Course Notes:

Algebraic Geometry – Schemes 1

Fall 2018-19

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Contents

1. Basics

Lecture 1, 17 Oct 2018

Before starting with the actual material, lets us go quickly over some basic ideas that we will need. I hope all these are familiar to all students; if not, then we will have to see how to close the gaps.

The first few weeks will be on geometry in general, but from the point of view of *locally ringed spaces*.

Everybody needs to know a sufficient amount of elementary topology. Some algebraic topology will be required (homology, cohomology and fundamental groups).

Categories, functors and natural transformations will be used a lot. I am assuming that all students have already been exposed to these notions. For instance, all should understand this statement:

- Let Top_* and Grp be the categories of pointed topological spaces and of groups, respectively. The fundamental group is a functor

$$\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}\,.$$

If not, then we will have to see how to close this gap. (Maybe go over material from [Ye3].)

Differential geometry will serve as an introductory model for locally ringed spaces. (A preparation for the more complicated schemes.) Everybody should have some knowledge on this topic (C^∞ manifolds and maps between them, tangent bundles, etc.) Knowledge of complex analytic geometry will be very useful.

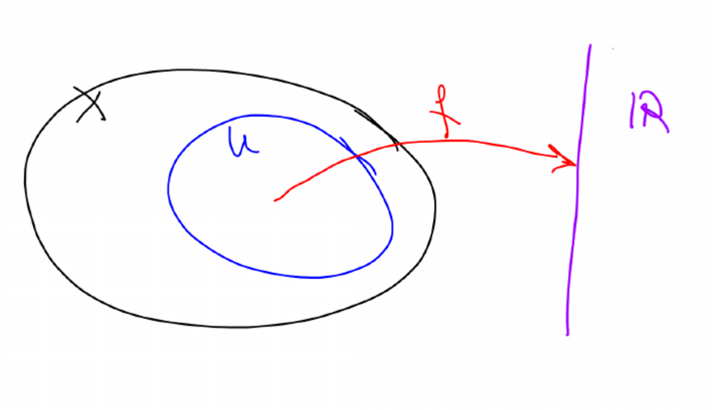
2. Sheaves of Functions on Topological Spaces

Consider a topological space X. We do not make any conditions on X, especially we don't assume X is Hausdorff. But at first you can pretend, to help intuition, that X is a topological subspace of $\mathbb{R}^n$ (with its usual topology).

Given an open subset $U \subseteq X$, consider the continous functions

$$f : U \to \mathbb{R}.$$

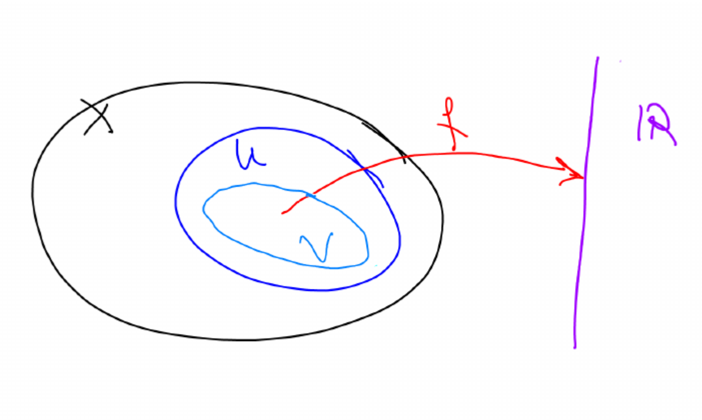
Let us denote this set of functions by $\Gamma(U, \mathcal{O}_X)$.



We know that $\Gamma(U, \mathcal{O}_X)$ is a commutative $\mathbb{R}$-ring.

Let $V \subseteq U$ be a smaller open set. We get a continous function

$$f|_V : V \to \mathbb{R}.$$

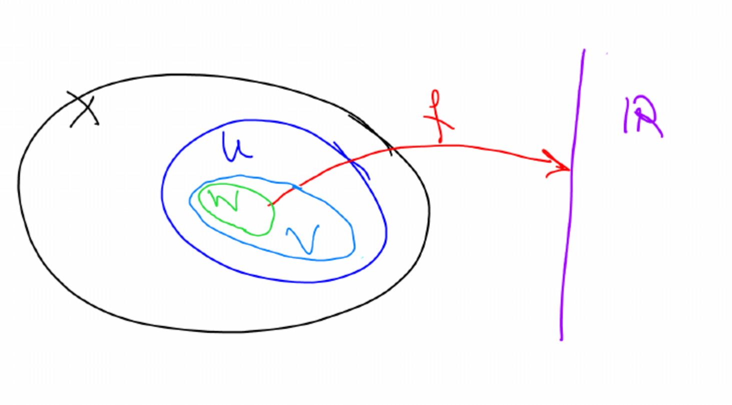


The opration $f \mapsto f|_V$ is a ring homomorphism

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X).$$

If $W \subseteq V$ is another smaller open set, then of course

$$(f|_V)|_W = f|_W.$$



We see that the restriction homomorphisms satisfy

$$\mathrm{rest}_{W/V} \circ \mathrm{rest}_{V/U} = \mathrm{rest}_{W/U} : \Gamma(U, \mathcal{O}_X) \to \Gamma(W, \mathcal{O}_X).$$

This means that $\mathcal{O}_X$ *is a presheaf of* $\mathbb{R}$*-rings on* X.

Here is a categorical interpretation of this statement. Let $\mathsf{Rng}_{\mathrm{c}}/\mathbb{R}$ be the category of commutative $\mathbb{R}$-rings.

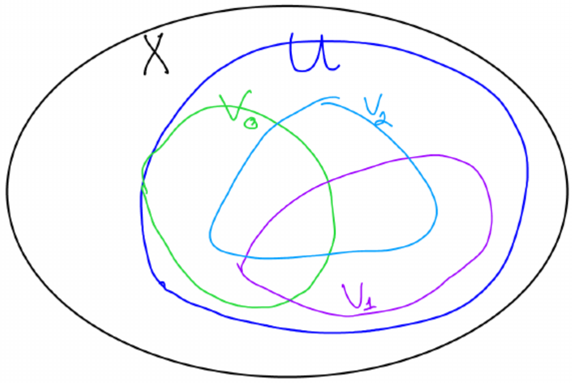
Let $\mathsf{Open}(X)$ be the category of open sets of X, where the morphisms are inclusions. Thus if $V \subseteq U$ then there is one arrow $V \to U$; and if $V \not\subseteq U$ then there are no arrows $V \to U$. The presheaf $\mathcal{O}_X$ is a functor

$$\Gamma(-, \mathcal{O}_X) : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Rng}_{\mathrm{c}}/\mathbb{R}.$$

But in fact much more is true.

Suppose $U \subseteq X$ is an open set, and we are given an open covering

$$U = \bigcup_{i \in I} V_i.$$



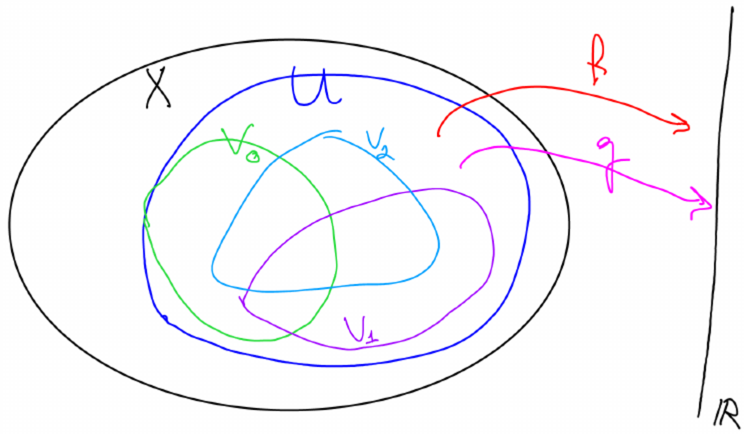
Let $f, g \in \Gamma(U, \mathcal{O}_X)$, i.e.

$$f, g : U \to \mathbb{R},$$

and assume that

$$f|_{V_i} = g|_{V_i}$$

for all i.



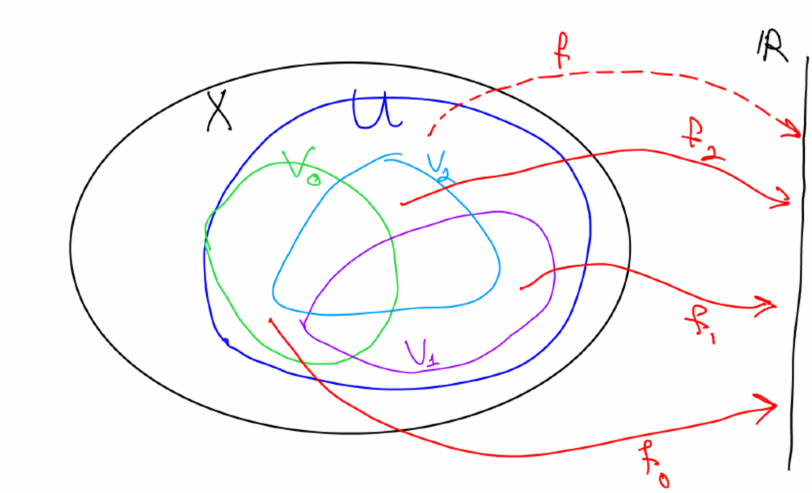
Then of course $f = g$.
Now assume that we are given

$$f_i \in \Gamma(V_i, \mathcal{O}_X)$$

such that

$$f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$$

for all i, j.



Because the various f_i agree on double intersections, there is a function

$$f : U \to \mathbb{R}$$

such that

$$f|_{U_i} = f_i.$$

Of course this function f is unique (by the previous discussion). But also f is continous. This is because continuity is a local property, and on each of the open sets U_i we know that f is continous.

Thus

$$f \in \Gamma(U, \mathcal{O}_X).$$

Let us summarize these two further properties of $\mathcal{O}_X$:

(a) Let $U \subseteq X$ be an open set, let $U = \bigcup_{i \in I} V_i$ an open covering, and let

$$f, g \in \Gamma(U, \mathcal{O}_X)$$

be such that $f|_{V_i} = g|_{V_i}$ for all i. Then $f = g$.

(b) Let $U \subseteq X$ be an open set, let $U = \bigcup_{i \in I} V_i$ be an open covering, and let

$$f_i \in \Gamma(V_i, \mathcal{O}_X)$$

be such that

$$f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$$

for all i, j. Then there exists

$$f \in \Gamma(U, \mathcal{O}_X)$$

such that $f|_{V_i} = f_i$ for all i.

These are the *sheaf axioms*. They tell us that $\mathcal{O}_X$ is a *sheaf of rings on* X.

Because rings have underlying abelian groups, axioms (a) and (b) can be stated in terms of *exact sequences*.

($*$) For every open set $U \subseteq X$ and every open covering $U = \bigcup_{i \in I} V_i$ the sequence of abelian groups

$$0 \to \Gamma(U, \mathcal{O}_X) \xrightarrow{\rho} \prod_{i \in I} \Gamma(V_i, \mathcal{O}_X) \xrightarrow{\delta^0 - \delta^1} \prod_{j,k \in I} \Gamma(V_j \cap V_k, \mathcal{O}_X)$$

is exact.

Here ρ is the product on all $i \in I$ of the restriction homomorphisms

$$\mathrm{rest}_{V_i/U} : \Gamma(U, \mathcal{O}_X) \to \Gamma(V_i, \mathcal{O}_X).$$

The homomorphism δ^1 is the product on all $i = j \in I$ of the product on all $k \in I$ of

$$\mathrm{rest}_{V_j \cap V_k/V_j} : \Gamma(V_j, \mathcal{O}_X) \to \Gamma(V_j \cap V_k, \mathcal{O}_X).$$

And the homomorphism δ^0 is the product on all $i = k \in I$ of the product on all $j \in I$ of of

$$\mathrm{rest}_{V_j \cap V_k/V_k} : \Gamma(V_k, \mathcal{O}_X) \to \Gamma(V_j \cap V_k, \mathcal{O}_X).$$

Exercise 2.1. Prove that condition $(*)$ is equivalent to condition $\big(\text{(a) and (b)}\big)$.

The next exercise gives a variation of what we did above.

Exercise 2.2. Let X be a differentiable manifold (of type C^∞). For every open set $U \subseteq X$ let $\Gamma(U, \mathcal{O}_X)$ be the set of differentiable functions $f : U \to \mathbb{R}$.

Prove that the assignment

$$U \mapsto \Gamma(U, \mathcal{O}_X)$$

is a sheaf of $\mathbb{R}$-rings on X. The sheaf $\mathcal{O}_X$ is called the *sheaf of differentiable functions on* X.

Exercise 2.3. If you know about real or complex analytic manifolds, state and prove the corresponding analogue of Exercise 2.2.

Exercise 2.4. This exercise is for those who know the algebraic geometry of varieties. Let $\mathbb{K}$ be an algebraically closed field, and let X be an algebraic variety over $\mathbb{K}$. For every (Zariski) open set $U \subseteq X$ let $\Gamma(U, \mathcal{O}_X)$ be ring of algebraic functions on U. Prove that the assignment

$$U \mapsto \Gamma(U, \mathcal{O}_X)$$

is a sheaf of $\mathbb{K}$-rings on X. The sheaf $\mathcal{O}_X$ is called the *sheaf of algebraic functions on* X.

3. Sheaves on Topological Spaces

Until now we only saw ring valued sheaves. Here are some variations.

Definition 3.1. Let X be a topological space. A *presheaf of groups on* X is a functor

$$\mathcal{G} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Grp},$$

where Grp is the category of groups.

Concretely, the presheaf $\mathcal{G}$ is the data of a group $\Gamma(U, \mathcal{G})$ for every open set $U \subseteq X$, called the *group of sections of* $\mathcal{G}$ *over* U, and a group homomorphism

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathcal{G}) \to \Gamma(V, \mathcal{G})$$

for every inclusion $V \subseteq U$, such that

$$\mathrm{rest}_{W/U} = \mathrm{rest}_{W/V} \circ \mathrm{rest}_{V/U}$$

for every double inclusion $W \subseteq V \subseteq U$. And of course

$$\mathrm{rest}_{U/U} = \mathrm{id}_{\Gamma(U,\mathcal{G})}$$

for every U.

We often use the abbreviation

$$g|_V := \mathrm{rest}_{V/U}(g) \in \Gamma(V, \mathcal{G}) \qquad (3.2)$$

for a presheaf $\mathcal{G}$, an inclusion of open sets $V \subseteq U$, and a section $g \in \Gamma(U, \mathcal{G})$.

Definition 3.3. Let X be a topological space. A *sheaf of groups on X* is a presheaf of groups $\mathcal{G}$ on X that satisfies the two sheaf axioms:

(a) Let $U \subseteq X$ be an open set, let $U = \bigcup_{i \in I} V_i$ be an open covering, and let $g, h \in \Gamma(U, \mathcal{G})$ be sections such that $g|_{V_i} = h|_{V_i}$ for all i. Then $g = h$.

(b) Let $U \subseteq X$ be an open set, let $U = \bigcup_{i \in I} V_i$ be an open covering, and let $g_i \in \Gamma(V_i, \mathcal{G})$ be sections such that
$$g_i|_{V_i \cap V_j} = g_j|_{V_i \cap V_j}$$
for all i, j. Then there exists a section $g \in \Gamma(U, \mathcal{O}_X)$ such that
$$g|_{V_i} = g_i$$
for all i.

Recall that a *topological group* is a topological space G, that is also a group, such that the operations of multiplication and inversion are continous. Namely
$$\mathrm{mult} : G \times G \to G$$
and
$$\mathrm{inv} : G \to G$$
are continous functions.

Example 3.4. Let X be a topological space and G a topological group. For every open set $U \subseteq X$ define
$$\Gamma(U, \mathcal{G}) := \{\text{continous functions } g : U \to G\}.$$
I claim that $\mathcal{G}$ is a *sheaf of groups on X*.

That $\mathcal{G}$ is a presheaf is obvious. Sheaf axiom (a) is also clear, because for every point $x \in U$ we can find some i such that $x \in V_i$, and hence we have
$$g(x) = g|_{V_i}(x) = g'|_{V_i}(x) = g'(x).$$
Thus $g = g'$.

Axiom (b) is also easy to verify. The values $g_i(x)$ at a point $x \in U$ are equal, for all $i \in I$ such that $x \in V_i$. So there is a function $g : U \to G$. Because continuity is a local property, and $g|_{V_i} = g_i$, we see that g is continous. Thus $g \in \Gamma(U, \mathcal{G})$.

Exercise 3.5. Let X be a topological space. Let $G := \mathrm{GL}_n(\mathbb{K})$ for some positive integer n, with the usual topology (by the embedding $\mathrm{GL}_n(\mathbb{K}) \subseteq \mathbb{R}^{n^2}$). So G is a topological group.

Let $\mathcal{G}$ be the sheaf of groups on X from Example 3.4 for this choice of G. And let $\mathcal{O}_X$ be the sheaf of continous real valued function on X. Prove that for every open set $U \subseteq X$ there is a group isomorphism
$$\Gamma(U, \mathcal{G}) \cong \mathrm{GL}_n(\Gamma(U, \mathcal{O}_X)),$$
that respects the restriction homomorphisms.

Definition 3.6. Let X be a topological space.

(1) A *presheaf of abelian groups on X* is a functor
$$\mathcal{G} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Ab},$$
where Ab is the category of abelian groups.

(2) A *sheaf of abelian groups on X* is a presheaf of abelian groups $\mathcal{G}$ that satisfies the sheaf axioms (a) and (b) from Definition 3.3.

It is not hard to see that a sheaf of abelian groups $\mathcal{G}$ is the same as a sheaf of groups $\mathcal{G}$ such that each $\Gamma(U, \mathcal{G})$ is abelian.

Definition 3.7. Let X be a topological space.

(1) A *presheaf of commutative rings on X* is a functor

$$\mathcal{A} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Rng}_{\mathrm{c}},$$

where $\mathsf{Rng}_{\mathrm{c}}$ is the category of commutative rings.

(2) A *sheaf of commutative rings on X* is a presheaf of commutative rings $\mathcal{A}$ that satisfies the sheaf axioms (a) and (b) from Definition 3.3.

If $\mathcal{A}$ is a sheaf of commutative rings, and we forget the multiplication of $\mathcal{A}$, then we obtain a sheaf of abelian groups.

