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Prop Let $F: \text{Mod } A \rightarrow \text{Mod } A$ be an A -bim. Funct.

TFAC: (i) F is exact
(ii) F preserves qu-isoms.

[will not prove this - not needed]

Example F not exact, $0 \rightarrow M^0 \xrightarrow{\psi} M^1 \xrightarrow{\psi} M^2 \rightarrow 0$

and ex. seq. s.t. $0 \rightarrow F(M^0) \xrightarrow{F(\psi)} F(M^1) \xrightarrow{F(\psi)} F(M^2) \rightarrow 0$

not exact, say at $F(M^i)$. Letting $d^0 := 0$ and

skip $d^i := \psi$, we get a complex (M, d) , and

$H(M) = 0$. Consider the zero complex $N = \bigoplus N^i$,

$N^i = 0$. Then the zero map $\psi: N \rightarrow N$ is

a qu-isom. Now $F(N)$ is also the zero complex,

so $H(F(N)) = 0$. But $H^i(F(M)) \neq 0$, because

$F(M)$ not exact at $F(M^i)$. Hence

$$H(F(\psi)): H(F(N)) \rightarrow H(F(M))$$

is not a qu-isom.

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However, additive functors do preserve hom. equ!

Proposition Let $F: \underline{\text{Mod}} A \rightarrow \underline{\text{Mod}} A$ be an A -linear functor. If $\varphi: M \rightarrow N$ is a homotopy equivalence in $\underline{\text{C}}(\underline{\text{Mod}} A)$, then $F(\varphi): F(M) \rightarrow F(N)$ is a homotopy equivalence. Therefore $F(\varphi)$ is a quasi-isomorphism.

proof. Say $\psi: N \rightarrow M$ is a homot. inverse of φ , and $\eta: \varphi \circ \varphi \rightarrow \mathbb{1}_M$, $\zeta: \psi \circ \psi \rightarrow \mathbb{1}_N$ are homotopies. Then $F(\eta): F(\varphi \circ \varphi) \rightarrow F(\mathbb{1}_M)$ is a degree -1 hom., and it satisfies

$$\begin{aligned} d_{F(M)} \circ F(\eta) + F(\eta) \circ d_{F(M)} &= F(d_M) \circ F(\eta) + F(\eta) \circ F(d_M) \\ &= F(d_M \circ \varphi + \varphi \circ d_M) = F(\varphi \circ \varphi - \mathbb{1}_M) \\ &= F(\psi) \circ F(\varphi) - \mathbb{1}_{F(M)}. \end{aligned}$$

We see that $F(\eta): F(\varphi) \circ F(\varphi) \rightarrow \mathbb{1}_{F(M)}$ is a homotopy. Likewise $F(\zeta): F(\psi) \circ F(\psi) \rightarrow \mathbb{1}_{F(N)}$ is a homotopy.

□

(154) Projective Resolutions

A is a comm. ring.

Def Let $M \in \text{Mod } A$. A projective resolution of M is an exact sequence

$$\dots \rightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{\gamma} M \rightarrow 0 \rightarrow \dots$$

of A -modules, such that all the modules P^i are projective.

Given a proj. res. as above, let P be the complex

$$P := (\dots \rightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} 0 \rightarrow \dots).$$

We also view M as a complex, concentrated in degree 0:

$$\tilde{M} = (\dots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \dots).$$

The hom. $\gamma: P^0 \rightarrow M$ becomes a hom. of complexes

$$\tilde{\gamma}: P \rightarrow \tilde{M}.$$

$$\begin{array}{ccccccc} \dots & \rightarrow & P^{-2} & \xrightarrow{d^{-2}} & P^{-1} & \xrightarrow{d^{-1}} & P^0 & \xrightarrow{d^0} & 0 & \rightarrow & \dots \\ & & \circ \downarrow & & \circ \downarrow & & \gamma \downarrow & & \circ \downarrow & & \\ \dots & \rightarrow & 0 & \xrightarrow{\circ} & 0 & \xrightarrow{\circ} & M & \xrightarrow{\circ} & 0 & \rightarrow & \dots \end{array}$$

Indeed, $\gamma \circ d^{-1} = 0 = 0 \circ \circ$

Now all elements of P^0 are cocycles, and

$$H^0(P) = P^0 / d(P^{-1}).$$



The hom.

$H^0(\tilde{\gamma}) : H^0(P) \rightarrow H^0(\tilde{M}) = M$ sends a coh. class $\gamma(P) \in H^0(P)$ to $\gamma(P) \in M$. Since the seq.

$$P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{\gamma} M \rightarrow 0$$

is exact, it follows that $H^0(\tilde{\gamma})$ is an isom. We have proved half of:

Prop. Let $M \in \text{Mod } A$. Let $P \in \underline{C}(\text{Mod } A)$ be a complex s.t. P^i are projective for all i , and $P^i = 0$ for $i > 0$. Let $\gamma : P^0 \rightarrow M$ be an A -lin. hom. TFAE:

(i) The seq.

$$\dots \rightarrow P^{-1} \xrightarrow{d_P^{-1}} P^0 \xrightarrow{\gamma} M \rightarrow 0 \rightarrow \dots$$

is a proj. res. of M .

