take an element $a \in A$ that's not fixed by γ, then the element $e := 1 \in M = A(\gamma)$ has

$$e \cdot^{\gamma} a = 1 \cdot \gamma(a) = \gamma(a) \in M = A(\gamma),$$

but

$$a \cdot e = a \cdot 1 = a \in M = A(\gamma).$$

Let A be a ring, $M \in \mathsf{M}(A^{\mathrm{op}})$, $N \in \mathsf{M}(A)$ and $V \in \mathsf{M}(\mathbb{K})$. An *$A$-bilinear function*

$$\beta : M \times N \to V$$

is a $\mathbb{K}$-bilinear function s.t.

$$\beta(m \cdot a, n) = \beta(m, a \cdot n)$$

for all $a \in A$, $m \in M$ and $n \in N$.

Definition 3.7. Let A be a ring, $M \in \mathsf{M}(A^{\mathrm{op}})$ and $N \in \mathsf{M}(A)$. Let $R \subseteq M \otimes N$ be the $\mathbb{K}$-submodule generated by the elements

$$(m \cdot a) \otimes n - m \otimes (a \cdot n)$$

for all $a \in A$, $m \in M$ and $n \in N$. Define the $\mathbb{K}$-module

$$M \otimes_A N := (M \otimes N)/R$$

and the function

$$\beta_{\mathrm{u},A} : M \times N \to M \otimes_A N, \quad (m, n) \mapsto m \otimes n + R.$$

Proposition 3.8. *In the situation of Def* 3.7, *$\beta_{\mathrm{u},A}$ is an A-bilinear function.*

Moreover, it is the universal A-bilinear function, in the following sense: given an A-bilinear function

$$\beta : M \times N \to V$$

there is a unique $\mathbb{K}$-linear homomorphism

$$\phi : M \otimes_A N \to V$$

s.t. $\beta = \phi \circ \beta_{\mathrm{u},A}$.

Exercise 3.9. Prove this prop.

The universal bilinear function $\beta_{\mathrm{u},A}$ will be written as follows:

$$m \otimes n := \beta_{\mathrm{u},A}(m,n) \in M \otimes_A N. \tag{3.10}$$

As usual, such an element is called a pure tensor. Note that $M \otimes_A N$ is generated, as a $\mathbb{K}$-module, by the pure tensors.

Proposition 3.11. *Let A, B, C be rings, and let $M \in \mathsf{M}(A \otimes B^{\mathrm{op}})$ and $N \in \mathsf{M}(B \otimes C^{\mathrm{op}})$ be modules. Then the $\mathbb{K}$-module $M \otimes_B N$ has a unique $(A \otimes C^{\mathrm{op}})$-module structure s.t.*

$$(a \otimes \mathrm{op}(c)) \cdot (m \otimes n) = (a \cdot m) \otimes (n \cdot c).$$

Proof. An element $a \in A$ gives rise to a B^{op}-linear endomorphism $\lambda_a : M \to M$, $\lambda_a(m) := a \cdot m$. We then get a function

$$(\lambda_a, \mathrm{id}_N) : M \times N \to M \times N.$$

The composition

$$\beta_{\mathrm{u},B} \circ (\lambda_a, \mathrm{id}_N) : M \times N \to M \otimes_B N$$

is B-bilinear. Hence it induces a unique $\mathbb{K}$-linear homomorphism

$$\lambda_a \otimes \mathrm{id}_N : M \otimes_B N \to M \otimes_B N,$$

and this satisfies

$$(\lambda_a \otimes \mathrm{id}_N)(m \otimes n) = (a \cdot m) \otimes n$$

on pure tensors.

For an element $u \in M \otimes_B N$ we define

$$a \cdot u := (\lambda_a \otimes \mathrm{id}_N)(u) \in M \otimes_B N.$$

It is easy to verify that this is an A-module structure on $M \otimes_B N$.

Similarly we define a C^{op}-module structure on $M \otimes_B N$ such that

$$\mathrm{op}(c) \cdot (m \otimes n) = m \otimes (n \cdot c).$$

Finally it is clear that the actions of A and C^{op} commute with each other. Uniqueness is also clear. □

Proposition 3.12. *Let A, B, C, D be rings, and let $L \in \mathsf{M}(A \otimes B^{\mathrm{op}})$, $M \in \mathsf{M}(B \otimes C^{\mathrm{op}})$ and $N \in \mathsf{M}(C \otimes D^{\mathrm{op}})$ be modules. There is a unique isomorphism*

$$(L \otimes_B M) \otimes_C N \xrightarrow{\simeq} L \otimes_B (M \otimes_C N)$$

is $\mathsf{M}(A \otimes D^{\mathrm{op}})$ s.t.

$$(l \otimes m) \otimes n \mapsto l \otimes (m \otimes n)$$

for all $l \in L$, $m \in M$ and $n \in M$.