Example 3.8. Let X be a topological space and let A be a commutative ring. The *constant sheaf of rings on X with values in A* is the sheaf A_X defined as follows. Put on A the discrete topology. Then for every $U \subseteq X$ open we let

$$\Gamma(U, A_X) := \{\text{continous functions } g : U \to A\}.$$

Exercise 3.9. Take a nonzero commutative ring A, say $A := \mathbb{Z}$. Calculate the ring $\Gamma(X, A_X)$ for these choices of X :

(1) $X := \mathbb{R}$ with the classical topology.
(2) $X := \mathbb{N}$ with the discrete topology.

4. Stalks

A *directed set* is a partially ordered set I such that for every $i, j \in I$ there exists some $k \in I$ with $i, j \leq k$. We can view the directed set I as a category, with a single arrow $r_{i,j} : i \to j$ if $i \leq j$, and no arrows otherwise.

A *direct system* in a category C, indexed by a directed set I, is a functor

$$C : I \to \mathsf{C}, \quad i \mapsto C(i) = C_i, \quad r_{i,j} \mapsto C(r_{i,j}).$$

We usually denote such a direct system by $\{C_i\}_{i \in I}$, leaving the $r_{i,j}$ implicit.

A *direct limit* of a direct system $\{C_i\}_{i \in I}$ is an object $C_\infty \in \mathsf{C}$, together with a collection of morphisms $f_i : C_i \to C_\infty$, such that the diagram

$$\begin{tikzcd} C_i \arrow[d, "C(r_{i,j})"'] \arrow[dr, "f_i"] & \\ C_j \arrow[r, "f_j"] & C_\infty \end{tikzcd}$$

is commutative for every $i \to j$, and such that

$$\big(C_\infty, \{f_i\}_{i \in I}\big)$$

is universal for this property. We write

$$\lim_{i \to} C_i := C_\infty.$$

The categories Grp, Ab and $\mathsf{Rng}_{\mathrm{c}}$ have direct limits. Here is the construction:

$$\lim_{i \to} C_i = \Big(\coprod\nolimits_{i \in I} C_i\Big) \,/ \sim\,,$$

where $\sim$ is the relation $c_i \sim c_j$ for $c_i \in C_i$ and $c_j \in C_j$ whenever there are arrows $i, j \to k$ such that

$$C(r_{i,k})(c_i) = C(r_{j,k})(c_j) \in C_k.$$

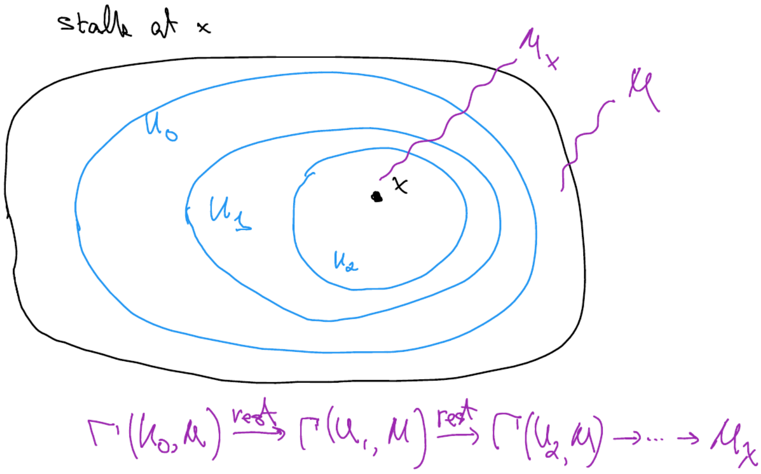
Let X be a topological space. For a point $x \in X$ let $\mathsf{Open}(X, x)$ be the set of open neighborhoods of x, made into a category by inclusions. Then $\mathsf{Open}(X, x)^{\mathrm{op}}$ is a directed set.

Definition 4.1. Let $\mathcal{M}$ be a presheaf of abelian groups on a topological space X. Let $x \in X$ be a point. The *stalk of* $\mathcal{M}$ *at* x is the abelian group

$$\mathcal{M}_x := \lim_{U \to} \Gamma(U, \mathcal{M}),$$

where the direct limit is on $U \in \mathsf{Open}(X, x)^{\mathrm{op}}$.

Likewise for a presheaf of groups and for a sheaf of commutative rings.



Exercise 4.2. With the assumptions of Exercise 3.9(1, 2), calculate the stalks $(A_X)_x$ for a point $x \in X$.

Lecture 2, 24 Oct 2018

5. Morphisms of Sheaves

We will mostly work with sheaves of abelian groups; but things are the same for sheaves in $\mathsf{Rng_c}$, Grp and Set.

First we talk about morphisms of presheaves.

Definition 5.1. Let $\mathcal{M}$ and $\mathcal{N}$ be presheaves of abelian groups on a topological space X. A *morphism* of presheaves of abelian groups

$$\phi : \mathcal{M} \to \mathcal{N}$$

is a collection

$$\phi = \big\{\Gamma(U, \phi)\big\}_{U \in \mathsf{Open}(X)}$$

of homomorphisms of abelian groups

$$\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N}),$$

such that the diagrams

$$\begin{tikzcd} \Gamma(U, \mathcal{M}) \arrow[r] \arrow[d, "\mathrm{rest}_{V/U}"'] & \Gamma(U, \mathcal{N}) \arrow[d, "\mathrm{rest}_{V/U}"] \\ \Gamma(V, \mathcal{M}) \arrow[r, "\Gamma(V, \phi)"] & \Gamma(V, \mathcal{N}) \end{tikzcd}$$

are commutative for all inclusions $V \subseteq U$.

The category of presheaves of abelian groups on X us denoted by $\mathsf{PAb}\, X$

In other words, a morphism of presheaves $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism of functors (natural transformation)

$$(\mathsf{Open}\, X)^{\mathrm{op}} \to \mathsf{Ab}\,.$$

Given a morphism $\phi : \mathcal{M} \to \mathcal{N}$ and a point $x \in X$, there is a group homomorphism

$$\phi_x : \mathcal{M}_x \to \mathcal{N}_x$$

in the stalks.

We say that ϕ is *injective* (respect. *surjective*) if for every open set U the homomorphism $\Gamma(U, \phi)$ is injective (respect. surjective).

Let $\mathcal{M}$ be a presheaf. A *subpresheaf* of $\mathcal{M}$ is a presheaf $\mathcal{M}'$ such that

$$\Gamma(U, \mathcal{M}') \subseteq \Gamma(U, \mathcal{M})$$

for every U, and they have the same restriction homomorphisms. The inclusion $\mathcal{M}' \to \mathcal{M}$ is an injective morphism of presheaves.

Recall that the sheaves on X form a subset of the presheaves on X – these are presheaves that satisfy the sheaf axioms.

Definition 5.2. Let $\mathcal{M}$ and $\mathcal{N}$ be sheaves of abelian groups on a topological space X. A *morphism* of sheaves of abelian groups $\phi : \mathcal{M} \to \mathcal{N}$ is just a morphism of presheaves.

Thus the category $\mathsf{Ab}\, X$ of sheaves of abelian groups on X is a full subcategory of $\mathsf{PAb}\, X$.

Likewise for groups, rings and sets: there are full embeddings

$$\mathsf{Grp}\, X \subseteq \mathsf{PGrp}\, X,$$

$$\mathsf{Rng}\, X \subseteq \mathsf{PRng}\, X$$

and

$$\mathsf{Set}\,X \subseteq \mathsf{PSet}\,X$$

of the categories of sheaves in the corresponding categories of presheaves.

It will be convenient to be a bit ambiguous sometimes - we shall talk about a *morphism* of presheaves or sheaves, meaning any of the four kinds (Ab, Grp, Rng or Set). For this we introduce the symbolic notation

$$\mathsf{Sh}\,X \subseteq \mathsf{PSh}\,X.$$

(Unless this turns out to be too confusing – then we will abolish it.)

Let $\phi : \mathcal{M} \to \mathcal{N}$ be a morphism of sheaves on X. Given a point $x \in X$ there is an induced morphism

$$\phi_x : \mathcal{M}_x \to \mathcal{N}_x. \tag{5.3}$$

Definition 5.4. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a morphism of sheaves on X.

(1) We call ϕ an *injective sheaf morphism* if every point x the morphism ϕ_x is injective.
(2) We call ϕ a *surjective sheaf morphism* if every point x the morphism ϕ_x is surjective.

Proposition 5.5. *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a morphism of sheaves. The following conditions are equivalent.*

(i) *ϕ is an injective sheaf morphism.*
(ii) *ϕ is an injective presheaf morphism.*

Exercise 5.6. Prove the last proposition.

A *subsheaf* of a sheaf $\mathcal{M}$ is a subpresheaf $\mathcal{M}' \subseteq \mathcal{M}$ which is itself a sheaf. The inclusion $\mathcal{M}' \to \mathcal{M}$ is an injective morphism of sheaves.

A presheaf $\mathcal{M}$ is called a *separated presheaf* if it satisfies sheaf axiom (a).

Exercise 5.7. Suppose $\mathcal{M}$ is a sheaf and $\mathcal{M}' \subseteq \mathcal{M}$ is a subpresheaf. Show that $\mathcal{M}'$ is a separated presheaf.

Proposition 5.5 is false for surjections! See Exercise 5.10 below. Instead we have:

Proposition 5.8. *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a morphism of sheaves. The following conditions are equivalent.*

(i) *ϕ is a surjective sheaf morphism.*
(ii) *For evey open set $U \subseteq X$ and every section $n \in \Gamma(U, \mathcal{N})$ there is an open covering $U = \bigcup_{i \in I} V_i$ and sections $m_i \in \Gamma(V_i, \mathcal{M})$ such that*

$$\Gamma(V_i, \phi)(m_i) = n|_{V_i}$$

in $\Gamma(V_i, \mathcal{N})$.

Exercise 5.9. Prove this proposition.

Exercise 5.10. Find an example of a sheaf homomorphism $\phi : \mathcal{M} \to \mathcal{N}$ on a space X with this property: ϕ is a surjection of sheaves, but it is not a surjection of presheaves. (This could be hard; we will see examples later.)

Update. In class today such an example was proposed by Guy. The topological space was $X := \mathbb{C} - \{0\}$, the sheaf $\mathcal{M}$ was the sheaf of holomorphic (i.e. analytic) $\mathbb{C}$-valued functions on X, the sheaf $\mathcal{N}$ was the subsheaf of nonzero functions, and $\phi : \mathcal{M} \to \mathcal{N}$ was $f \mapsto \exp(f)$. We can view these a morphism in $\mathsf{Ab}\,X$. Try to understand why

$$\Gamma(X, \phi) : \Gamma(X, \mathcal{M}) \to \Gamma(X, \mathcal{N})$$

is not surjective; so ϕ is not surjective as a morphism of presheaves. But $\phi : \mathcal{M} \to \mathcal{N}$ is surjective as a morphism of sheaves. (Hint: a logarithm is defined on each contractible open set in X.)

Exercise 5.11. Let $\mathcal{M}$ be a sheaf on X. What is $\Gamma(\varnothing, \mathcal{M})$?

Proposition 5.12. *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a morphism of sheaves on X. The following are equivalent*:

(i) *ϕ is an isomorphism of sheaves, i.e. an isomorphism in the category of sheaves.*

(ii) *For every point $x \in X$ the morphism on stalks*

$$\phi_x : \mathcal{M}_x \to \mathcal{N}_x$$

is bijective.

(iii) *For every open set $U \subseteq X$ the morphism*

$$\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})$$

is bijective.

Note that condition (ii) above says that ϕ is both injective and surjective, see Definition 5.4.

Exercise 5.13. Prove this proposition.

6. Sheafification

[**comment**: (181104) made this into a new section; small changes below]

Recall that by a (pre)sheaf, and a morphism of (pre)sheaves, we mean of the four kinds: abelian groups, groups, rings or sets. (Later we will also talk about sheave of $\mathcal{A}$-modules, where $\mathcal{A}$ is a sheaf of rings.) We shall use the generic notation $\mathsf{C}(X) \subseteq \mathsf{PC}(X)$, where $\mathsf{C} = \mathsf{Set}, \mathsf{Grp}, \mathsf{Ab}, \mathsf{Rng}$. So when $\mathsf{C} = \mathsf{Ab}$ this stands for $\mathsf{Ab}(X) \subseteq \mathsf{PAb}(X)$, etc.

Theorem 6.1 (Sheafification)**.** *Let $\mathcal{M}$ be a presheaf with values in C on a topological space X. There is a sheaf $\mathrm{Sh}(\mathcal{M})$ on X, with a morphism of presheaves*

$$\tau_{\mathcal{M}} : \mathcal{M} \to \mathrm{Sh}(\mathcal{M}),$$

having this universal property:

(S) *For every pair $(\mathcal{N}, \phi)$, consisting of a sheaf $\mathcal{N}$ and a morphism of presheaves*

$$\phi : \mathcal{M} \to \mathcal{N},$$

there is a unique morphism of sheaves

$$\phi' : \mathrm{Sh}(\mathcal{M}) \to \mathcal{N}$$

such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\tau_{\mathcal{M}}"] \arrow[dr, "\phi"'] & \mathrm{Sh}(\mathcal{M}) \arrow[d, "\phi'"] \\ & \mathcal{N} \end{tikzcd}$$

in $\mathsf{PC}(X)$ is commutative.

The pair $\big(\mathrm{Sh}(\mathcal{M}), \tau_{\mathcal{M}}\big)$ *is called the* sheafification of $\mathcal{M}$.

Let us note, before proving the theorem, that:

Proposition 6.2. *The sheafification $\big(\mathrm{Sh}(\mathcal{M}), \tau_{\mathcal{M}}\big)$ of $\mathcal{M}$ is unique, up to a unique isomorphism.*

Exercise 6.3. Prove this proposition.

Corollary 6.4. *In $\mathcal{M}$ is a sheaf then* $(\mathrm{Sh}(\mathcal{M}), \tau_{\mathcal{M}}) = (\mathcal{M}, \mathrm{id})$; *i.e. uniquely isomorphic.*

Exercise 6.5. Prove this corollary.

We need an auxilliary construction. Given a presheaf $\mathcal{M}$, let us define the presheaf $\mathrm{GSh}(\mathcal{M})$ as follows: for every open set U we take

$$\Gamma(U, \mathrm{GSh}(\mathcal{M})) := \prod_{x \in U} \mathcal{M}_x,$$

the product on all stalks. For an open subset $V \subseteq U$ we define

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathrm{GSh}(\mathcal{M})) \to \Gamma(V, \mathrm{GSh}(\mathcal{M}))$$

to be the projection

$$\begin{aligned} \Gamma(U, \mathrm{GSh}(\mathcal{M})) &= \prod_{x \in U} \mathcal{M}_x = \Big(\prod_{x \in V} \mathcal{M}_x \Big) \times \Big(\prod_{x \in U - V} \mathcal{M}_x \Big) \\ &\xrightarrow{\mathrm{pr}} \prod_{x \in V} \mathcal{M}_x = \Gamma(V, \mathrm{GSh}(\mathcal{M})). \end{aligned} \tag{6.6}$$

Note that a section $m \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$ looks like this:

$$m = \{m_x\}_{x \in U}, \quad m_x \in \mathcal{M}_x. \tag{6.7}$$

Lemma 6.8. *Let $\mathcal{M}$ be a presheaf.*

(1) *The presheaf $\mathrm{GSh}(\mathcal{M})$ is a sheaf.*
(2) *There is a presheaf morphism*

$$\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{GSh}(\mathcal{M}).$$

(3) *If the presheaf $\mathcal{M}$ is separated, then the morphism $\gamma_{\mathcal{M}}$ is injective.*
(4) *For every inclusion $V \subseteq U$ of open sets, the morphism*

$$\Gamma(U, \mathrm{GSh}(\mathcal{M})) \to \Gamma(V, \mathrm{GSh}(\mathcal{M}))$$

is surjective.

A sheaf satisfying (4) above is called a *flasque sheaf*.

Proof. (1) Let $U = \bigcup_{i \in I} V_i$ be an open covering.

Let's verify axiom (a) of Definition 3.3 for this covering. Let $m, n \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$ be sections such that $m|_{V_i} = n|_{V_i}$ in

$$\Gamma(V_i, \mathrm{GSh}(\mathcal{M})) = \prod_{x \in V_i} \mathcal{M}_x$$

for all i. This means that the stalks satisfy

$$m_x = n_x \in \mathcal{M}_x$$

for all $x \in V_i$. But for every $x \in U$ there is some i such that $x \in V_i$. We see that

$$m_x = n_x \in \mathcal{M}_x$$

for all $x \in U$. By formula (6.7) we conclude that $m = n$.

Now we shall verify axiom (b) of Definition 3.3 for this covering. So we are given a collection $\{m_i\}_{i \in I}$ of sections

$$m_i \in \Gamma(V_i, \mathrm{GSh}(\mathcal{M}))$$

satisfying

$$m_i|_{V_i \cap V_j} = m_j|_{V_i \cap V_j} \tag{6.9}$$

for all $i, j \in I$. Let's write

$$m_i = \{m_{i,x}\}_{x \in V_i}, \quad m_{i,x} \in \mathcal{M}_x.$$

From (6.9) we see that $m_{i,x} = m_{j,x}$ for all $x \in V_i \cap V_j$. Hence for every $x \in U$ we can define

$$m_x := m_{i,x} \in \mathcal{M}_x$$

where i is some index such that $x \in V_i$, and this does not depend on the choice of i. We obtain a section

$$m := \{m_x\}_{x \in U} \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$$

which satisfies

$$m|_{V_i} = m_i$$

for all i.

(2) For every open set $U \subseteq X$, a section $m \in \Gamma(U, \mathcal{M})$ and a point $x \in U$ let $m_x \in \mathcal{M}_x$ be the image of m under the canonical homomorphism

$$\Gamma(U, \mathcal{M}) \to \mathcal{M}_x.$$

We get a section

$$\{m_x\}_{x \in U} \in \Gamma(U, \mathrm{GSh}(\mathcal{M})).$$

It is easy to see that this construction respects restrictions, so it is a morphism of presheaves.

(3) Exercise (see below).

(4) This is clear from fromula (6.6). □

Exercise 6.10. Prove item (3) above.

Definition 6.11. We call $\mathrm{GSh}(\mathcal{M})$ the *Godement sheaf associated to* $\mathcal{M}$

Exercise 6.12. Show that

$$\mathrm{GSh} : \mathsf{PC}(X) \to \mathsf{C}(X)$$

is a functor, and

$$\gamma : \mathrm{Id} \to \mathrm{GSh}$$

is a morphism of functors from $\mathsf{C}(X)$ to itself.