(ii) Let $\tilde{M} := (\dots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \dots)$ and $\tilde{\gamma} := \{\gamma^i\}$, $\gamma^i = \gamma$, $\gamma^i = 0$ for $i \neq 0$. Then

$$\tilde{\gamma} : P \rightarrow \tilde{M}$$

is a quasi-isom. of complexes.

Exercise: Finish proof; i.e. (ii) \Rightarrow (i).

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Prop. Let M be an A -module. There exists a projective resolution $P \rightarrow M$.

Pr. Actually we will produce a free resolution. Choose a surjection $\gamma: P^0 \rightarrow M$ from some free A -mod. P^0 . Then choose a subj.

$d^{-1}: P^{-1} \rightarrow \text{Ker}(\gamma)$ from some free A -mod. P^{-1} . Proceed recursively. \square

Theorem

Let $\psi: M \rightarrow N$ be a hom. in $\text{Mod } A$, and let $\gamma: P \rightarrow M$, $\zeta: Q \rightarrow N$ be projective resolutions.

(1) There is a hom. of complexes

$$\tilde{\varphi}: P \rightarrow Q$$

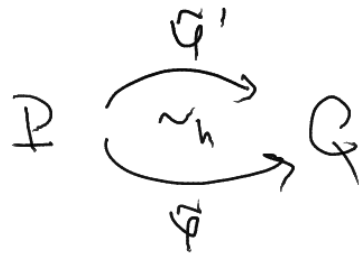
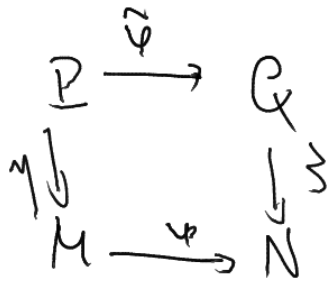
s.t.

$$\zeta \circ \tilde{\varphi} = \psi \circ \gamma.$$

(2) Suppose $\tilde{\varphi}': P \rightarrow Q$ is another hom. of complexes s.t. $\zeta \circ \tilde{\varphi}' = \psi \circ \gamma$.

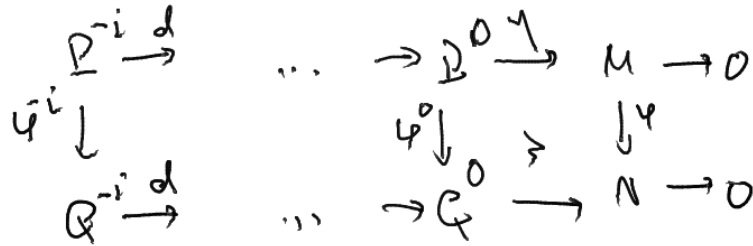
Then $\tilde{\varphi}'$ and $\tilde{\varphi}$ are homotopic.

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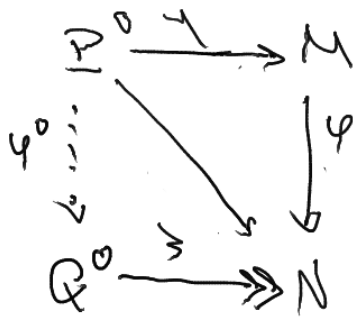


proof. (1) We define maps $\varphi^i: P^i \rightarrow Q^i$ recursively

st. diag's commute:



$i=0$:

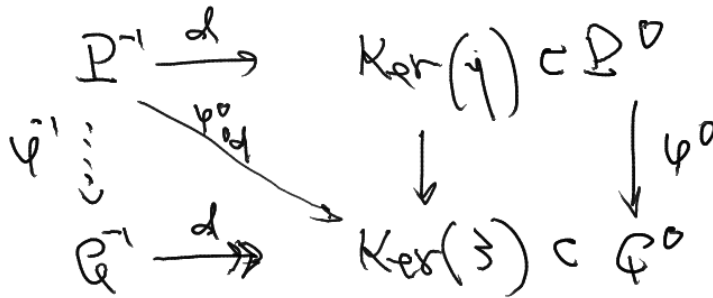


$\varphi_0 \gamma: P^0 \rightarrow N$, P^0 proj.