Proposition 3.13. *Let A, B, C be rings. The formula*

$$(M, N) \mapsto M \otimes_B N$$

is a $\mathbb{K}$-bilinear bifunctor

$$\mathsf{M}(A \otimes B^{\mathrm{op}}) \times \mathsf{M}(B \otimes C^{\mathrm{op}}) \to \mathsf{M}(A \otimes C^{\mathrm{op}}).$$

Exercise 3.14. Prove Props. 3.12 and 3.13.

Example 3.15. This example is about *invertible bimodules*. We will return to this concept when we talk about Morita Theory (next week?).

When $B = A$ we say *A-bimodule* instead of A-A-bimodule. It is useful to introduce the *enveloping ring* of A (relative to $\mathbb{K}$):

$$A^{\mathrm{en}} := A \otimes A^{\mathrm{op}}.$$

Thus the A-bimodules are the objects of $\mathsf{M}(A^{\mathrm{en}})$.

Given $P, Q \in \mathsf{M}(A^{\mathrm{en}})$, we have

$$P \otimes_A Q \in \mathsf{M}(A^{\mathrm{en}}).$$

In this way we have a beautiful "internal operation"; the experts call it a *monoidal category*. See [Mac2].

Suppose $P \in \mathsf{M}(A^{\mathrm{en}})$ has this property: there exists some $Q \in \mathsf{M}(A^{\mathrm{en}})$ s.t.

$$P \otimes_A Q \cong A \cong Q \otimes_A P$$

in $\mathsf{M}(A^{\mathrm{en}})$. Then we say that P is an *invertible bimodule*, and that Q is a quasi-inverse of P (and vice versa).

Define the set

$$\mathrm{Pic}_{\mathbb{K}}(A) := \frac{\{\text{invertible } A\text{-bimodules}\}}{\text{ismorphism}},$$

where the isomorphisms are in the category $\mathsf{M}(A^{\mathrm{en}})$. Let $[P]$ denote the isomorphism class of P.

Proposition 3.17 shows that $\mathrm{Pic}_{\mathbb{K}}(A)$ is a group, with operation

$$[P] \cdot [Q] := [P \otimes_A Q]$$

and unit $[A]$. This is called the *NC Picard group of A* (relative to $\mathbb{K}$).

In case A is commutative, we can consider central invertible A-bimodules (that are just A-modules in the commutative sense). These form the *commutative Picard group of A*, with notation $\mathrm{Pic}_A(A)$. (It is the familiar Picard group from algebraic geometry and number theory.)

For commutative A, the NC Picard group ahs this semi-direct product structure

$$\mathrm{Pic}_{\mathbb{K}}(A) \cong \mathrm{Aut}_{\mathbb{K}}(A) \ltimes \mathrm{Pic}_A(A),$$

where $\mathrm{Aut}_{\mathbb{K}}(A)$ is the group of $\mathbb{K}$-ring automorphisms (the Galois group).

These results are quite hard to prove.

There is also the derived Picard group $\operatorname{DPic}_{\mathbb{K}}(A)$, that I discovered in 1997, simultaneously with Rouquier and Zimmermann. See Chap 14 of [Yek3].

Exercise 3.16. Suppose $P, Q, Q' \in \mathsf{M}(A^{\mathrm{en}})$ satisfy

$$P \otimes_A Q \cong A \cong Q' \otimes_A P$$

in $\mathsf{M}(A^{\mathrm{en}})$. Prove that $Q \cong Q'$.

Conclude that P is invertible.

Conclude that the quasi-inverse of P is unique up to isomorphism.

Exercise 3.17. Suppose $P, Q \in \mathsf{M}(A^{\mathrm{en}})$ are invertible bimodules. Prove that $P \otimes_A Q$ is invertible.

Exercise 3.18 (Very hard!)**.** Suppose $P \in \mathsf{M}(A^{\mathrm{en}})$ is an invertible bimodule. Prove that its quasi-inverse Q is

$$Q \cong \operatorname{Hom}_A(P, A) \cong \operatorname{Hom}_{A^{\mathrm{op}}}(P, A).$$

comment: End of Lecture 4, 1 April 2020. Lecture via ZOOM.

comment: (200401) For next time....

Now to the NC Hom operation.