Definition 6.13. Let $\mathcal{M}$ be a presheaf, and let $U \subseteq X$ be an open set. A section

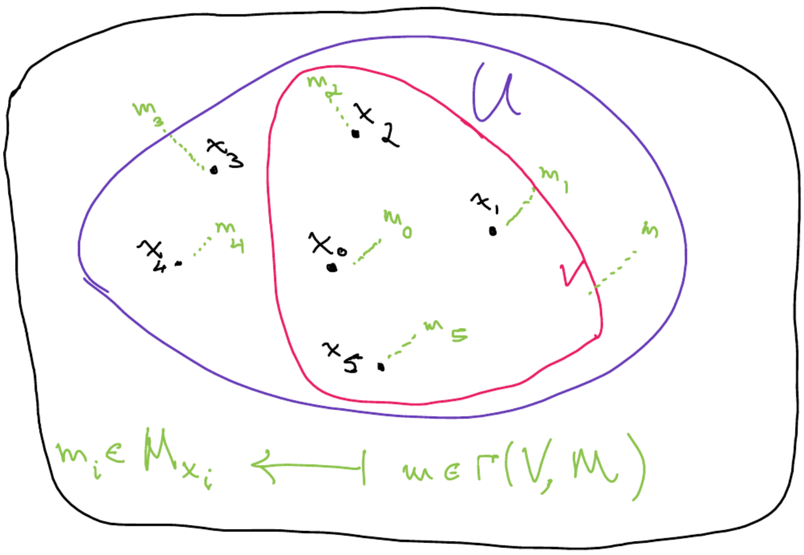
$$m \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$$

is called a *geometric section* if there is an open covering $U = \bigcup_{i \in I} V_i$ and sections $m_i \in \Gamma(V_i, (\mathcal{M}))$, such that for every $x \in V_i$ the morphism

$$\Gamma(V_i, (\mathcal{M})) \to \mathcal{M}_x$$

sends $m_i \mapsto m_x$.

See picture below.



We refer to the data $(\{V_i\}_{i\in I}, \{m_i\}_{i\in I})$ as evidence for the geometricity of m.

Lemma 6.14. *Let $\mathcal{M}$ be a presheaf, let $U \subseteq X$ be an open set, and let*

$$m = \{m_y\}_{y\in U} \in \Gamma(U, \mathrm{GSh}(\mathcal{M})).$$

The following conditions are equivalent:

(i) *m is a geometric section.*

(ii) *For every point $x \in X$ there is an open set V s.t. $x \in V \subseteq U$, and a section $m' \in \Gamma(V, \mathcal{M})$, s.t. $m' \mapsto m_y$ for every $y \in V$.*

Exercise 6.15. Prove this lemma.

Lecture 3, 31 Oct 2018

Lemma 6.16. *Let $\mathcal{M}$ be a presheaf on X. The assignment*

$$\mathrm{Sh}(\mathcal{M}) : U \mapsto \{\text{geometric sections of } \Gamma(U, \mathrm{GSh}(\mathcal{M}))\}$$

is a subsheaf of $\mathrm{GSh}(\mathcal{M})$.

Proof. Step 1. Here we prove that $\mathrm{Sh}(\mathcal{M})$ is a subpresheaf of $\mathrm{GSh}(\mathcal{M})$. Namely that for an inclusion $V \subseteq U$, the morphism

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathrm{GSh}(\mathcal{M})) \to \Gamma(V, \mathrm{GSh}(\mathcal{M}))$$

sends geometric sections to geometric sections.

So let $m \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$ be a geometric section, and let $m|_V \in \Gamma(V, \mathrm{GSh}(\mathcal{M}))$ be its restriction to V. Talk a point $x \in V$. By Lemma 6.14 there is evidence for m at x : an open set W such that $x \in W \subseteq U$, and a section $m' \in \Gamma(W, \mathcal{M})$ such that $m' \mapsto m_y$ for all $y \in W$. Then the pair $(W \cap V, m'|_{W \cap V})$ is evidence for $m|_V$ at x. We see that $m|_V$ is a geometric section.

Step 2. Because $\mathrm{Sh}(\mathcal{M})$ is a subpresheaf of the sheaf $\mathrm{GSh}(\mathcal{M})$, it is automatically separated (axiom (a) holds); see Exercise 5.7.

Now for axiom (b). Let $U = \bigcup_{i \in I} V_i$ be an open covering of an open set, and let $m_i \in \Gamma(V_i, \mathrm{Sh}(\mathcal{M}))$ be a collection of sections that agree on double intersections. Let $m \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$ be the unique section such that $m|_{V_i} = m_i$. Like in step 1, we see that m is a geometric section, namely $m \in \Gamma(U, \mathrm{Sh}(\mathcal{M}))$. □

Remark 6.17. Here is a useful heuristic for the inclusion of sheaves

$$\mathrm{Sh}(\mathcal{M})) \subseteq \mathrm{GSh}(\mathcal{M})).$$

We can pretend that the elements of $\Gamma(U, \mathrm{GSh}(\mathcal{M}))$ are "arbitrary functions" on U, and the elements of $\Gamma(U, \mathrm{Sh}(\mathcal{M}))$ are the "continous functions".

Lemma 6.18. *If $\mathcal{M}$ is a presheaf of abelian groups, then*

$$\mathrm{Sh}(\mathcal{M}) \subseteq \mathrm{GSh}(\mathcal{M})$$

is a subsheaf of abelian groups. Likewise for a presheaf of groups or rings.

Exercise 6.19. Prove this lemma.

Lemma 6.20. *The assignment $\mathcal{M} \mapsto \mathrm{Sh}(\mathcal{M})$ is a functor $\mathsf{PC}(X) \to \mathsf{C}(X)$.*

Exercise 6.21. Prove this lemma.

[**comment**: (date 181104) new lemma next – was part of proof of thm]

Lemma 6.22. *There is a morphism*

$$\tau : \mathrm{Id} \to \mathrm{Sh}$$

of functors from $\mathsf{PC}(X)$ to itself, such that for every presheaf $\mathcal{M}$ the diagram such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\tau_{\mathcal{M}}"] \arrow[dr, "\gamma_{\mathcal{M}}"'] & \mathrm{Sh}(\mathcal{M}) \arrow[d, "\subseteq"] \\ & \mathrm{GSh}(\mathcal{M}) \end{tikzcd}$$

in $\mathsf{PC}(X)$ is commutative.

Proof. Take an open set U. For each section $m \in \Gamma(U, \mathcal{M})$, the section

$$\gamma_{\mathcal{M}}(m) \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$$

is geometric section – the pair (U, m) is a tautological evidence. We define

$$\tau_{\mathcal{M}}(m) := \gamma_{\mathcal{M}}(m) \in \Gamma(U, \mathrm{Sh}(\mathcal{M})) \subseteq \Gamma(U, \mathrm{GSh}(\mathcal{M})).$$

□

[**comment**: (date 181104) new lemma next – was Exer 5.36 in prev version]

Lemma 6.23. *Let $\mathcal{M}$ be a presheaf on X and let $x \in X$ be a point. Then the function on stalks*

$$(\tau_{\mathcal{M}})_x : \mathcal{M}_x \to \mathrm{Sh}(\mathcal{M})_x$$

induced by $\tau_{\mathcal{M}}$ is bijective.

Proof. Injectivity: For every open set U containing x there is a canonical morphism

$$\Gamma(U, \mathrm{GSh}(\mathcal{M})) = \prod_{y \in U} \mathcal{M}_y \to \mathcal{M}_x.$$

So there are canonical morphisms

$$\Gamma(U, \mathcal{M}) \xrightarrow{\Gamma(U, \tau_{\mathcal{M}})} \Gamma(U, \mathrm{Sh}(\mathcal{M})) \to \Gamma(U, \mathrm{GSh}(\mathcal{M})) \to \mathcal{M}_x.$$

Passing to the direct limit over all $U \ni x$ we get a commutative diagram

$$\begin{tikzcd} \mathcal{M}_x \arrow[r, "(\tau_{\mathcal{M}})_x"] \arrow[rrr, bend left, "\mathrm{id}"] & \mathrm{Sh}(\mathcal{M})_x \arrow[r] & \mathrm{GSh}(\mathcal{M})_x \arrow[r] & \mathcal{M}_x \end{tikzcd}$$

Hence $(\tau_{\mathcal{M}})_x$ in injective.

Surjectivity: Take a germ $m_x \in \mathrm{Sh}(\mathcal{M})_x$. It is represented by some section $m \in \Gamma(U, \mathrm{Sh}(\mathcal{M}))$. This means that $m \in \Gamma(U, \mathrm{GSh}(\mathcal{M}))$ is a geometric section. So there is evidence for m at x : there is an open set V and a section $m' \in \Gamma(V, \mathcal{M})$ such that $x \in V \subseteq U$ and $m' = m|_U$. But then the germ $m'_x \in \mathcal{M}_x$ satisfies $(\tau_{\mathcal{M}})_x(m'_x) = m_x$. □

[**comment**: (date 181104) new lemma next]

Lemma 6.24. *Let $\mathcal{M}$ be a presheaf on X. The morphism of sheaves*

$$\mathrm{GSh}(\tau_{\mathcal{M}}) : \mathrm{GSh}(\mathcal{M}) \to \mathrm{GSh}(\mathrm{Sh}(\mathcal{M}))$$

is an isomorphism.

Proof. This is immediate from Lemma 6.23. □

[**comment**: (date 181104) many changes in proof below]
[**comment**: (date 181107) new, improved (?) notation]

A change of notation: for a "legitimate category" C, i.e. $\mathsf{C} = \mathsf{Set}, \mathsf{Grp}, \mathsf{Ab}, \mathsf{Rng}$ or $\mathsf{Mod}\, A$ for a ring A, we write C_X for the category of sheaves with values in C, and $\mathsf{C}_X^{\mathrm{pre}}$ for the category of presheaves with values in it.

Proof of Theorem 6.1. We will prove that the pair $\big(\mathrm{Sh}(\mathcal{M}), \tau_{\mathcal{M}}\big)$ defined in Lemmas 6.16 and 6.22 has the required properties.

Let $\phi : \mathcal{M} \to \mathcal{N}$ be a morphism to a sheaf $\mathcal{N}$. We get the solid commutative diagram

(6.25)
$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\phi"] \arrow[d, "\tau_{\mathcal{M}}"'] \arrow[dd, bend right, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \arrow[d, "\tau_{\mathcal{N}}"] \arrow[dd, bend left, "\gamma_{\mathcal{N}}"] \\
\mathrm{Sh}(\mathcal{M}) \arrow[r, "\mathrm{Sh}(\phi)"] \arrow[d, "\subseteq"] \arrow[ur, dashed, "\phi'"] & \mathrm{Sh}(\mathcal{N}) \arrow[d, "\subseteq"] \\
\mathrm{GSh}(\mathcal{M}) \arrow[r, "\mathrm{GSh}(\phi)"] & \mathrm{GSh}(\mathcal{N})
\end{tikzcd}$$

in $\mathsf{C}_X^{\mathrm{pre}}$. Because $\tau_{\mathcal{N}}$ is an isomorphism (see Corollary 6.4), there is a unique morphism

$$\phi' : \mathrm{Sh}(\mathcal{M}) \to \mathcal{N}$$

that makes the diagram commutative, namely

$$\phi' := \tau_{\mathcal{N}}^{-1} \circ \mathrm{Sh}(\phi). \tag{6.26}$$

It remains to verify the uniqueness of ϕ'. So let $\phi'' : \mathrm{Sh}(\mathcal{M}) \to \mathcal{N}$ be any morphism in $\mathsf{C}_X^{\mathrm{pre}}$ s.t. $\phi'' \circ \tau_{\mathcal{M}} = \phi$. We need to prove that $\phi'' = \phi'$. Define

$$\psi'' := \tau_{\mathcal{N}} \circ \phi'' : \mathrm{Sh}(\mathcal{M}) \to \mathrm{Sh}(\mathcal{N}).$$

In view of (6.26), it suffices to prove that $\psi'' = \mathrm{Sh}(\phi)$.

We have this commutative diagram

(6.27)
$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\phi"] \arrow[d, "\tau_{\mathcal{M}}"'] & \mathcal{N} \arrow[d, "\tau_{\mathcal{N}}", "\cong"'] \\
\mathrm{Sh}(\mathcal{M}) \arrow[r, "\psi''"] & \mathrm{Sh}(\mathcal{N})
\end{tikzcd}$$

in $\mathsf{C}_X^{\mathrm{pre}}$. Passing to stalks at each point $x \in X$ we get a commutative diagram

(6.28)
$$\begin{tikzcd}
\mathcal{M}_x \arrow[r, "\phi_x"] \arrow[d, "(\tau_{\mathcal{M}})_x"', "\cong"] & \mathcal{N}_x \arrow[d, "(\tau_{\mathcal{N}})_x", "\cong"'] \\
\mathrm{Sh}(\mathcal{M})_x \arrow[r, "\psi''_x"] & \mathrm{Sh}(\mathcal{N})_x
\end{tikzcd}$$

in C. The right vertical arrow is an isomorphism by Lemma 6.23. The commutativity of diagram (6.28), together with Lemma 6.24, say that

$$\mathrm{GSh}(\phi) = \mathrm{GSh}(\psi'') : \mathrm{GSh}(\mathcal{M}) \to \mathrm{GSh}(\mathcal{N}).$$

[**comment**: (181107) small change below] We end up with this commutative diagram

(6.29)
$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "\phi"] \arrow[d, "\tau_{\mathcal{M}}"'] \arrow[dd, bend right=60, "\gamma_{\mathcal{M}}"'] & & \mathcal{N} \arrow[d, "\tau_{\mathcal{N}}", "\cong"'] \arrow[dd, bend left=60, "\gamma_{\mathcal{N}}"] \\
\mathrm{Sh}(\mathcal{M}) \arrow[rr, "\psi''"] \arrow[d, "\subseteq"'] & & \mathrm{Sh}(\mathcal{N}) \arrow[d, "\subseteq"] \\
\mathrm{GSh}(\mathcal{M}) \arrow[rr, "\mathrm{GSh}(\phi)=\mathrm{GSh}(\psi'')"] & & \mathrm{GSh}(\mathcal{N})
\end{tikzcd}$$

in $\mathsf{C}_X^{\mathrm{pre}}$. Comparing the bottom square in this diagram to the bottom square in diagram (6.25), and noting that $\mathrm{Sh}(\mathcal{N}) \rightarrowtail \mathrm{GSh}(\mathcal{N})$ is a monomorphism, we conclude that $\psi'' = \mathrm{Sh}(\phi)$, as required. □

Exercise 6.30. Let X be a topological space and M an abelian group (or a ring, etc.). Define $\mathcal{M}$ to be the *constant presheaf* with values in M, namely

$$\Gamma(U, \mathcal{M}) := M$$

for every open set U. Prove that the sheafification of $\mathcal{M}$ is

$$\mathrm{Sh}(\mathcal{M}) = M_X,$$

the *constant sheaf* with values in M.

Exercise 6.31. Consider $X := \mathbb{R}$ with its classical topology, let $\mathcal{M} := \mathbb{Z}_X$, the constant sheaf with values in $\mathbb{Z}$.

(1) Let $U \subseteq X$ be a connected open set (i.e. a nonempty open interval). Calculate $\Gamma(U, \mathcal{M})$ and $\Gamma(U, \mathrm{GSh}(\mathcal{M}))$. Conclude that

$$\Gamma(U, \mathcal{M}) \subsetneq \Gamma(U, \mathrm{GSh}(\mathcal{M})).$$

(2) Conclude that for every point $x \in X$,

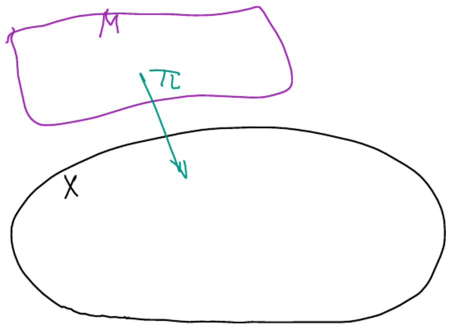
$$\mathcal{M}_x \subsetneq \mathrm{GSh}(\mathcal{M})_x.$$

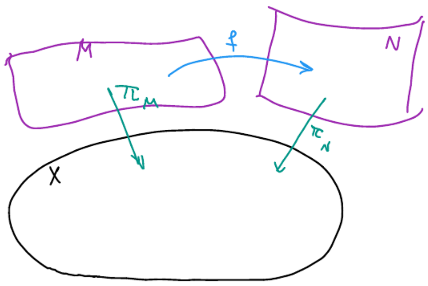
Exercise 6.32. Consider $X := \widehat{\mathbb{Z}}_p$ with its p-adic topology. This is a totally disconnected compact Hausdorff topological space. Let $\mathcal{A} := \mathbb{K}_X$, the constant sheaf with values in a ring $\mathbb{K}$. Calculate $\Gamma(X, \mathcal{A})$.

7. Gluing Sheaves and Morphisms between Them

As a prelude to this abstract theory, today in class we saw two "geometric" versions.

Let X be a topological space (the base), and let $\pi : M \to X$ be a map of spaces (a continous function). We call (M, π) and X-space. A morphism of X-spaces $f : M \to N$ is a map f such that $\pi_N \circ f = \pi_M$. See Figures below.





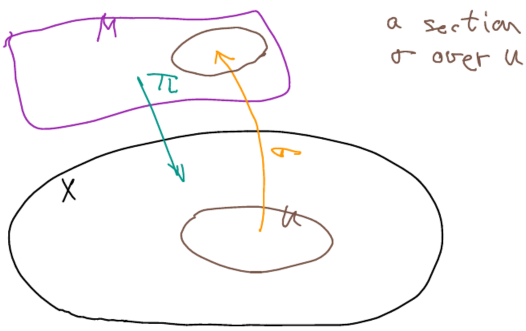
Given an open set $U \subseteq X$, a *section of M over U* is a map

$$\sigma : U \to M$$

such that

$$\pi \circ \sigma = \mathrm{id}_U .$$

I.e. σ is a map of X-spaces. We denote by $\Gamma(U, M)$ the set of sections over of M over U. See Figure:



The assignment

$$\mathcal{M} : U \mapsto \Gamma(U, M)$$

is a sheaf of sets on X. We call $\mathcal{M}$ the *sheaf of sections of M*.

A map $f : M \to N$ of X-spaces induces a morphism

$$\phi : \mathcal{M} \to \mathcal{N}$$

on the sheaves of sections.

The first geometric tale was on *gluing maps of X-spaces*. We are given X-spaces $\pi_M : M \to X$ and $\pi_N : N \to X$, an open covering $X = U = \bigcup_{i \in I} U_i$, and for every i a maps of X-spaces

$$f_i : \pi_M^{-1}(U_i) \to \pi_N^{-1}(U_i).$$

The condition is that

$$f_i|_{\pi_M^{-1}(U_i \cap U_j)} = f_j|_{\pi_M^{-1}(U_i \cap U_j)}.$$

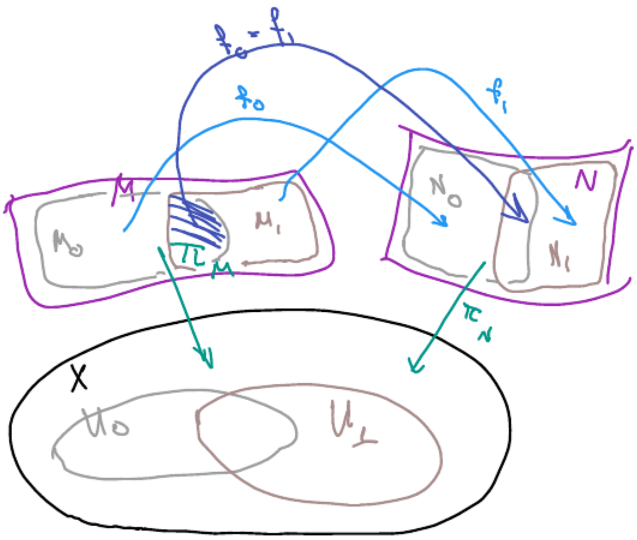
Then there is a unique map of X-spaces

$$f : M \to N$$

such that

$$f|_{\pi_M^{-1}(U_i)} = f_i$$

for all i. The reason: basic topology. See Figure below.



The second geometric tale today was on *gluing X-spaces*. We are given an open covering $X = \bigcup_{i \in I} U_i$, and for every i a U_i-space

$$\pi_i : M_i \to U_i,$$

for every i, j an isomorphism

$$f_{i,j} : \pi_i^{-1}(U_i \cap U_j) \xrightarrow{\simeq} \pi_j^{-1}(U_i \cap U_j)$$

of X-spaces. The condition is that

$$f_{j,k}|_{\pi_M^{-1}(U_i \cap U_j \cap U_k)} \circ f_{i,j}|_{\pi_M^{-1}(U_i \cap U_j \cap U_k)} = f_{i,k}|_{\pi_M^{-1}(U_i \cap U_j \cap U_k)}$$

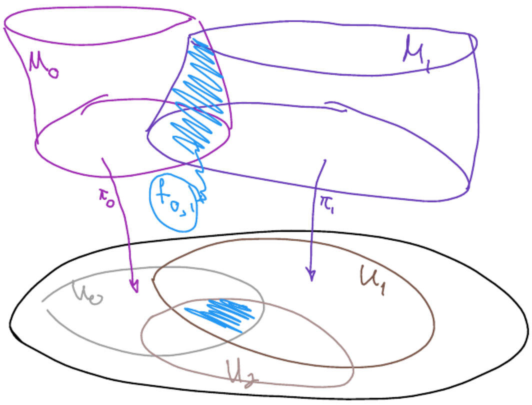
for all i, j, k. Then there is an X-space $\pi : M \to X$, with isomorphisms of X-spaces

$$f_i : \pi^{-1}(U_i) \xrightarrow{\simeq} M_i$$

such that

$$f_{i,j} \circ f_i|_{\pi^{-1}(U_i \cap U_j)} = f_j|_{\pi^{-1}(U_i \cap U_j)}.$$

Again, the proof is just basic topology, with complicated bookkeeping. A partial figure is:



Exercise 7.1. Draw a full picture of this gluing procedure, with these 3 open sets.

By the first tale the X-space M that we get here is unique up to a unique isomorphism.

The theorems that we want are abstract versions of the concrete geometric constructions above.

Definition 7.2. Let $\mathcal{M}$ be a sheaf on a space X and let $U \subseteq X$ be an open set. The *restriction of $\mathcal{M}$ to U* is the sheaf $\mathcal{M}|_U$ on U such that

$$\Gamma(\mathcal{M}|_U, V) := \Gamma(\mathcal{M}, V)$$

for every open set $V \subseteq U$, and

$$\mathrm{rest}^{\mathcal{M}|_U}_{W/V} := \mathrm{rest}^{\mathcal{M}}_{W/V}$$

for every $W \subseteq V \subseteq U$ open.

Theorem 7.3 (Gluing Sheaf Morphisms). *Let $\mathcal{M}$ and $\mathcal{N}$ be sheaves on a topological space X, let $X = \bigcup_{i \in I} U_i$ be an open covering, and let*

$$\phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}$$

be morphisms of sheaves satisfying the condition

$$\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j} : \mathcal{M}|_{U_i \cap U_j} \to \mathcal{N}|_{U_i \cap U_j}.$$

Then there is a unique morphism of sheaves

$$\phi : \mathcal{M} \to \mathcal{N}$$

such that

$$\phi|_{U_i} = \phi_i : \mathcal{M}|_{U_i} \to \mathcal{N}|_{U_i}.$$

Theorem 7.4 (Gluing Sheaves). *Let X be a topological space, let $X = \bigcup_{i \in I} U_i$ be an open covering, for every i let $\mathcal{M}_i$ be a sheaf on U_i, and for every i, j let*

$$\phi_{i,j} : \mathcal{M}_i|_{U_i \cap U_j} \xrightarrow{\simeq} \mathcal{M}_j|_{U_i \cap U_j}$$

be an isomorphism of sheaves on $U_i \cap U_j$. The condition is that

$$\phi_{j,k}|_{U_i \cap U_j \cap U_k} \circ \phi_{i,j}|_{U_i \cap U_j \cap U_k} = \phi_{i,k}|_{U_i \cap U_j \cap U_k}$$

as isomorphisms

$$\mathcal{M}_i|_{U_i \cap U_j \cap U_k} \xrightarrow{\simeq} \mathcal{M}_k|_{U_i \cap U_j \cap U_k},$$

for all i, j, k.

Then there is a sheaf $\mathcal{M}$ on X, together with isomorphisms

$$\phi_i : \mathcal{M}|_{U_i} \xrightarrow{\simeq} \mathcal{M}_i,$$

such that

$$\phi_{i,j} \circ \phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j} : \mathcal{M}|_{U_i \cap U_j} \xrightarrow{\simeq} \mathcal{M}_j|_{U_i \cap U_j} .$$

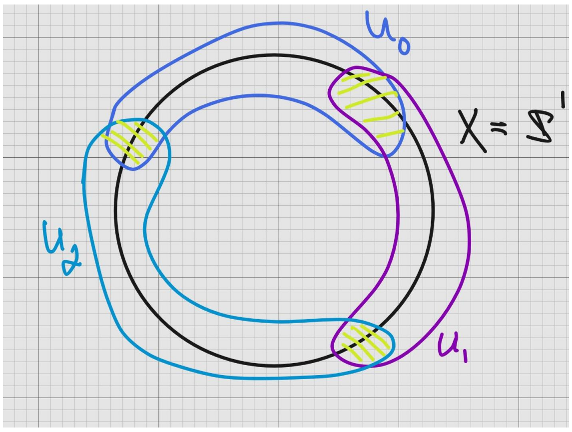
Moreover, that sheaf $\mathcal{M}$, with the collection of isomorphisms $\{\phi_i\}$, is unique up to a unique isomorphism.

We will give a full proof next week.

Lecture 4, 7 Nov 2018

I owe you a nice example of the topological – or geometric – gluing.

Example 7.5. The base space is $X = \mathbf{S}^1$, the circle. We take the covering $X = \bigcup_{i \in I} U_i$ with $I = \{0, 1, 2\}$ shown below.



Let $Z := [-1, 1] \subseteq \mathbb{R}$, the closed line segment. So $X \times Z$ is the ordinary, untwisted, band. Let $\phi : Z \to Z$ be the homeomorphism (or better yet, diffeomorphism) $\psi(z) := -z$. (If you don't know about manifolds with boundary and their diffeomorphisms, then take take Z to be the open line segment.)

[**comment**: (21Nov2018) In the book [Lee] there is a good discussion of manifolds with boundary.]

For $i \in I$ we define the space (or differentiable manifold)

$$M_i := U_i \times Z,$$

with the obvious map

$$\pi_i : M_i \to U_i.$$

The gluing data (what will soon be called the 1-cochain...) $\{\phi_{i,j}\}$ is

$$\phi_{i,j} := \mathrm{id} \times \mathrm{id} : (U_i \cap U_j) \times Z \to (U_i \cap U_j) \times Z$$

for $(i, j) \in \{(0, 1), (1, 2)\}$, and

$$\phi_{0,2} := \mathrm{id} \times \psi : (U_0 \cap U_2) \times Z \to (U_0 \cap U_2) \times Z.$$

These are extended (i.e. for $j \leq i$) by $\phi_{i,j} := \phi_{j,i}^{-1}$ and $\phi_{i,i} := \mathrm{id}$.

Because the triple intersections are empty, the condition

$$\phi_{j,k}|_{U_i \cap U_j \cap U_k} \circ \phi_{i,j}|_{U_i \cap U_j \cap U_k} = \phi_{i,k}|_{U_i \cap U_j \cap U_k}$$

is satisfied automatically.

The resulting X-space (or manifold over X) M is the *Mobius band* of course.

What invariant tells us that M is not homeomorphic (or diffeomorphic) to $X \times Z$?

The only one I know is *orientability*. It is easier to explain in the differentiable case (but still not easy). Here is a sketchy explanation...

In the differentiable version, the manifold M has its tangent bundle $\mathrm{T}M$. This is a rank 2 (real differentiable) vector bundle, that is glued by very similar formulas (the differentials of the $\{\phi_{i,j}\}$). Indeed, for every i the tangent bundle of M_i is trivial:

$$\mathrm{T}M_i \cong M_i \times \mathbb{R}^2,$$

and in the fiber direction $\mathbb{R}^2$ we glue by $(1, \pm 1)$.

For the ordinary band the tangent bundle is trivial:

$$\mathrm{T}(X \times Z) \cong X \times Z \times \mathbb{R}^2.$$

But not so for M. Still, why?

Here is what we do.

First, in general, for a rank d vector bundle $p : E \to M$ on M we have its *frame bundle* $\mathcal{F}_E$, that is a sheaf of sets on M. Over every open set $V \subseteq M$ we define $\Gamma(V, \mathcal{F}_E)$ to be the set of vector bundle isomorphisms

$$\sigma : V \times \mathbb{R}^d \xrightarrow{\simeq} p^{-1}(V). \tag{7.6}$$

This is a sheaf, and the set $\Gamma(V, \mathcal{F}_E)$ is either empty (if E is not trivial above V), or it is isomorphic as a set to

$$\Gamma(V, V \times \mathrm{GL}_d(\mathbb{K})) \cong \mathrm{Hom}_{\mathsf{Mfld}}(V, \mathrm{GL}_d(\mathbb{K})) \cong \Gamma(V, \mathrm{GL}_d(\mathcal{O}_M)).$$

Here $V \times \mathrm{GL}_d(\mathbb{K})$ is the bundle over V, and we look at sections of it; these are the same as morphisms $V \to \mathrm{GL}_d(\mathbb{K})$ in the category Mfld of differentiable real manifolds; and also as the sections on V of the sheaf of groups $\mathrm{GL}_d(\mathcal{O}_M)$, where $\mathcal{O}_M$ is the sheaf of differentiable manifolds. If there is one frame σ, then we get all other frames by the action of $V \times \mathrm{GL}_d(\mathbb{K})$ on $V \times \mathbb{R}^d$.

[**comment**: (21Nov2018) some changes below]

Now the topological group $\mathrm{GL}_d(\mathbb{K})$ has two connected components (according to the determinant). Hence for a small connected open set V (small enough so that $\Gamma(V, \mathcal{F}_E) \neq \varnothing$) the space $p^{-1}(V)$ has two connected components – see (7.6). Let $\mathrm{conn}(\mathcal{F}_E)$ be the sheaf of sets on M associated to the presheaf

$$V \mapsto \boldsymbol{\pi}_0(\Gamma(V, \mathcal{F}_E)),$$

the set of connected components. This is a *locally constant sheaf of sets*: it is locally isomorphic to the constant sheaf of sets $\{1, -1\}$. (On small open sets $V \subseteq M$ this is the constant sheaf.)

There are two options: either the sheaf $\mathrm{conn}(\mathcal{F}_E)$ is the constant sheaf, or it is not. This is detected by the *monodromy representation*.

Suppose $\mathcal{S}$ is a locally constant sheaf of sets on a path connected space Y, that's locally isomorphic to the constant sheaf $\{1, -1\}$. Then there is a representation

$$\rho_{\mathcal{S}} : \boldsymbol{\pi}_1(M) \to G,$$

where G is the 2-element group, seen as permutations of $\{1, -1\}$. The monodromy $\rho_{\mathcal{S}}$ is either trivial; and then $\mathcal{S}$ is the constant sheaf, and

$$\Gamma(Y, \mathcal{S}) = \{1, -1\};$$

or $\rho_{\mathcal{S}}$ is not trivial, and then

$$\Gamma(Y, \mathcal{S}) = \varnothing.$$

Getting back to Mobius, the explicit gluing that we made shows that $\rho_{\mathcal{S}}$, for $Y := M$, $E := \mathrm{T}M$ and $\mathcal{S} := \mathrm{conn}(\mathcal{F}_E)$, is not trivial!

Geometrically this says that the manifold M is not orientable – an orientation of M is by definition a global section of $\mathrm{conn}(\mathcal{F}_E)$. So either there are two or none. An orientation is what we need to integrate on a manifold (to get a consistent sign for the Jacobian matrix).

By "legitimate category", or "very concrete category" we mean a category C that admits infinite products, infinite direct limits, finite fiber products, and a faithful functor

$$F : \mathsf{C} \to \mathsf{Set}$$

that respects the previous constructions. As we know, the categories Set, Grp, Ab, Rng and $\mathsf{Mod}\, A$ for a ring A, all have these good properties. (Warning: F might not respect initial objects and epimorphisms.)

We write C_X for the category of sheaves with values in C, and $\mathsf{C}_X^{\mathrm{pre}}$ for the category of presheaves with values in it.

Another general fact on sheaves, related to Definition 5.4.
[**comment**: (21Nov2018) There was a mistake earlier in item (1) below. It is now correct. The proof is given here: Proposition 7.18.]

Proposition 7.7. *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a morphism in C_X.*

(1) *Assume $\mathsf{C} = \mathsf{Ab}$. The morphism ϕ is surjective iff it is a categorical epimorphism in C_X.*
(2) *ϕ is injective iff it is a categorical monomorphism in C_X.*

Exercise 7.8. Prove this proposition.

Before proceeding, I see that there is something we talked about in class that was not typed in the notes. This is the *equalizer diagram* formulation of the sheaf axioms.

Recall that an equalizer sequence (also called a cartesian sequence) in a category C is a diagram

$$C_0 \xrightarrow{\epsilon} C_1 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} C_2$$

such that when we write it like this:

$$\begin{array}{ccc} C_0 & \xrightarrow{\epsilon} & C_1 \\ {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \delta^0} \\ C_1 & \xrightarrow{\delta^1} & C_2 \end{array}$$

is a *cartesian diagram*, or synonymously a *pullback diagram*, or equivalently that

$$C_0 \cong C_1 \times_{C_2} C_1,$$

the fibered product. The pair (C_0, ϵ) is somtimes called the *kernel* of $C_1 \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} C_2$.

In Set we know that

$$\epsilon : C_0 \xrightarrow{\simeq} \big\{ c \in C_1 \mid \delta^0(c) = \delta^1(c) \big\}.$$

Hence it is the same when C is a very concrete category (the forgetful functor F repects fiber products).

Exercise 7.9. Prove that the kernel ϵ is a monomorphism in C.

We have seen that:

Proposition 7.10. *A presheaf $\mathcal{M} \in \mathsf{C}_X^{\mathrm{pre}}$ is a sheaf iff for every open set $U \subseteq X$ and every open covering $U = \bigcup_{i \in I} V_i$ the diagram*

$$\Gamma(U, \mathcal{M}) \xrightarrow{\epsilon} \prod_{i \in I} \Gamma(V_i, \mathcal{M}) \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} \prod_{j,k \in I} \Gamma(V_j \cap V_k, \mathcal{M})$$

is an equalizer sequence in C.

Here ϵ is the product on all $i \in I$ of the restriction morphisms

$$\mathrm{rest}_{V_i/U} : \Gamma(U, \mathcal{M}) \to \Gamma(V_i, \mathcal{M}).$$

The morphism δ^1 is the product on all $i = j \in I$ of the product on all $k \in I$ of

$$\mathrm{rest}_{V_j \cap V_k/V_j} : \Gamma(V_j, \mathcal{M}) \to \Gamma(V_j \cap V_k, \mathcal{M}).$$

And the morphism δ^0 is the product on all $i = k \in I$ of the product on all $j \in I$ of

$$\mathrm{rest}_{V_j \cap V_k/V_k} : \Gamma(V_k, \mathcal{M}) \to \Gamma(V_j \cap V_k, \mathcal{M}).$$

We now provide proofs of the gluing theorems.

Proof of Theorem 7.3: *gluing sheaf morphisms.* Let $V \subseteq X$ be an open set. Defining $V_i := V \cap U_i$, we get an open covering $V = \bigcup_{i \in I} V_i$. Consider the following solid diagram

$$\begin{tikzcd}
\Gamma(V, \mathcal{M}) \arrow[r, "\epsilon"] \arrow[d, dashed, "\Gamma(V,\phi)"] & \prod_{i \in I} \Gamma(V_i, \mathcal{M}) \arrow[r, shift left, "\delta^0"] \arrow[r, shift right, "\delta^1"'] \arrow[d, "\{\Gamma(V_i,\phi_i)\}"] & \prod_{j,k \in I} \Gamma(V_j \cap V_k, \mathcal{M}) \arrow[d, "\{\Gamma(V_j \cap V_k,\phi_k)\}", "\{\Gamma(V_j \cap V_k,\phi_j)\} ="'] \\
\Gamma(V, \mathcal{N}) \arrow[r, "\epsilon"] & \prod_{i \in I} \Gamma(V_i, \mathcal{N}) \arrow[r, shift left, "\delta^0"] \arrow[r, shift right, "\delta^1"'] & \prod_{j,k \in I} \Gamma(V_j \cap V_k, \mathcal{N})
\end{tikzcd} \tag{7.11}$$

in the category C. This is commutative, by the compatibilty condition

$$\phi_j|_{U_j \cap U_k} = \phi_k|_{U_j \cap U_k}.$$

Therefore there is a unique morphism $\Gamma(V, \phi)$ on the dashed vertical arrow.