Proposition 3.19. *Let A, B, C be rings, and let $M \in \mathsf{M}(B \otimes A^{\mathrm{op}})$ and $N \in \mathsf{M}(B \otimes C^{\mathrm{op}})$ be modules. Then the $\mathbb{K}$-module $\mathrm{Hom}_B(M, N)$ has unique structure of an $(A \otimes C^{\mathrm{op}})$-module s.t.*

$$((a \otimes \mathrm{op}(c)) \cdot \phi)(m) = \phi(m \cdot a) \cdot c$$

for all $a \in A$, $c \in C$, $\phi \in \mathrm{Hom}_B(M, N)$ and $m \in M$.

Proposition 3.20. *Let A, B, C be rings. The formula*

$$(M, N) \mapsto \mathrm{Hom}_B(M, N)$$

is a $\mathbb{K}$-bilinear bifunctor

$$\mathsf{M}(B \otimes A^{\mathrm{op}})^{\mathrm{op}} \times \mathsf{M}(B \otimes C^{\mathrm{op}}) \to \mathsf{M}(A \otimes C^{\mathrm{op}}).$$

Exercise 3.21. Prove Props. 3.19 and 3.20.

Theorem 3.22. *Let P be a B-A-bimodule. Consider the $\mathbb{K}$-linear functors*

$$G_P := P \otimes_A (-) : \mathsf{M}(A) \to \mathsf{M}(B)$$

and

$$F_P := \mathrm{Hom}_A(P, -) : \mathsf{M}(B) \to \mathsf{M}(A).$$

These functors are adjoint: G_P is left adjoint to F_P.

Proof. The proof is basically the same as that of Proposition 1.28. So I will just stress the important NC aspects.

We have to construct a $\mathbb{K}$-linear isomorphism

$$\alpha_{M,N} : \mathrm{Hom}_A\big(M, \mathrm{Hom}_B(P, N)\big) \xrightarrow{\simeq} \mathrm{Hom}_B\big(P \otimes_A M, N\big) \tag{3.23}$$

that's functorial in $M \in \mathsf{M}(A)$ and $N \in \mathsf{M}(B)$.

We define $\alpha_{M,N}$ as follows: for

$$\phi \in \mathrm{Hom}_A\big(M, \mathrm{Hom}_B(P, N)\big),$$

$p \in P$ and $m \in M$, we let

$$\alpha_{M,N}(\phi)(p \otimes m) := \phi(m)(p) \in N.$$

We need to check a few things.

First: that

$$\alpha_{M,N}(\phi)((p \cdot a) \otimes m) = \alpha_{M,N}(\phi)(p \otimes (a \cdot m))$$

for all $a \in A$. But

$$\phi(m)(p \cdot a) = a \cdot (\phi(m)(p)) = \phi(a \cdot m)(p)$$

by the definition of the various actions, so the assertion is true.

Second: that

$$\alpha_{M,N}(\phi)(b \cdot (p \otimes m)) = b \cdot (\alpha_{M,N}(\phi)(p \otimes m)).$$

This is proved Similarly.

The $\mathbb{K}$-linear homomorphism

$$\beta_{M,N} : \operatorname{Hom}_B\big(P \otimes_A M, N\big) \to \operatorname{Hom}_A\big(M, \operatorname{Hom}_B(P, N)\big),$$

which is inverse to $\alpha_{M,N}$, is just like in the proof of Proposition 1.28. □

4. Linear Equivalences and Exactness

Now we study equivalences for linear functors.

$\mathbb{K}$-linear functors were defined in [Yek1, Sectin 9], and exact additive functors were defined in [Yek1, Section 13].

Let $\phi : M \to N$ be a homomorphism in $\mathsf{M}(A)$. Recall that a *kernel* of ϕ is a pair (K, k), where $k : K \to M$ is a homomorphism such that $\phi \circ k = 0$, and it is universal for this property, i.e. every homomorphism $k' : K' \to M$ s.t. $\phi \circ k' = 0$ factor uniquely through k.

Such a pair (K, k) is unique up to a unique isomorphism, so we call it *the kernel of* ϕ.

The standard choice for the kernel is

$$K := \{m \in M \mid \phi(m) = 0\} \subseteq M,$$

and k is the inclusion.

The concept of *cokernel* is dual to it. Let $\phi : M \to N$ be as above. A *cokernel* of ϕ is a pair (C, c), where $c : N \to C$ is a homomorphism such that $c \circ \phi = 0$, and it is universal for this property, i.e. every homomorphism $c' : N \to K'$ s.t. $c' \circ \phi = 0$ factor uniquely through c.

Again, the pair (C, c) is unique up to a unique isomorphism, so we call it *the cokernel of* ϕ.