As the open set V varies, we obtain a morphism of sheaves

$$\phi : \mathcal{M} \to \mathcal{N}.$$

If $V \subseteq U_i$ for some index i, then $V_i = V$, and therefore by the commutativity of the left square in (7.11) – and neglecting all indices other than i – we see that

$$\Gamma(V, \phi) = \Gamma(V_i, \phi_i).$$

This means that

$$\phi|_{U_i} = \phi_i.$$

The uniqueness of this ϕ is also because it is the only morphism that makes (7.11) commutative. □

Proof of Theorem 7.4: *gluing sheaves.* Recall that we are given an open covering $X = \bigcup_{i \in I} U_i$, a sheaf $\mathcal{M}_i$ on U_i, and an isomorphism

$$\phi_{i,j} : \mathcal{M}_i|_{U_i \cap U_j} \xrightarrow{\simeq} \mathcal{M}_j|_{U_i \cap U_j}$$

for every i, j. The condition is that

$$\phi_{j,k}|_{U_i \cap U_j \cap U_k} \circ \phi_{i,j}|_{U_i \cap U_j \cap U_k} = \phi_{i,k}|_{U_i \cap U_j \cap U_k}.$$

Take a point $x \in X$. Let us denote by $\mathcal{M}_{i,x}$ the stalk of $\mathcal{M}_i$ at x. There is an object $\mathcal{M}_x \in \mathsf{C}$, together with an isomorphism

$$\phi_{i,x} : \mathcal{M}_{i,x} \xrightarrow{\simeq} \mathcal{M}_x$$

for every i, such that

$$\phi_{j,k,x} \circ \phi_{i,j,x} = \phi_{i,k,x}.$$

Moreover, the object $\mathcal{M}_x$, with its collection of isomorphisms $\{\phi_{i,x}\}$, is unique (up to a unique isomorphism).

Let us define the sheaf $\widehat{\mathcal{M}}$ on X as follows:

$$\Gamma(V, \widehat{\mathcal{M}}) := \prod_{x \in V} \mathcal{M}_x .$$

(This will eventually be the Godement sheaf of $\mathcal{M}$.) On every U_i there is a morphism of sheaves

$$\widehat{\phi}_i : \mathcal{M}_i \to \widehat{\mathcal{M}}|_{U_i} ,$$

and it gives rise to an isomorphism of sheaves

$$\mathrm{GSh}(\mathcal{M}_i) \xrightarrow{\simeq} \widehat{\mathcal{M}}|_{U_i} . \tag{7.12}$$

A section

$$\{m_x\}_{x \in V} \in \Gamma(V, \widehat{\mathcal{M}})$$

will be called geometric relative to the collection $\{\mathcal{M}_i\}$ if for every $x \in V$ there is an open set W s.t. $x \in W \subseteq V \cap U_i$ for some i, and a section $m \in \Gamma(W, \mathcal{M}_i)$, s.t. $\phi_{i,y}(m) = m_y \in \mathcal{M}_y$ for all $y \in W$.

Now let $\mathcal{M}$ be the subpresheaf of $\widehat{\mathcal{M}}$ defined as follows:

$$\Gamma(V, \mathcal{M}) \subseteq \Gamma(V, \widehat{\mathcal{M}})$$

is the subset of all geometric sections, in the relative sense as above. As we already know from previous calculations, $\mathcal{M}$ is a subsheaf of $\widehat{\mathcal{M}}$; and $\widehat{\mathcal{M}} \cong \mathrm{GSh}(\mathcal{M})$.

For every index i, the isomorphism (7.12) identifies $\mathcal{M}_i$ with $\mathcal{M}|_{U_i}$, as the subsheaves of geometric sections of $\widehat{\mathcal{M}}|_{U_i}$. This is the isomorphism

$$\phi_i : \mathcal{M}|_{U_i} \xrightarrow{\simeq} \mathcal{M}_i$$

that we want. By construction these satisfy

$$\phi_{i,j} \circ \phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j} .$$

The uniqueness (up to unique isomorphism) of $\mathcal{M}$ is a consequence of Theorem 7.3, and is left as an exercise. $\square$

Exercise 7.13. Finish the proof above.

Lecture 5, 21 Nov 2018

[**comment**: (21Nov2018) (1) No lecture 14 Nov. (2) There are corrections above.]

I did not mean to introduce the next definition now, but it is needed for the proof of Proposition 7.18 (the correction of Prop 7.7(1)).

Note that for $\mathcal{M}, \mathcal{N} \in \mathsf{Ab}_X^{\mathrm{pre}}$ and an open set $U \subseteq X$ the set of morphisms $\mathcal{M}|_U \to \mathcal{N}|_U$ in $\mathsf{Ab}_U^{\mathrm{pre}}$ is itself an abelian group. We denote it by

$$\mathrm{Hom}_{\mathsf{Ab}_U^{\mathrm{pre}}}(\mathcal{M}|_U, \mathcal{N}|_U).$$

Also recall that Ab_U (sheaves) is a full subcategory of $\mathsf{Ab}_U^{\mathrm{pre}}$ (presehaves).

Definition 7.14. Let X be a topological space and let $\phi : \mathcal{M} \to \mathcal{N}$ be a homomorphism in $\mathsf{Ab}_X^{\mathrm{pre}}$. The *cokernel* of ϕ is the presheaf

$$\mathrm{Coker}^{\mathrm{pre}}(\phi) \in \mathsf{Ab}_X^{\mathrm{pre}}$$

defined by

$$\mathrm{Coker}^{\mathrm{pre}}(\phi)(U) := \mathrm{Coker}\big(\Gamma(U,\phi) : \Gamma(U,\mathcal{M}) \to \Gamma(U,\mathcal{N})\big).$$

Definition 7.15. Let X be a topological space and let $\phi : \mathcal{M} \to \mathcal{N}$ be a homomorphism in Ab_X. The *cokernel* of ϕ is the sheaf

$$\mathrm{Coker}(\phi) := \mathrm{Sh}(\mathrm{Coker}^{\mathrm{pre}}(\phi)) \in \mathsf{Ab}_X .$$

There is a canonical homomorphism

$$\pi : \mathcal{N} \to \mathrm{Coker}(\phi)$$

in Ab_X that's induced from the homomorphism

$$\mathcal{N} \to \mathrm{Coker}^{\mathrm{pre}}(\phi)$$

in $\mathsf{Ab}_X^{\mathrm{pre}}$.

Proposition 7.16. *Let $\phi : \mathcal{M} \to \mathcal{N}$ and be a homomorphism in Ab_X.*

(1) *The canonical homomorphism $\pi : \mathcal{N} \to \mathrm{Coker}(\phi)$ in Ab_X is surjective.*
(2) *The composition $\pi \circ \phi$ is the zero homomorphism.*
(3) *Let $\psi : \mathcal{N} \to \mathcal{P}$ be a homomorphism in Ab_X such that $\psi \circ \phi = 0$. Then ψ factors uniquely through $\mathrm{Coker}(\phi)$. Namely there is a unique morphism*

$$\bar{\psi} : \mathrm{Coker}(\phi) \to \mathcal{P}$$

in Ab_X such that

$$\psi = \bar{\psi} \circ \pi.$$

(4) *For every point $x \in X$ the sequence*

$$\mathcal{M}_x \xrightarrow{\phi_x} \mathcal{N}_x \xrightarrow{\pi_x} \mathrm{Coker}(\phi)_x \to 0$$

in Ab is exact. In other words, π_x induces an isomorphism

$$\mathrm{Coker}(\phi_x) \xrightarrow{\simeq} \mathrm{Coker}(\phi)_x.$$

Exercise 7.17. Prove Proposition 7.16.

Recall that the trivial abelian group is denoted by 0. It is both the initial and terminal object of the category Ab. Given a space X, let $0_X \in \mathsf{Ab}_X$ be the constant sheaf with values in the group 0. This sheaf is both the initial and terminal object of the category Ab_X.

Proposition 7.18. *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a homomorphism in Ab_X. The following conditions are equivalent*:

(i) *ϕ is surjective (Definition 5.4).*
(ii) *ϕ is a categorical epimorphism in Ab_X.*
(iii) *The sheaf $\mathrm{Coker}(\phi)$ is the constant sheaf 0_X.*

Proof.
(i) ⇒ (ii): This is easy, and was done in class by Yotam.

(ii) ⇒ (iii): Let $\mathcal{P} := \mathrm{Coker}(\phi)$. We consider these two homomorphisms in Ab_X :

$$\psi_0, \psi_1 : \mathcal{N} \to \mathcal{P},$$

$\psi_0 := 0$ and $\psi_1 := \pi$. These both satisfy

$$\psi_i \circ \phi = 0.$$

But ϕ is a categorical epimorphism, so we must have $\psi_0 = \psi_1$. This means that $\pi = 0$. According to Proposition 7.16(1) the homomorphism π is surjective. Hence $\mathcal{P} = 0_X$.

(iii) ⇒ (i): We are given that $\mathrm{Coker}(\phi) = 0_X$, so for every point $x \in X$ the stalk is

$$\mathrm{Coker}(\phi)_x = 0 \in \mathsf{Ab}\,.$$

Using 7.16(4) we conclude that

$$\phi_x : \mathcal{M}_x \to \mathcal{N}_x$$

is surjective. Thus ϕ is a surjection of sheaves. □

8. Vector Bundles

We are going to discuss real vector bundles on spaces and manifolds. This will take us a step closer to a modern understanding of geometry. Later we will use similar ideas for schemes.

Convention 8.1. We shall work in one of the following categories:

- The category Top of topological spaces and continous maps between them. Here $\mathbb{K} = \mathbb{R}$, the field of real numbers.
- The category Mfld of real differentiable manifolds and differentiable maps between them, where by differentiable we mean of class C^∞. Here $\mathbb{K} = \mathbb{R}$.
- The category Var of quasi-projective algebraic varieties over an algebraically closed field $\mathbb{K}$.

We shall denote by Sp any of these categories, and refer to an object of Sp as a *space*.

The category Sp has finite products. These products respect the forgetful functor

$$\mathsf{Sp} \to \mathsf{Set}\,.$$

Warning: the forgetful functor

$$\mathsf{Var} \to \mathsf{Top}$$

does not respect products. On the other hand, the forgetful functor

$$\mathsf{Mfld} \to \mathsf{Top}$$

does respect products.

In Top and Var we have fiber products: given $f : Y \to X$ and $g : Z \to X$ the fiber product is the closed subset

$$Y \times_X Z = \{(y, z) \mid f(y) = g(z)\} \subseteq Y \times Z \tag{8.2}$$

with the induced structure of a closed subspace.

Fiber products in Mfld are more delicate, because the closed subset in (8.2) is not a submanifold in general.

Recall that a map $f : Y \to X$ in Mfld is called a *submersion* if for every point $y \in Y$ the linear map on tangent spaces

$$\mathrm{d}(f)_y : \mathrm{T}_y\, Y \to \mathrm{T}_{f(y)}\, X$$

is surjective. (See [Lee].)

Example 8.3. The inclusion $f : U \to X$ of an open subset is a submersion in Mfld.

Exercise 8.4. Prove that if $f : U \to X$ is the inclusion of an open set in Sp, then for every $g : Z \to X$ in Sp the fiber product exists, and it is

$$U \times_X Z \cong g^{-1}(U) \subseteq Z.$$

Example 8.5. A submersion $f : Y \to X$ in Mfld of relative dimension 0 (i.e. $\dim(Y) = \dim(X)$) is a *local diffeomorphism*.

Lemma 8.6. *Given maps $f : Y \to X$ and $g : Z \to X$ in Mfld such that either f or g is a submersion, the closed subset*

$$\{(y, z) \mid f(y) = g(z)\} \subseteq Y \times Z$$

is a submanifold. It is the categorical fiber product $Y \times_X Z$ in Mfld.

Exercise 8.7. Prove the lemma. (Hint: use the Implicit Function Theorem.)

Remark 8.8. From a contemporary point of view, Mfld is the wrong category. It is a relic from decades ago, and it is "deficient".

There actually is a theory of "differentiable spaces", very recent, due to D. Joyce. It is new and not many people are aware of it. The manifolds are the nonsingular objects in the category of differentiable spaces.

Remark 8.9. What are the "manifolds" in Var ? These are the *nonsingular* varieties. They have tangent spaces, and one can talk about submersions in Var. But these have another name: *smooth maps of varieties*. Lemma 8.6 holds in Var. See [Har, Prop III.10.4].

A smooth map f of relative dimension 0 is called an *étale map*. I is usually not a local isomorphism!

In algebraic geometry we also talk about smooth maps $f : Y \to X$ between singular varieties.

If we are lucky (it is a matter of time) we will be able to talk about smooth maps between schemes.

Next is a nonstandard definition.

Definition 8.10. A map $Y \to X$ in Sp is called Sp-*fibered*, and Y is called Sp-*fibered over* X, if for every $Z \to X$ in Sp the fibered product $Y \times_X Z$ exists in Sp.

Thus when $\mathsf{Sp} = \mathsf{Top}$ or $\mathsf{Sp} = \mathsf{Var}$ this is an empty condition (all maps are Sp-fibered), and when $\mathsf{Sp} = \mathsf{Mfld}$ all submersions a fibered (by Lemma 8.6).

For every $n \geq 0$ we have the *affine n-dimensional space* $\mathbf{A}^n(\mathbb{K})$. As a set it is $\mathbb{R}^n$. It is viewed as an object of Sp, either as a topological space with the usual metric topology, or as a differentiable manifold with the usual differentiable structure, or as an algebraic variety with the Zariski topology with the usual algebro-geometric structure, as the case may be.

For $n = 0$, $\mathbf{A}^0(\mathbb{K})$ is a single point, that we denote by 0. Note that $\mathbf{A}^0(\mathbb{K})$ is the terminal object of Sp.

For $n = 1$, $\mathbf{A}^1(\mathbb{K})$ is a *commutative ring object in* Sp. There are maps

$$\begin{gathered} \mathrm{add} : \mathbf{A}^1(\mathbb{K}) \times \mathbf{A}^1(\mathbb{K}) \to \mathbf{A}^1(\mathbb{K}), \quad \mathrm{add}(a, b) := a + b, \\ \mathrm{mult} : \mathbf{A}^1(\mathbb{K}) \times \mathbf{A}^1(\mathbb{K}) \to \mathbf{A}^1(\mathbb{K}), \quad \mathrm{mult}(a, b) := a \cdot b, \\ 0 : \mathbf{A}^0(\mathbb{K}) \to \mathbf{A}^1(\mathbb{K}), \quad 0(0) := 0_{\mathbb{K}} \in \mathbb{K}, \\ 1 : \mathbf{A}^0(\mathbb{K}) \to \mathbf{A}^1(\mathbb{K}), \quad 1(0) := 1_{\mathbb{K}} \in \mathbb{K} \end{gathered} \tag{8.11}$$

in Sp that satisfy the axioms of a commutative ring. After applying the forgetful functor to Set we recover the familiar operations of the ring $\mathbb{K}$.

For other values of $n \in \mathbb{N}$ the space $\mathbf{A}^n(\mathbb{K})$ is an $\mathbf{A}^1(\mathbb{K})$-*module in* Sp, i.e. literally a *vector space*. This just means that there are maps

$$\begin{gathered} \mathrm{add} : \mathbf{A}^n(\mathbb{K}) \times \mathbf{A}^n(\mathbb{K}) \to \mathbf{A}^n(\mathbb{K}), \quad \mathrm{add}(v, w) := v + w \in \mathbb{K}^n, \\ \mathrm{mult} : \mathbf{A}^1(\mathbb{K}) \times \mathbf{A}^n(\mathbb{K}) \to \mathbf{A}^n(\mathbb{K}), \quad \mathrm{mult}(a, v) := a \cdot v \in \mathbb{K}^n, \\ 0 : \mathbf{A}^0(\mathbb{K}) \to \mathbf{A}^n(\mathbb{K}), \quad 0(0) := 0_{\mathbb{K}^n} \in \mathbb{K}^n \end{gathered} \tag{8.12}$$

in Sp that satisfy the axioms of a module (for the ring structure (8.11). After applying the forgetful functor to Set we recover the familiar operations of the $\mathbb{K}$-module $\mathbb{K}^n$.

The idea of a vector bundle is meant to provide a relative version of (8.12): *a collection of vector spaces that move – continously or smoothly or algebraically – over a base X.*

Fixing a space $X \in \mathsf{Sp}$ (the base), we can talk about the category Sp/X of X-*spaces*, or *spaces over X*. An object of Sp/X is a pair (Y, π_Y) where $\pi_Y : Y \to X$ is a map in Sp, called the *structure map*. A morphism

$$g : (Y, \pi_Y) \to (Z, \pi_Z)$$

in Sp/X is a map $g : Y \to Z$ satisfying

$$\pi_Z \circ g = \pi_Y .$$

In a commutative diagram:

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "\pi_Y"'] & & Z \arrow[dl, "\pi_Z"] \\ & X & \end{tikzcd}$$

We shall often keep π_Y implicit.

We need to know about finite products in Sp/X. These are just the fibered products in Sp. To be precise:

$$(Y, \pi_Y) \times (Z, \pi_Z) = (Y \times_X Z, \pi), \tag{8.13}$$

where

$$Y \times_X Z \subseteq X \times Y$$

is the fibered product in Sp (if it exists); and

$$\pi(y, z) := \pi_Y(y) = \pi_Z(z) \in X$$

for $(y, z) \in Y \times_X Z$.

Thus in the cases $\mathsf{Sp} = \mathsf{Top}$ and $\mathsf{Sp} = \mathsf{Var}$ all finite products exist in Sp/X.

In the case $\mathsf{Sp} = \mathsf{Mfld}$, in many important instances at least one of the maps π_Y or π_Z will be a submersion, so the finite product will exist. Here is a typical good situation in this case. Given any space Y_0, consider the product

$$Y := X \times Y_0 \tag{8.14}$$

Define

$$\pi_Y : Y = X \times Y_0 \to X \tag{8.15}$$

to be the projection on the first coordinate. This is a submersion, so Y is Sp-fibered over X.

Lemma 8.16. *Let* $(U, \pi_U) \in \mathsf{Sp}/X$, *let* $Y_0 \in \mathsf{Sp}$, *and let* $(Y, \pi_Y) \in \mathsf{Sp}/X$ *be as in* (8.14) *and* (8.15).

(1) *There is an isomorphism of sets*

$$\mathrm{Hom}_{\mathsf{Sp}/X}(U, Y) = \mathrm{Hom}_{\mathsf{Sp}/X}(U, X \times Y_0) \cong \mathrm{Hom}_{\mathsf{Sp}}(U, Y_0).$$

It is functorial in U *and* Y_0.

(2) *There is an isomorphism*

$$U \times_X Y \cong U \times Y_0$$

in Sp. *It is functorial in* U *and* Y_0.

Exercise 8.17. Prove the lemma. Give explicit formulas for the isomorphisms.

Before approaching the vectors bundles, let's talk about more general (and less structured) bundles.

Definition 8.18. Let $X \in \mathsf{Sp}$. A *fiber bundle* over X is an object (Y, π_Y) in Sp/X with this property: there is a space Z, an open covering $X = \bigcup_{i \in I} U_i$, and isomorphisms

$$\phi_i : U_i \times Z \xrightarrow{\simeq} \pi_Y^{-1}(U_i)$$

in Sp/U_i. The space Z is called the *fiber*, and the isomorphisms ϕ_i are called *local trivializations*.

Example 8.19. The Möbius band is a fiber bundle in Top (or, if we choose the open version, in Mfld). The base is $X = \mathbf{S}^1$, and the fiber is $Z = [-1, 1]$ (or $Z = (-1, 1)$).

Proposition 8.20. *Let* (Y, π_Y) *be a fiber bundle over* X *with fiber* Z.

(1) *If* $\mathsf{Sp} = \mathsf{Mfld}$ *then* π_Y *is a submersion.*
(2) *The object* $(Y, \pi_Y) \in \mathsf{Sp}/X$ *is* Sp*-fibered.*
(3) *For every point* $x \in X$ *the fiber*

$$\pi_Y^{-1}(x) \cong \{x\} \times_X Y$$

is an object of Sp, *and it is isomorphic (noncanonically) to* Z.

Exercise 8.21. Prove the proposition.

Lecture 6, 28 Nov 2018

Recall that Sp is either Top, Mfld or Var, and correspondingly the field $\mathbb{K}$ is either $\mathbb{R}$, $\mathbb{R}$ or algebraically closed.

Last time we talked about the *vector spaces* $\mathbf{A}^n(\mathbb{K})$, that are $\mathbf{A}^1(\mathbb{K})$-modules in Sp. We also talked about *fiber bundles* in Sp. Now we are going to combine these notions.

Definition 8.22. We fix a base $X \in \mathsf{Sp}$. For every $n \in \mathbb{N}$ consider the X-space

$$\mathbf{A}^n(X) := X \times \mathbf{A}^n(\mathbb{K}) \tag{8.23}$$

with structure map

$$\pi_{\mathbf{A}^n(X)}(x, v) := x,$$

the projection on the first coordinate.

Of course $\mathbf{A}^0(X) = X$.

For $n = 1$, $\mathbf{A}^1(X)$ is a commutative ring object in Sp/X. This structure is induced from the commutative ring structure of $\mathbf{A}^1(\mathbb{K})$ in formula (8.11). Here it is explicitly. As mentioned above (in Prop 8.20(2)) the fiber product

$$\mathbf{A}^1(X) \times_X \mathbf{A}^1(X) \subseteq \mathbf{A}^1(X) \times \mathbf{A}^1(X)$$

exists in Sp. (In fact it is isomorphic to $\mathbf{A}^2(X)$, but that's not helpful here.) The points in $\mathbf{A}^1(X)$ are pairs $\big((x,a),(x,b)\big)$ with $x \in X$ and $a, b \in \mathbb{R}$. The operations are:

$$\begin{gathered} \mathrm{add} : \mathbf{A}^1(X) \times_X \mathbf{A}^1(X) \to \mathbf{A}^1(X), \quad \mathrm{add}\big((x,a),(x,b)\big) := (x, a+b), \\ \mathrm{mult} : \mathbf{A}^1(X) \times_X \mathbf{A}^1(X) \to \mathbf{A}^1(X), \quad \mathrm{mult}\big((x,a),(x,b)\big) := (x, a \cdot b), \\ 0_X : \mathbf{A}^0(X) \to \mathbf{A}^1(X), \quad 0(x) := (x, 0), \\ 1_X : \mathbf{A}^0(X) \to \mathbf{A}^1(X), \quad 1(x) := (x, 1). \end{gathered} \tag{8.24}$$

These are maps in Sp/X.

For any value $n \in \mathbb{N}$, the space $\mathbf{A}^n(X)$ is an $\mathbf{A}^1(X)$-module object in Sp/X. This structure is induced from formula (8.12). Again in explicit terms: the fiber product

$$\mathbf{A}^n(X) \times_X \mathbf{A}^n(X) \subseteq \mathbf{A}^n(X) \times \mathbf{A}^n(X)$$

exists in Sp. The points in it are $\big((x,v),(x,w)\big)$ with $x \in X$ and $v, w \in \mathbb{R}^n$. The operations are:

$$\begin{gathered} \mathrm{add} : \mathbf{A}^n(X) \times_X \mathbf{A}^n(X) \to \mathbf{A}^n(X), \quad \mathrm{add}\big((x,v),(x,w)\big) := (x, v+w), \\ \mathrm{mult} : \mathbf{A}^1(X) \times_X \mathbf{A}^n(X) \to \mathbf{A}^n(X), \quad \mathrm{mult}\big((x,a),(x,v)\big) := (x, a \cdot v), \\ 0_X : \mathbf{A}^0(X) \to \mathbf{A}^n(X), \quad 0(x) := (x, 0). \end{gathered} \tag{8.25}$$

The map 0_X is called the *zero section*.

Definition 8.26. Let n be a natural number. The *standard trivial rank n vector bundle over X* is the space

$$\mathbf{A}^n(X) = X \times \mathbf{A}^n(\mathbb{K})$$

with the operations defined in (8.25).

Definition 8.27. Let $X \in \mathsf{Sp}$ and $n \in \mathbb{N}$. A *rank n vector bundle* over X is an Sp-fibered object (E, π_E) in Sp/X, with these maps

$$\mathrm{add}_E : E \times_X E \to E,$$

$$\mathrm{mult}_E : \mathbf{A}^1(X) \times_X E \to E,$$

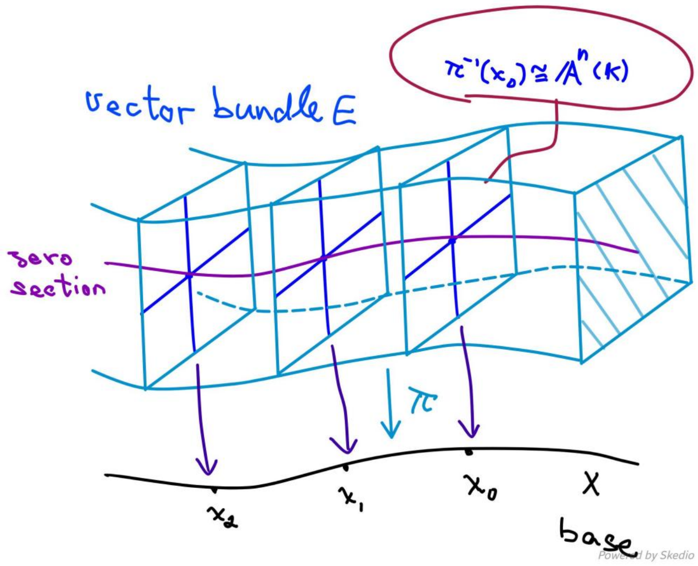


FIGURE 1. A rank n vector bundle E over X

$$0_E : X \to E$$

in Sp/X that are called the operations. The conditions are:

(i) The operations make E into an $\mathbf{A}^1(X)$-module in Sp/X, i.e. the module axioms are satisfied.

(ii) Local triviality: there is an open covering $X = \bigcup_{i \in I} U_i$, and for every i an isomorphism

$$\phi_i : \mathbf{A}^n(U_i) \xrightarrow{\simeq} \pi_E^{-1}(U_i)$$

of $\mathbf{A}^1(U_i)$-modules, i.e. an isomorphism in Sp/U_i that respects the operations.

See Figure 1.

Notice that for every point $x \in X$ the fiber

$$E(x) := \pi_E^{-1}(x) \cong \{x\} \times_X E \tag{8.28}$$

is isomorphic to $\mathbf{A}^n(\mathbb{R})$ as $\mathbb{K}$-modules.

Example 8.29. Assume $\mathsf{Sp} = \mathsf{Mfld}$. Let X be an n-dimensional manifold, an object of Mfld. The *tangent bundle* $\mathrm{T}X$ is a rank n vector bundle on X.

Exercise 8.30. If you don't understand the example above, then read about it (in [Lee] or another book) and write a proof.

Remark 8.31. This is an elaboration of Example 8.29. Unfortunately you won't find this material in [Lee], and probably not in any other textbook on differential geometry. It is "extra material". We might do the algebraic version next semester (for schemes).

Let X be an n-dimensional differentiable manifold (so $\mathsf{Sp} = \mathsf{Mfld}$ and $\mathbb{K} = \mathbb{R}$). The tangent bundle is $\mathrm{T}X$, and let $\mathcal{T}_X$ be the sheaf of section of $\mathrm{T}X$, i.e.

$$\Gamma(U, \mathcal{T}_X) := \Gamma(U, \mathrm{T}X) = \mathrm{Hom}_{\mathsf{Sp}/X}(U, \mathrm{T}X)$$

for every open set $U \subseteq X$. It is standard to refer to a section $\partial \in \Gamma(U, \mathrm{T}X)$ as a *vector field* on U.

This is not mysterious, at least when $X = \mathbf{A}^n(\mathbb{R})$. Let $t_1, \ldots, t_n$ be the coordinate functions. Then there are the constant vector fields

$$\partial_i := \frac{\partial}{\partial t_i},$$

and every vector field ∂ can be written uniquely as

$$\partial = \sum_{i=1}^{n} f_i \cdot \partial_i$$

with $f_i \in \Gamma(X, \mathcal{O}_X)$. Moreover, ∂ is a derivation of $A := \Gamma(X, \mathcal{O}_X)$, as defined in the next paragraph, and

$$f_i = \partial(t_i).$$

Suppose A is a commutative $\mathbb{R}$-ring. A *derivation* of A (relative to $\mathbb{R}$) is an $\mathbb{R}$-linear homomorphism

$$\partial : A \to A$$

satisfying the *Leibniz rule*

$$\partial(a \cdot b) = \partial(a) \cdot b + a \cdot \partial(b).$$

The set of derivations of A is $\operatorname{Der}_{\mathbb{R}}(A)$.

This can be done for sheaves. Consider the sheaf of differentiable functions $\mathcal{O}_X$ on X. For an open set $U \subseteq X$ we can look at derivations

$$\partial : \mathcal{O}_U \to \mathcal{O}_U,$$

namely $\mathbb{R}$-linear sheaf homomorphisms on U that satisfy the Leibniz rule on every open set $U' \subseteq U$ (or equivalently, in all stalks at $x \in U$). These form an $\mathbb{R}$-module

$$\operatorname{Der}_{\mathbb{R}}(\mathcal{O}_U) \subseteq \operatorname{Hom}_{\mathbb{R}_X}(\mathcal{O}_U, \mathcal{O}_U).$$

As U moves we get a sheaf $\mathcal{D}er_{\mathbb{R}}(\mathcal{O}_X)$ such that

$$\Gamma(U, \mathcal{D}er_{\mathbb{R}}(\mathcal{O}_X)) = \operatorname{Der}_{\mathbb{R}}(\mathcal{O}_U).$$

Theorem. There is a canonical isomorphism of sheaves of $\mathbb{R}$-modules

$$\mathcal{D}er_{\mathbb{R}}(\mathcal{O}_X) \cong \mathcal{T}_X.$$

Proposition 8.32. *Let X be a space in Sp, and let E, F be vector bundles on X, of ranks m, n respectively.*

(1) *There is a vector bundle $E \oplus F$ on X, whose fibers are*

$$(E \oplus F)(x) = E(x) \oplus F(x).$$

(2) *There is a vector bundle $E \otimes F$ on X, whose fibers are*

$$(E \otimes F)(x) = E(x) \otimes_{\mathbb{K}} F(x).$$

(3) *There is a vector bundle* $\operatorname{Hom}(E, F)$ *on X, whose fibers are*

$$\operatorname{Hom}(E, F)(x) = \operatorname{Hom}_{\mathbb{K}}(E(x), F(x)).$$

(4) *There is a vector bundle $\bigwedge^k(E)$ on X, whose fibers are*

$$\textstyle\bigwedge^k(E)(x) = \bigwedge^k_{\mathbb{K}}(E(x)),$$

the k-the exterior power, for $0 \leq k \leq m$.

(5) *There is a vector bundle* $\operatorname{Sym}^k(E)$ *on X, whose fibers are*

$$\operatorname{Sym}^k(E)(x) = \operatorname{Sym}^k_{\mathbb{K}}(E(x)),$$

the k-the symmetric power, for $0 \leq k$.

Exercise 8.33. Prove this proposition. What are the ranks of the various vector bundles there?

As a prelude to next week's lecture, do the following exercise.

Exercise 8.34. Let E be a vector bundle over X, let $U \subseteq X$ be an open set, and let

$$\sigma, \tau : U \to E$$

be sections, i.e.

$$\sigma, \tau \in \mathrm{Hom}_{\mathsf{Sp}/X}(U, E).$$

Then there is a unique section

$$\sigma + \tau \in \mathrm{Hom}_{\mathsf{Sp}/X}(U, E)$$

such that for every point $x \in U$ we have

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x) \in E(x).$$

(Hint: this is easy if you understand the definitions well.)

Lecture 7, 5 Dec 2018

Last time we defined vector bundles. Recall that Sp is either Top, Mfld or Var, and correspondingly the field $\mathbb{K}$ is either $\mathbb{R}$, $\mathbb{R}$ or algebraically closed.

Definition 8.35. Let X be a space in Sp, and let E and F be vector bundles on X. A map of vector bundles

$$\phi : E \to F$$

is a map ϕ in Sp/X that respects the operations add and mult.

We denote by Vec/X the category of vector bundles on X.

We could require ϕ to respect the zero sections, but that is a consequence of the operation mult.

Note that the category Vec/X is deficient: given $\phi : E \to F$, there usually aren't vector bundles E' and F', and maps $\alpha : E' \to E$ and $\beta : F \to F'$, such that in all fibers the sequence of $\mathbb{K}$-modules

$$0 \to E'(x) \xrightarrow{\alpha(x)} E(x) \xrightarrow{\phi(x)} F(x) \xrightarrow{\beta(x)} F'(x) \to 0$$

is exact.

Exercise 8.36. Find an example of the phenomenon above, and explain what goes wrong. (See Proposition 9.14 for another point of view.)

This weakness of Vec/X is an important reason to work with sheaves, as we now do.

9. Ringed Spaces and Sheaves of Modules

We now take a break from vector bundles.

Definition 9.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space X, and $\mathcal{O}_X$ is a sheaf of commutative rings on X.

Every space $X \in \mathsf{Sp}$, see Convention 8.1, is actually a ringed space.

- When $\mathsf{Sp} = \mathsf{Top}$, $\mathcal{O}_X$ is the sheaf of continous $\mathbb{R}$-valued functions.
- When $\mathsf{Sp} = \mathsf{Mfld}$, $\mathcal{O}_X$ is the sheaf of differentiable $\mathbb{R}$-valued functions.
- When $\mathsf{Sp} = \mathsf{Var}$, $\mathcal{O}_X$ is the sheaf of algebraic $\mathbb{K}$-valued functions.

In all these cases the formula for $\mathcal{O}_X$ is this:

$$\Gamma(U, \mathcal{O}_X) = \operatorname{Hom}_{\mathsf{Sp}}(U, \mathbf{A}^1(\mathbb{K})),$$

and the ring structure comes from that of $\mathbf{A}^1(\mathbb{K})$ in (8.11).

Definition 9.2. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules on X*, or an *$\mathcal{O}_X$-module*, is a sheaf of abelian groups $\mathcal{M}$ on X, together with a structure of a $\Gamma(U, \mathcal{O}_X)$-module on $\Gamma(U, \mathcal{M})$ for every open set $U \subseteq X$, such that for every inclusion of open sets $V \subseteq U$ the homomorphism

$$\operatorname{rest}^{\mathcal{M}}_{V/U} : \Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$

is $\Gamma(U, \mathcal{O}_X)$-linear.

Definition 9.3. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules. A homomorphism of $\mathcal{O}_X$-modules from $\mathcal{M}$ to $\mathcal{N}$ is a homomorphism of sheaves of abelian groups

$$\phi : \mathcal{M} \to \mathcal{N}$$

such that for every open set $U \subseteq X$ the homomorphism

$$\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})$$

is $\Gamma(U, \mathcal{O}_X)$-linear.

The category of $\mathcal{O}_X$-modules is denoted by $\mathsf{Mod}\, \mathcal{O}_X$.

There are forgetful functors

$$\mathsf{Mod}\, \mathcal{O}_X \to \mathsf{Mod}\, \mathbb{K}_X \to \mathsf{Mod}\, \mathbb{Z}_X = \mathsf{Ab}_X .$$

Definition 9.4. A sequence of homomorphisms

$$\cdots \to \mathcal{M}^i \xrightarrow{\phi^i} \mathcal{M}^{i+1} \xrightarrow{\phi^{i+1}} \mathcal{M}^{i+1} \to \cdots$$

in $\mathsf{Mod}\, \mathcal{O}_X$ is called *exact* if for every point $x \in X$ the sequence of homomorphisms

$$\cdots \to \mathcal{M}^i_x \xrightarrow{\phi^i_x} \mathcal{M}^{i+1}_x \xrightarrow{\phi^{i+1}_x} \mathcal{M}^{i+1}_x \to \cdots$$

in $\mathsf{Mod}\, \mathcal{O}_{X,x}$ is exact.

Example 9.5. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a homomorphism in $\mathsf{Mod}\, \mathcal{O}_X$.

(1) ϕ is surjective iff

$$\mathcal{M} \xrightarrow{\phi} \mathcal{N} \to 0$$

is exact. Here 0 is the zero sheaf.

(2) ϕ is injective iff

$$0 \to \mathcal{M} \xrightarrow{\phi} \mathcal{N}$$

is exact.

Exactness can be checked also in $\mathsf{Mod}\, \mathbb{K}_X$ or Ab_X.

In Definitions 7.14 and 7.15 we learned about the cokernel of a sheaf homomorphism in Ab_X. It works also for homomorphisms in $\mathsf{Mod}\, \mathcal{O}_X$. Namely $\operatorname{Coker}(\phi)$ is the sheaf associated to the presheaf

$$\operatorname{Coker}^{\mathrm{pre}} : U \mapsto \operatorname{Coker}\big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\big).$$

Definition 9.6. Given a homomorphisms $\phi : \mathcal{M} \to \mathcal{N}$ in $\mathsf{Mod}\, \mathcal{O}_X$, its *kernel* is the $\mathcal{O}_X$-module $\operatorname{Ker}(\phi)$ such that for every open set U

$$\Gamma(U, \operatorname{Ker}(\phi)) := \operatorname{Ker}\big(\Gamma(U, \phi) : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\big).$$

Exercise 9.7.