The standard choice for the cokernel is $C := N/\operatorname{Im}(\phi)$, and c is the canonical projection.

Let A and B be rings, and let

$$F : \mathsf{M}(A) \to \mathsf{M}(B)$$

be an linear functor.

We say that F *preserves kernels* if for every homomorphism $\phi : M_0 \to M_1$ in $\mathsf{M}(A)$, with kernel $k : K \to M_0$, the homomorphism

$$F(k) : F(K) \to F(M_0)$$

in $\mathsf{M}(B)$ is the kernel of the homomorphism

$$F(\phi) : F(M_0) \to F(M_1)$$

in $\mathsf{M}(B)$.

We say that F *preserves cokernels* if for every homomorphism $\phi : M_0 \to M_1$ in $\mathsf{M}(A)$, with cokernel $c : M_1 \to C$, the homomorphism

$$F(c) : F(M_1) \to F(C)$$

in $\mathsf{M}(B)$ is the cokernel of the homomorphism

$$F(\phi) : F(M_0) \to F(M_1).$$

Lemma 4.1. *Let A and B be rings, and let*

$$F : \mathsf{M}(A) \to \mathsf{M}(B)$$

be an additive functor. TFAE:

(1) *F is exact.*
(2) *F preserves kernels and cokernels.*

Exercise 4.2. Prove the lemma.

Theorem 4.3. *Let A and B be rings, and let*

$$F : \mathsf{M}(A) \to \mathsf{M}(B)$$

be a linear functor, which is an equivalence of unstructured categories.

(1) *Let $G : \mathsf{M}(B) \to \mathsf{M}(A)$ be a quasi-inverse of F. Then G is a linear functor.*
(2) *The functor F is exact.*

Proof.

(1) We must prove that for every $N_0, N_1 \in \mathsf{M}(B)$ the function

$$G_{N_0,N_1} : \operatorname{Hom}_B(N_0, N_1) \to \operatorname{Hom}_A\big(G(N_0), G(N_1)\big)$$

is $\mathbb{K}$-linear. Note that this is a bijection.

Looking at [Yek1, Definition 13.28] we see that there is an isomorphism of functors

$$\zeta : F \circ G \xrightarrow{\simeq} \operatorname{Id}_{\mathsf{M}(B)} .$$

Let's examine this diagram of bijections:

(4.4)
$$\begin{array}{ccc} \operatorname{Hom}_B(N_0, N_1) & \xrightarrow[\simeq]{G_{N_0,N_1}} & \operatorname{Hom}_A\big(G(N_0), G(N_1)\big) \\ \Big\downarrow{\scriptstyle (\mathrm{Id}_{\mathsf{M}(B)})_{N_0,N_1}} & & \simeq \Big\downarrow{\scriptstyle F_{G(N_0),G(N_1)}} \\ \operatorname{Hom}_B(N_0, N_1) & \xleftarrow[\simeq]{\operatorname{Hom}(\zeta^{-1}_{N_0}, \zeta_{N_1})} & \operatorname{Hom}_B\big((F \circ G)(N_0), (F \circ G)(N_1)\big) \end{array}$$

It is commutative because ζ is a morphism of functors. (Exercise)

The bijection $F_{-,-}$ is $\mathbb{K}$-linear because F is a $\mathbb{K}$-linear functor.

The bijection $(\mathrm{Id}_{\mathsf{M}(B)})_{N_0,N_1}$ is the identity, so it is $\mathbb{K}$-linear.

The bijection $\operatorname{Hom}(\zeta^{-1}_{N_0}, \zeta_{N_1})$ $\mathbb{K}$-linear, because $\mathsf{M}(B)$ is a $\mathbb{K}$-linear category.

We conclude that G_{N_0,N_1} is $\mathbb{K}$-linear.

(2) It is enough to prove that F preserves kernels and cokernels. This is by Lemma 4.1.

Consider a homomorphism $\phi : M_0 \to M_1$. Let $K := \operatorname{Ker}(\phi)$.

We must prove that

$$F(K) = \operatorname{Ker}\big(F(\phi) : F(M_0) \to F(M_1)\big).$$

Suppose $\psi : N \to F(M_0)$ is a homomorphism in $\mathsf{M}(B)$ such that $F(\phi) \circ \psi = 0$.

Applying G we get

$$G(F(\phi)) \circ G(\psi) = 0$$

as homomorphisms $G(N) \to G(F(M_0)) \to G(F(M_1))$.

Using the isomorphism ???

comment: (200323) finish !!!!

□

Exercise 4.5. Verify that diagram (4.4).

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