(1) Verify that the sheaf $\operatorname{Coker}(\phi)$ is an $\mathcal{O}_X$-module.

(2) Verify that $\operatorname{Ker}(\phi)$ is a sheaf, and it is an $\mathcal{O}_X$-module.

Proposition 9.8. *Given a homomorphism $\phi : \mathcal{M} \to \mathcal{N}$ in $\mathsf{Mod}\, \mathcal{O}_X$, the sequence*

$$0 \to \operatorname{Ker}(\phi) \to \mathcal{M} \xrightarrow{\phi} \mathcal{N} \to \operatorname{Coker}(\phi) \to 0$$

in $\mathsf{Mod}\, \mathcal{O}_X$ is exact.

Proof. By definition for every open set U we have an exact sequence of abelian groups

$$(\mathbf{E}_U) \qquad 0 \to \Gamma(U, \operatorname{Ker}(\phi)) \to \Gamma(U, \mathcal{M}) \xrightarrow{\Gamma(U,\phi)} \Gamma(U, \mathcal{N}) \to \Gamma(U, \operatorname{Coker}^{\mathrm{pre}}(\phi)) \to 0.$$

Choose some point $x \in X$. Passing to the direct limit on all exact sequences $\mathbf{E}_U$, where U is an open neighborhood of x, we get an exact sequence

$$(\mathbf{E}_x) \qquad 0 \to \operatorname{Ker}(\phi)_x \to \mathcal{M}_x \xrightarrow{\phi_x} \mathcal{N}_x \to \operatorname{Coker}(\phi)_x \to 0.$$

Here we are using the fact that the direct limit of a direct system of exact sequences is exact (see below), and that

$$\mathrm{Coker}^{\mathrm{pre}}(\phi)_x \to \mathrm{Coker}(\phi)_x$$

is bijective. □

Theorem 9.9. *Let $\{\mathbf{E}_i\}_{i \in I}$ be a direct system of sequences of A-modules, and let*

$$\mathbf{E} := \lim_{i \to} \mathbf{E}_i .$$

If all the sequences $\mathbf{E}_i$ are exact, then so is $\mathbf{E}$.

Exercise 9.10. Prove this theorem. (Unless you already know it.)

Exercise 9.11. State and prove the categorical properties of Ker and Coker inside $\mathsf{Mod}\, \mathcal{O}_X$. (Cf. Prop 7.16(3).)

For an open set $U \subseteq X$ there is a functor

$$\mathrm{Rest}_{U/X} : \mathsf{Mod}\, \mathcal{O}_X \to \mathsf{Mod}\, \mathcal{O}_X|_U , \quad \mathcal{M} \mapsto \mathcal{M}|_U .$$

Definition 9.12. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules $\mathcal{E}$ on X is called *locally free of rank n*, for some $n \in \mathbb{N}$, if there is an open covering $\boldsymbol{U} = \{U_i\}_{i \in I}$ of X, and for every i there is an isomorphism

$$\phi_i : \mathcal{O}_{U_i}^{\oplus n} \xrightarrow{\simeq} \mathcal{E}|_{U_i}$$

of $\mathcal{O}_{U_i}$-modules. We say that the covering $\boldsymbol{U}$ *trivializes* $\mathcal{E}$, and that the collection $\{\phi_i\}_{i \in I}$ is a *trivialization* of $\mathcal{E}$ on $\boldsymbol{U}$.

Just like in module categories, an exact sequence

$$0 \to \mathcal{L} \xrightarrow{\phi} \mathcal{M} \xrightarrow{\psi} \mathcal{N} \to 0$$

in $\mathsf{Mod}\, \mathcal{O}_X$ is called split if there is a homomorphism $\tau : \mathcal{N} \to \mathcal{M}$ such that $\psi \circ \tau = \mathrm{id}_{\mathcal{N}}$.

Exercise 9.13. Suppose

$$(\heartsuit) \qquad 0 \to \mathcal{L} \xrightarrow{\phi} \mathcal{M} \xrightarrow{\psi} \mathcal{N} \to 0$$

is a split exact sequence in $\mathsf{Mod}\, \mathcal{O}_X$. Show that

$$\mathcal{M} \cong \mathcal{L} \oplus \mathcal{N} .$$

Proposition 9.14. *Let*

$$(\mathbf{E}) \qquad 0 \to \mathcal{L} \xrightarrow{\phi} \mathcal{M} \xrightarrow{\psi} \mathcal{N} \to 0$$

be an exact sequence in $\mathsf{Mod}\, \mathcal{O}_X$. Assume that $\mathcal{N}$ is locally free of finite rank. Then $(\dagger)$ is locally split; namely every point $x \in X$ has an open neighborhood U such that the sequence of $\mathcal{O}_U$-modules

$$0 \to \mathcal{L}|_U \xrightarrow{\phi} \mathcal{M}|_U \xrightarrow{\psi} \mathcal{N}|_U \to 0$$

is split.

Exercise 9.15. Prove this proposition.

We now return to vector bundles.

Let X be a space and (Y, π_Y) an X-space. For an open set $U \subseteq X$ let us write

$$(9.16) \qquad \Gamma(U, Y) := \mathrm{Hom}_{\mathsf{Sp}/X}(U, Y),$$

the set of sections $\sigma : U \to Y$ in Sp/X of $\pi_Y : Y \to X$ over U.

Lemma 9.17. *The assignment*

$$U \mapsto \Gamma(U, Y)$$

is a sheaf of sets on X. It is called the sheaf of sections of Y, and is denoted by $\mathrm{ShSe}_X(Y)$.

Exercise 9.18. Prove the lemma.

Proposition 9.19. *Let $X \in \mathsf{Sp}$. There is a canonical isomorphism of sheaves of rings between $\mathcal{O}_X$ and the sheaf* $\mathrm{ShSe}_X(\mathbf{A}^1(X))$ *of sections of the ring bundle* $\mathbf{A}^1(X)$.

Proof. Recall that

$$\mathbf{A}^1(X) = X \times \mathbf{A}^1(\mathbb{K}).$$

As we saw in Lemma 8.16(1), for an open set $U \subseteq X$ there is an isomorphism

$$\Gamma(U, \mathbf{A}^1(X)) \cong \mathrm{Hom}_{\mathsf{Sp}}(U, \mathbf{A}^1(\mathbb{K})).$$

And by definition

$$\Gamma(U, \mathcal{O}_X) = \mathrm{Hom}_{\mathsf{Sp}}(U, \mathbf{A}^1(\mathbb{K})).$$

We see that $\mathcal{O}_X$ is the sheaf of sections of the bundle $\mathbf{A}^1(X)$. The ring structure in both cases comes from that of $\mathbf{A}^1(\mathbb{K})$. ◻

Proposition 9.20. *Let $X \in \mathsf{Sp}$, and let (E, π_E) be a rank n vector bundle on X. Let $\mathcal{E} :=$ $\mathrm{ShSe}_X(E)$, the sheaf of sections of E. Then $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank n.*

Proof. The operations on E give $\mathcal{E}$ a structure of an $\mathcal{O}_X$-module. If $\boldsymbol{U} = \{U_i\}_{i \in I}$ is an open covering that trivializes E, and $\{\phi_i\}_{i \in I}$ is a trivialization, then this same data trivializes $\mathcal{E}$, so $\mathcal{E}$ is locally free of rank n. ◻

Exercise 9.21. Let E be a vector bundle on X in Sp, with associated locally free sheaf $\mathcal{E}$. Show that for every point $x \in X$ there is a canonical isomorphism of $\mathbb{K}$-modules

$$E(x) \cong \mathbb{K} \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x.$$

Here $E(x)$ is the fiber at x, and $\mathcal{E}_x$ is the stalk at x, which is a module over the ring $\mathcal{O}_{X,x}$.

Definition 9.22. Let $(X, \mathcal{O}_X)$ be a ringed space. The full subcategory of $\mathsf{Mod}\,\mathcal{O}_X$ on the finite rank locally free sheaves is denoted by $\mathsf{LFMod}\,\mathcal{O}_X$.

Theorem 9.23. *Let $X \in \mathsf{Sp}$. The assignment*

$$\mathrm{ShSe}_X : \mathsf{Vec}/X \to \mathsf{LFMod}\,\mathcal{O}_X$$

is an equivalence of categories.

The proof will be given next week.
For the proof we will need the next general result.

Theorem 9.24. *The following are equivalent for a functor $F : \mathsf{C} \to \mathsf{D}$.*

(i) *F is an equivalence.*
(ii) *F is fully faithful, and essentially surjective on objects.*

Exercise 9.25. Prove the theorem. (Hint: use the axiom of choice to construct a quasi-inverse of F.)

Lecture 8, 12 Dec 2018

Today's lecture turned out to be dedicated to reviewing earlier material and expanding it.

Our first topic is solving Exercise 9.7 For this we introduce the category $\mathsf{Mod}^{\mathrm{pre}}\,\mathcal{O}_X$ of presheaves of $\mathcal{O}_X$-modules. It is defined just like in Definitions 9.2 and 9.3, except the objects are presheaves.

Lemma 9.26. *Given $\mathcal{M} \in \mathsf{Mod}^{\mathrm{pre}}\,\mathcal{O}_X$, the sheaf of abelian groups* $\mathrm{Sh}(\mathcal{M})$ *has a unique structure of $\mathcal{O}_X$-modules such that $\tau_{\mathcal{M}} : \mathcal{M} \to \mathrm{Sh}(\mathcal{M})$ is $\mathcal{O}_X$-linear.*

Proof. Take an open set $U \subseteq X$. An element $f \in \Gamma(U, \mathcal{O}_X)$ induces, by multiplication, a homomorphism of presehaves

$$f \cdot (-) : \mathcal{M}|_U \to \mathcal{M}|_U.$$

This is viewed as a morphism in $\mathsf{Ab}_U^{\mathrm{pre}}$. Applying the functor Sh we get a morphism

$$f \cdot (-) : \mathrm{Sh}(\mathcal{M}|_U) \to \mathrm{Sh}(\mathcal{M}|_U)$$

in Ab_U. But $\mathrm{Sh}(\mathcal{M}|_U) = \mathrm{Sh}(\mathcal{M})|_U$, and so we obtain a homomorphism of abelian groups

$$f \cdot (-) : \Gamma(U, \mathrm{Sh}(\mathcal{M})) \to \Gamma(U, \mathrm{Sh}(\mathcal{M})).$$

A little checking shows that this gives $\Gamma(U, \mathrm{Sh}(\mathcal{M}))$ a structure of a $\Gamma(U, \mathcal{O}_X)$-module. As U changes we get an $\mathcal{O}_X$-module structure on $\mathrm{Sh}(\mathcal{M})$. And it respects $\tau_{\mathcal{M}}$.

The uniqueness can be checked in stalks. □

Lemma 9.27. *There is a functor*

$$\mathrm{Sh} : \mathsf{Mod}^{\mathrm{pre}}\,\mathcal{O}_X \to \mathsf{Mod}\,\mathcal{O}_X$$

that respects the forgetful functor to Ab_X.

This is an immediate consequence of the previous lemma.

Now back to our exercise. We are given a homomorphism $\phi : \mathcal{M} \to \mathcal{N}$ in $\mathsf{Mod}\,\mathcal{O}_X$. On every open set U we get an exact sequence

$$0 \to \Gamma(U, \mathrm{Ker}^{\mathrm{pre}}(\phi)) \to \Gamma(U, \mathcal{M}) \xrightarrow{\phi} \Gamma(U, \mathcal{N}) \to \Gamma(U, \mathrm{Coker}^{\mathrm{pre}}(\phi)) \to 0.$$

This makes both $\Gamma(U, \mathrm{Ker}^{\mathrm{pre}}(\phi))$ and $\Gamma(U, \mathrm{Coker}^{\mathrm{pre}}(\phi))$ into $\Gamma(U, \mathcal{O}_X)$-modules. As U varies we get

$$\mathrm{Ker}^{\mathrm{pre}}(\phi), \mathrm{Coker}^{\mathrm{pre}}(\phi)) \in \mathsf{Mod}^{\mathrm{pre}}\,\mathcal{O}_X.$$

But

$$\mathrm{Ker}(\phi) = \mathrm{Ker}^{\mathrm{pre}}(\phi),$$

so it is in $\mathsf{Mod}\,\mathcal{O}_X$. By Lemma 9.26 we know that

$$\mathrm{Ker}(\phi) \in \mathsf{Mod}\,\mathcal{O}_X.$$

This concludes Exercise 9.7.

– – –

Next we talked about Exercise 9.21. Here I forgot to tell you what is the $\mathbb{K}$-ring homomorphism $\mathcal{O}_{X,x} \to \mathbb{K}$. It comes from the next proposition.

Proposition 9.28. *Let $X \in \mathsf{Sp}$ and $x \in X$.*

(1) *There is a unique* $\mathbb{K}$*-ring homomorphism*

$$\mathrm{ev}_x : \mathcal{O}_{X,x} \to \mathbb{K}$$

such that for every open neighborhood U *of* x *and every function* $f \in \Gamma(U, \mathcal{O}_X)$, *with germ* $f_x \in \mathcal{O}_{X,x}$, *there is equality*

$$\mathrm{ev}_x(f_x) = f(x) \in \mathbb{K}.$$

(2) *Let* $\mathfrak{m}_x := \mathrm{Ker}(\mathrm{ev}_x)$. *Then* $\mathfrak{m}_x$ *is the only maximal ideal of the ring* $\mathcal{O}_{X,x}$, *and hence this is a local ring.*

We sometimes write

$$\boldsymbol{k}(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x,$$

and call it the *residue field* of x. In our three geometries we always have $\boldsymbol{k}(x) = \mathbb{K}$; but later, in schemes, this will usually not be the case.

Exercise 9.29. Prove this proposition.

Now I can again give you:

Exercise 9.30. Let $X \in \mathsf{Sp}$, and let E be a vector bundle on X, with associated locally free sheaf $\mathcal{E}$. Show that for every point $x \in X$ there is a canonical isomorphism of $\mathbb{K}$-modules

$$E(x) \cong \mathbb{K} \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x.$$

Here $E(x)$ is the fiber at x, and $\mathcal{E}_x$ is the stalk at x, which is a module over the ring $\mathcal{O}_{X,x}$. (I talked about this in class, but please write it in full.)

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The last topic is about Exer 8.36. Here is a possible solution (perhaps the simplest one). It is like what Yotam and Hezi wrote, but without the irrelevant stuff (the two extra components).

Take $X := \mathbf{A}^1(\mathbb{K}) \in \mathsf{Sp}$ and

$$E = F := \mathbf{A}^1(X) = X \times \mathbf{A}^1(\mathbb{K}) = \mathbf{A}^1(\mathbb{K}) \times \mathbf{A}^1(\mathbb{K}).$$

A point in E is a pair (x, e), where $x, e \in \mathbb{K}$. The projection is $\pi_E : E \to X$, $\pi_E(x, e) = x$. Likewise for F. The vector bundle map we choose is

$$\phi : E \to F, \quad \phi(x, e) := (x, x \cdot e).$$

If $x = 0$ then in the fiber the homomorphism $E(x) \to F(x)$ is 0. If $x \neq 0$ then the homomorphism $E(x) \to F(x)$ is bijective. We see that the kernels and cokernels have different ranks as x changes (either 0 or 1). If there were vector bundles E' and F', with maps $\alpha : E' \to E$ and $\beta : F \to F'$, such that the sequence

$$0 \to E'(x) \xrightarrow{\alpha} E(x) \xrightarrow{\phi} F(x) \xrightarrow{\beta} F'(x) \to 0$$

is exact for every x, then the ranks would be constant.

Let's now be smarter, and analyze this situation using sheaves. The sheaves of sections of E and F are $\mathcal{E}$ and $\mathcal{F}$ respectively. The homomorphism $\phi : \mathcal{E} \to \mathcal{F}$ in $\mathsf{Mod}\,\mathcal{O}_X$ has a kernel and a cokernel, say $\mathcal{K} := \mathrm{Ker}(\phi)$ and $\mathcal{C} := \mathrm{Coker}(\phi)$. They sit in this exact sequence

$$0 \to \mathcal{K} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\beta} \mathcal{C} \to 0 \tag{9.31}$$

in $\mathsf{Mod}\,\mathcal{O}_X$. We can calculate these sheaves.

Let us introduce the coordinate function

$$t : \mathbf{A}^1(\mathbb{K}) = X \to \mathbb{K},$$

which is nothing but the identity of $\mathbb{K}$ is disguise, i.e. $t(x) = x$. But it will be good to have notation for it, as a global section of $\mathcal{O}_X$. Indeed, as a homomorphism of $\mathcal{O}_X$-modules,

$$\phi : \mathcal{E} = \mathcal{O}_X \to \mathcal{F} = \mathcal{O}_X$$

is multiplication by t.

Claim 1. The homomorphism ϕ is injective (so $\mathcal{K} = 0$). Let's see why. Take a point $x \in X$. To prove that

$$\phi_x : \mathcal{E}_x = \mathcal{O}_{X,x} \to \mathcal{F}_x = \mathcal{O}_{X,x}$$

is injective is the same as to prove that the germ $t_x \in \mathcal{O}_{X,x}$ is not a zero-divisor. There are two cases. First assume $x \neq z$, where z is the origin (i.e. $t(z) = 0$). In this case t_x is invertible in $\mathcal{O}_{X,x}$, so a fortiori it is not a zero-divisor.

Now take $x = z$. Here $t_z \in \mathfrak{m}_z$, so it is not invertible. Let $f_z \in \mathcal{O}_{X,z}$ be a germ, and assume that $t_z \cdot f_z = 0$ in $\mathcal{O}_{X,z}$. We will show that $f_z = 0$. Now f_z has some representative $f \in \Gamma(U, \mathcal{O}_X)$ on a neighborhood U of z, and $f_z \cdot t_z = 0$ means that on some smaller neighborhood V of z we have $t|_V \cdot f|_V = 0$ in $\Gamma(V, \mathcal{O}_X)$. So $t(y) \cdot f(y) = 0$ for all $y \in V$. But $t(y) \neq 0$ for all $y \in V - \{z\}$, so $f(y) = 0$ for all $y \in V - \{z\}$. By continuity we get $f(z) = 0$ too. So $f = 0$ in $\Gamma(V, \mathcal{O}_X)$, and thus $f_z = 0$ in the stalk.

Claim 2. The sheaf $\mathcal{C}$ is supported on the closed set $Z := \{z\}$, so it is a "skyscraper sheaf". Indeed, by the calculations above for every x there is a short exact sequence

$$0 \to \mathcal{E}_x = \mathcal{O}_{X,x} \xrightarrow{t_x \cdot (-)} \mathcal{F}_x = \mathcal{O}_{X,x} \to \mathcal{C}_x \to 0.$$

So $\mathcal{C}_x = 0$ if $x \neq z$, and

$$\mathcal{C}_z = \mathcal{O}_{X,z}/(t_z) \neq 0.$$

The actual structure of the stalk $\mathcal{C}_x$ is clear only when $\mathsf{Sp} = \mathsf{Var}$. In this case the element t_z generates the maximal ideal $\mathfrak{m}_z$, so $\mathcal{C}_z = \boldsymbol{k}(z) = \mathbb{K}$. But in the other geometries there are rapidly decaying functions in $\mathfrak{m}_z$, like $\exp(-t^{-2})$, that are not divisible by t; and hence $\mathcal{C}_z$ is bigger.

From the two claims we see that for every $x \in X - \{z\}$ there is an exact sequence

$$0 \to \mathcal{E}_x \xrightarrow{t_x \cdot (-)} \mathcal{F}_x \to 0.$$

Passing to the fibers, i.e. tensoring with $\boldsymbol{k}(x) = \mathbb{K}$, we get an exact sequence

$$0 \to \mathcal{E}(x) \xrightarrow{t(x) \cdot (-)} \mathcal{F}(x) \to 0.$$

Just as expected.

But for $x = z$ something interesting happens. We have the short exact sequence of $\mathcal{O}_{X,z}$-modules

$$0 \to \mathcal{E}_z \xrightarrow{t_z \cdot (-)} \mathcal{F}_z \to \mathcal{C}_z \to 0.$$

Now the functor $\boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} (-)$ is right exact, so we get this exact sequence of $\boldsymbol{k}(z)$-modules:

$$\boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{E}_z \xrightarrow{t(z) \cdot (-)} \boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{E}_z \to \boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{C}_z \to 0.$$

To close it up to an exact sequence we use the long exact sequence of the left derived functor Tor, plus the fact that

$$\mathrm{Tor}_1^{\mathcal{O}_{X,z}}(\boldsymbol{k}(z), \mathcal{F}_z) = 0.$$

We obtain this exact sequence of $\boldsymbol{k}(z)$-modules:

(9.32)

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_{X,z}}(\boldsymbol{k}(z), \mathcal{C}_z) \to \boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{E}_z \xrightarrow{t(z) \cdot (-)} \boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{E}_z \to \boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{C}_z \to 0.$$

We need to figure out what are $\mathrm{Tor}_1^{\mathcal{O}_{X,z}}(\boldsymbol{k}(z), \mathcal{C}_z)$ and $\boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{C}_z$.

But we already know that

$$\boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{E}_z \cong \boldsymbol{k}(z)$$

and

$$\boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{F}_z \cong \boldsymbol{k}(z).$$

Also $t(z) = 0$. We conclude that

$$\boldsymbol{k}(z) \otimes_{\mathcal{O}_{X,z}} \mathcal{C}_z \cong \boldsymbol{k}(z)$$

and

$$\operatorname{Tor}_1^{\mathcal{O}_{X,z}}(\boldsymbol{k}(z), \mathcal{C}_z) \cong \boldsymbol{k}(z).$$

Thus (9.32) becomes this exact sequence

$$0 \to \boldsymbol{k}(z) \xrightarrow{\cong} \boldsymbol{k}(z) \xrightarrow{0} \boldsymbol{k}(z) \xrightarrow{\cong} \boldsymbol{k}(z) \to 0.$$

This agrees with what we saw in the more simplistic discussion above, but now we understand where the first $\boldsymbol{k}(z)$ comes from: it is the value of a derived functor.

Lecture 9, 19 Dec 2018

This lecture will start with a couple of general constructions on ringed spaces. After that we'll talk about homework.

Proposition 9.33. *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}\,\mathcal{O}_X$. There is an $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ such that*

$$\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) = \mathrm{Hom}_{\mathsf{Mod}\,\mathcal{O}_U}(\mathcal{M}|_U, \mathcal{N}|_U)$$

for every open set $U \subseteq X$.

Exercise 9.34. Prove Proposition 9.33.

It is convenient, and customary, to write

$$\mathrm{Hom}_X(\mathcal{M}, \mathcal{N}) := \mathrm{Hom}_{\mathsf{Mod}\,\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$$

and

$$\mathcal{H}om_X(\mathcal{M}, \mathcal{N}) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}).$$

Definition 9.35. Let $(X, \mathcal{O}_X)$ be a ringed space. Given $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}\,\mathcal{O}_X$ we let $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ be the $\mathcal{O}_X$-module associated to the presheaf

$$U \mapsto \Gamma(U, \mathcal{M}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{N}).$$

Exercise 9.36. In the setting of the definition above:

(1) Prove that for every point $x \in X$ there is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})_x \cong \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x.$$

(2) Show that is $\mathcal{M}$ and $\mathcal{N}$ are locally free $\mathcal{O}_X$-modules then so is $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$.

Exercise 9.37. Let $(X, \mathcal{O}_X)$ be a ringed space over $\mathbb{K}$, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Prove that $\mathcal{M}$ is a free $\mathcal{O}_X$-module of rank r iff

$$\mathcal{M} \cong \mathcal{O}_X \otimes_{\mathbb{K}_X} M_X,$$

where M is a free $\mathbb{K}$-module of rank r, and M_X is the associated constant sheaf.

The next exercisees were in an email last week.

Exercise 9.38. Find an example of a ringed space $(X, \mathcal{O}_X)$, and a short exact sequence $\mathbf{E}$ in $\mathsf{Mod}\,\mathcal{O}_X$ consisting of locally free sheaves, that is not split.

Exercise 9.39. Prove that when $(X, \mathcal{O}_X)$ is a compact manifold in Mfld, then every short exact sequence of locally free sheaves is split. (Hint: partitions of unity.) (This also works for $(X, \mathcal{O}_X)$ in Top, when X is a compact metric space.)

The next example is a solution of Exer 9.38.

Example 9.40. Here is an example of a short exact sequence

$$(\mathbf{E}) \qquad 0 \to \mathcal{L} \xrightarrow{\phi} \mathcal{M} \xrightarrow{\psi} \mathcal{N} \to 0$$

of locally free sheaves that is not globally split. This is in the setting of $\mathsf{Sp} = \mathsf{Var}$. The example will also take us on a tour into algebraic geometry.

The variety X is the projective line $\mathbf{P}^1(\mathbb{K})$. The homogeneous coordinates are t_0, t_1. Let

$$U_0 := \{t_0 \neq 0\} = \{(1, \lambda) \mid \lambda \in \mathbb{K}\} \subseteq \mathbf{P}^1(\mathbb{K})$$

and

$$U_1 := \{t_1 \neq 0\} = \{(\mu, 1) \mid \mu \in \mathbb{K}\} \subseteq \mathbf{P}^1(\mathbb{K}).$$

These are affine open sets, and

$$U_0 \cup U_1 = \mathbf{P}^1(\mathbb{K}).$$

Consider the rational function

$$t := t_1/t_0.$$

It has a zero of order 1 at the origin

$$x_0 = (1,0) \in U_0$$

and a pole of order 1 at infinity

$$x_\infty = (0,1) \in U_1.$$

At all other points t does not have a zero or a pole.

The rings of functions on the important affine open sets are

$$\Gamma(U_0, \mathcal{O}_X) = \mathbb{K}[t],$$
$$\Gamma(U_1, \mathcal{O}_X) = \mathbb{K}[t^{-1}],$$

and

$$\Gamma(U_0 \cap U_1, \mathcal{O}_X) = \mathbb{K}[t, t^{-1}].$$

The function field of X is the field

$$\boldsymbol{k}(X) = \mathbb{K}(t).$$

For every nonempty affine open set $U \subseteq X$ the field $\boldsymbol{k}(X)$ is the field of fractions of the integral domain $\Gamma(U, \mathcal{O}_X)$.

Let $\boldsymbol{k}(X)^\times$ be the set of nonzero elements of $\boldsymbol{k}(X)$, i.e. its multiplicative group. For a rational function $f \in \boldsymbol{k}(X)^\times$ and a point $x \in X$ we write

(9.41) $$\operatorname{ord}_x(f) := \text{order of vanishing of the function } f \text{ at the point } x\,.$$

Thus $\operatorname{ord}_x(f) > 0$ if f has a zero at x, and $\operatorname{ord}_x(f) < 0$ if f has a pole at x.

Note that

(9.42) $$\mathcal{O}_{X,x} = \big\{ f \in \boldsymbol{k}(X)^\times \mid \operatorname{ord}_x(f) \geq 0 \big\} \cup \{0\} \subseteq \boldsymbol{k}(X)$$

and

(9.43) $$\mathfrak{m}_x = \big\{ f \in \boldsymbol{k}(X)^\times \mid \operatorname{ord}_x(f) > 0 \big\} \cup \{0\} \subseteq \boldsymbol{k}(X).$$

We can consider $\boldsymbol{k}(X)$ as a constant sheaf of rings on X, and then

$$\mathcal{O}_X \subseteq \boldsymbol{k}(X).$$

This inclusion allows the following description: every nonempty open set $U \subseteq X$ we have

(9.44) $$\begin{aligned} \Gamma(U, \mathcal{O}_X) &= \big\{ f \in \boldsymbol{k}(X)^\times \mid f \text{ has no poles in } U \big\} \cup \{0\} \\ &= \bigcap_{x \in U} \mathcal{O}_{X,x}. \end{aligned}$$

For a nonzero rational function f and a subset $U \subseteq X$ we define the *divisor* to be the element

(9.45) $$\operatorname{div}_U(f) := \sum_{x \in U} \operatorname{ord}_x(f) \cdot x \in \bigoplus_{x \in U} \mathbb{Z} \cdot x = \mathrm{F}_{\mathrm{fin}}(U, \mathbb{Z}).$$

In the free $\mathbb{Z}$-module $\mathrm{F}_{\mathrm{fin}}(U, \mathbb{Z})$ we write $d \geq e$ if $d(x) \geq e(x)$ for all $x \in U$. Thus (9.44) can be rewriten as

$$\Gamma(U, \mathcal{O}_X) = \big\{ f \in \boldsymbol{k}(X)^\times \mid \operatorname{div}_U(f) \geq 0 \big\} \cup \{0\}.$$

For $V \subseteq U$ we have the restriction homomorphism:

$$\mathrm{F}_{\mathrm{fin}}(U, \mathbb{Z}) \to \mathrm{F}_{\mathrm{fin}}(V, \mathbb{Z}), \ e \mapsto e|_V\,.$$

For every $j \in \mathbb{Z}$ we define the subsheaf $\mathcal{O}(j) \subseteq \boldsymbol{k}(X)$ as follows:

$$\Gamma(U, \mathcal{O}(j)) := \left\{ f \in \boldsymbol{k}(X)^{\times} \mid \operatorname{div}_U(f) \geq (-j \cdot x_{\infty})|_U \right\} \cup \{0\}. \tag{9.46}$$

This means that a nonzero rational function f belongs to $\Gamma(U, \mathcal{O}(j))$ if these conditions hold:

- f has no poles in $U - \{x_{\infty}\}$.
- If $x_{\infty} \in U$, then f must vanish to order $\geq -j$ at x_{∞}. I.e. a pole of order at most j if $j \geq 0$, and a zero of order at least $-j$ if $j < 0$.

This is a locally free $\mathcal{O}_X$-module. In fact,

$$\mathcal{O}(j)|_{U_0} = \mathcal{O}_X|_{U_0} \subseteq \boldsymbol{k}(X) \tag{9.47}$$

and

$$\mathcal{O}(j)|_{U_1} = \mathcal{O}_X|_{U_1} \cdot t^j \subseteq \boldsymbol{k}(X). \tag{9.48}$$

It is clear that there are inclusions of $\mathcal{O}_X$-modules

$$\mathcal{O}(j) \subseteq \mathcal{O}(j+1) \subseteq \cdots \boldsymbol{k}(X).$$

This means that multiplication by the rational function $1 \in \boldsymbol{k}(X)$ belongs to the $\mathbb{K}$-module

$$\operatorname{Hom}_X(\mathcal{O}(j), \mathcal{O}(j+1)).$$

In fact:

$$\operatorname{Hom}_X(\mathcal{O}(j), \mathcal{O}(j+1)) = (\mathbb{K} \cdot 1) \oplus (\mathbb{K} \cdot t). \tag{9.49}$$

We can finally exhibit the exact sequence $\mathbf{E}$ in $\mathsf{Mod}\, \mathcal{O}_X$. It is this:

$$0 \to \mathcal{O}_X(-2) \xrightarrow{\left[\begin{smallmatrix} -1 \\ t \end{smallmatrix}\right] \cdot (-)} \mathcal{O}_X(-1)^{\oplus 2} \xrightarrow{[\, t \;\; 1\,] \cdot (-)} \mathcal{O}_X \to 0. \tag{$\mathbf{E}$}$$

Here we view the direct sums as column modules, and they are acted upon from the left by matrices of morphisms, using formula (9.49).

Exercise 9.50. This exercise is an elaboration of the last example.

(1) Explain how $X = \mathbf{P}^1(\mathbb{K})$ is the gluing of the ringed spaces $(U_0, \mathcal{O}_{U_0})$ and $(U_1, \mathcal{O}_{U_1})$.
(2) Prove formula (9.44).
(3) Prove that (9.46) defines an $\mathcal{O}_X$-submodule of $\boldsymbol{k}(X)$.
(4) Prove (9.47) and (9.48). Exhibit the gluing isomorphism for $\mathcal{O}(j)$ on $U_0 \cap U_1$.
(5) Find a $\mathbb{K}$-basis for $\Gamma(X, \mathcal{O}(j))$. (Hint: you should get

$$\operatorname{rank}_{\mathbb{K}}(\Gamma(X, \mathcal{O}(j))) = j + 1$$

for $j \geq 0$, and

$$\operatorname{rank}_{\mathbb{K}}(\Gamma(X, \mathcal{O}(j))) = 0$$

for $j < 0$.)
(6) Verify formulas (9.47), (9.48) and (9.49).
(7) Show that

$$\mathcal{O}(j) \otimes_{\mathcal{O}_X} \mathcal{O}(k) \cong \mathcal{O}(j+k)$$

and

$$\mathcal{H}om_X(\mathcal{O}(j), \mathcal{O}(k)) \cong \mathcal{O}(k-j).$$

(8) Verify that $\mathbf{E}$ is indeed an exact sequence. Do this by writing out $\mathbf{E}|_{U_i}$, for $i = 0, 1$, explicity as a sequence of free modules over the ring $\Gamma(U_i, \mathcal{O}_X)$, and show that $\mathbf{E}|_{U_i}$ is split exact.
(9) Use (5) to prove that $\mathbf{E}$ is not split in $\mathsf{Mod}\, \mathcal{O}_X$.

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Here is a complement on derivations, a follow up to Rem 8.31.

Example 9.51. Take $\mathsf{Sp} = \mathsf{Mfld}$. In Rem 8.31 we said that for a manifold X, the sheaf of sections of the tangent bundle $\mathrm{T}X$ is the sheaf $\mathcal{T}_X$ of derivations of $\mathcal{O}_X$.

When considered as a sheaf of $\mathbb{R}_X$-modules, the sheaf $\mathcal{T}_X$ has a Lie algebra structure. Namely for every open set U the $\mathbb{R}$-module $\Gamma(U, \mathcal{T}_X)$ is a Lie algebra, and the Lie brackets respect restriction to open sets. The formula is quite easy – see Exer 9.52 below.

Suppose G is a group object in Mfld. I.e. G is a *Lie group*. The identity element is $e \in G$. The tanget space $\mathrm{T}_e G$ is called the *Lie algebra* of G, with notation $\mathfrak{g}$. Let's see how $\mathfrak{g}$ gets a Lie algebra structure.

The tangent sheaf $\mathcal{T}_G$ is trivial. In fact there are two (in general distinct) trivializations of it. The group G acts on $\Gamma(G, \mathcal{T}_G)$ by left translations, and we denote by $\Gamma(G, \mathcal{T}_G)^G$ the $\mathbb{R}$-module of invariant sections, namely the left invariant vector fields on G. The canonical homomorphism

$$\Gamma(G, \mathcal{T}_G) \to \mathcal{T}_{G,e} \to \mathbb{R} \otimes_{\mathcal{O}_{G,e}} \mathcal{T}_{G,e} \cong \mathrm{T}_e G = \mathfrak{g}$$

induces a bijection

$$\Gamma(G, \mathcal{T}_G)^G \xrightarrow{\simeq} \mathfrak{g}.$$

In this way we get an embedding

$$\mathfrak{g} \rightarrowtail \Gamma(G, \mathcal{T}_G)$$

that can be viewed as an embedding of sheaves

$$\mathfrak{g}_G \rightarrowtail \mathcal{T}_G$$

where $\mathfrak{g}_G$ is the constant sheaf $\mathfrak{g}$ on G. It turns out that the induced $\mathcal{O}_G$-module homomorphism

$$\mathcal{O}_G \otimes_{\mathbb{R}} \mathfrak{g}_G \to \mathcal{T}_G$$

is an isomorphism. This shows that $\mathcal{T}_G$ is a free $\mathcal{O}_G$-module (see Exer 9.37). A calculation shows that the subsheaf $\mathfrak{g}_G$ of $\mathcal{T}_G$ is closed under the Lie bracket. Under the canonical isomorphism

$$\mathfrak{g} \cong (\mathfrak{g}_G)_e$$

there is a Lie algebra structure on $\mathfrak{g}$.

The other option is to look at right invariant vector fields on G.

Exercise 9.52. Now $(X, \mathcal{O}_X)$ is a ringed space over $\mathbb{K}$, for some commutative ring $\mathbb{K}$. By this we mean that $\mathcal{O}_X$ is a sheaf of $\mathbb{K}$-rings, or in other words, there is a given ring homomorphism $\mathbb{K}_X \to \mathcal{O}_X$. Consider the sheaf of rings $\mathcal{E}nd_{\mathbb{K}_X}(\mathcal{O}_X)$.

(1) Suppose

$$\phi, \psi \in \Gamma(U, \mathcal{E}nd_{\mathbb{K}_X}(\mathcal{O}_X))$$

are derivations of $\mathcal{O}_U = \mathcal{O}_X|_U$, for some open set U. Show that

$$[\phi, \psi] := \phi \circ \psi - \psi \circ \phi$$

is also a derivation of $\mathcal{O}_U$.

(2) Conclude that $\mathcal{D}er_{\mathbb{K}}(\mathcal{O}_X)$ is a sheaf of Lie algebras on X.

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