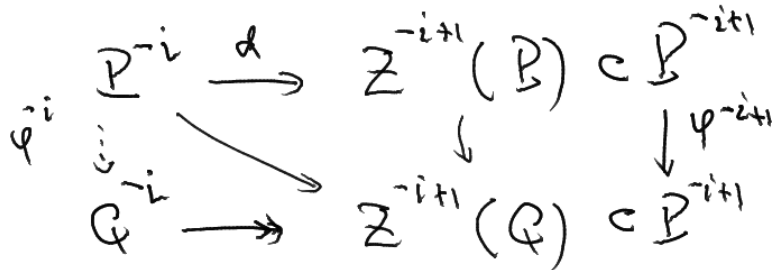
$\exists: Q^0 \rightarrow N$,

$\exists \varphi^0: P^0 \rightarrow Q^0$

$i=1$:

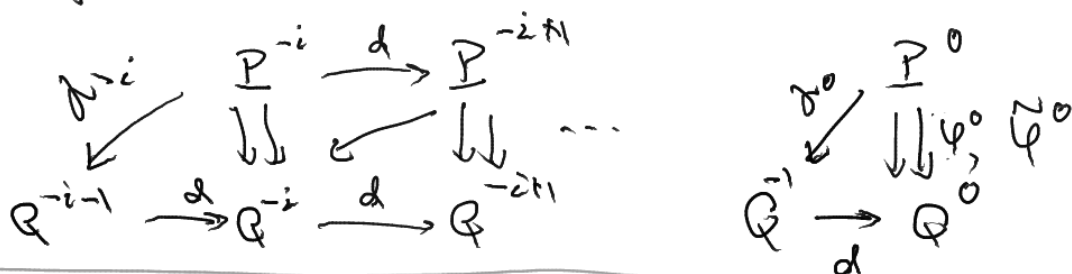


$i \geq 2$:



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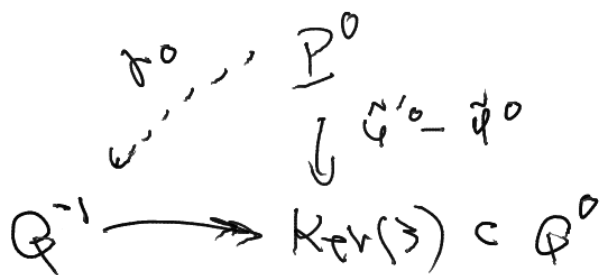
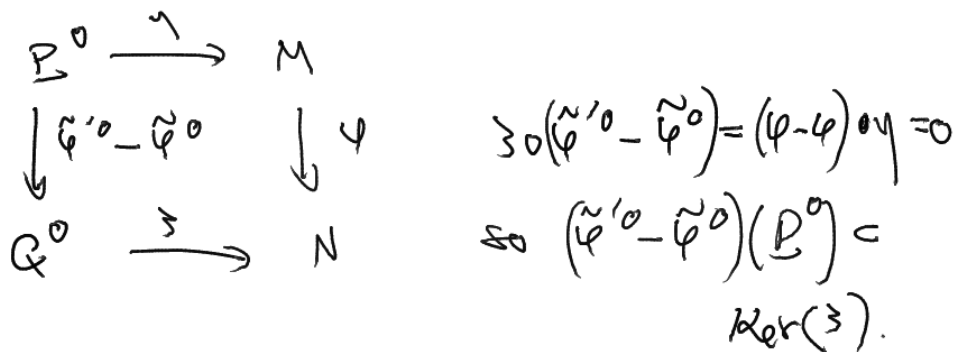
(2) Now we have $\tilde{\varphi}, \tilde{\varphi}': P \rightarrow Q$ that lift $\varphi: M \rightarrow N$. We have to construct a homotopy $\tilde{\gamma}: \tilde{\varphi} \rightarrow \tilde{\varphi}'$. We construct $\tilde{\gamma}^{-i}: P^{-i} \rightarrow Q^{-i-1}$ recursively for $i \geq 0$, s.t.



$\tilde{\gamma}^{-i} - \tilde{\gamma}^{-i} = d \tilde{\gamma}^{-i+1} + \tilde{\gamma}^{-i+1} \circ d : P^{-i} \rightarrow Q^{-i-1}$

For all $j \leq i$. Of course $\tilde{\gamma}^j = 0$ for $j < 0$ (since $P^j = 0$). Trivial holds for $j \leq -1$.

$i=0$:



P^0 proj $\Rightarrow \exists \tilde{\gamma}^0$. $\tilde{\gamma}^{-1} = 0$, so $\tilde{\gamma}$ holds for $j=0$.

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Now do $i=1$:

$$\delta^i = \tilde{\varphi}^i - \tilde{\varphi}^i$$

$$\begin{array}{ccccc} P^{-1} & \xrightarrow{d^{-1}} & P^0 & \rightarrow & 0 \\ \delta^1 \downarrow & \swarrow \gamma^0 & \downarrow \delta^0 & \downarrow & \delta^0 = d^0 \gamma^0 \\ Q^{-1} & \xrightarrow{d^{-1}} & Q^0 & \rightarrow & 0 \end{array} \quad \begin{array}{l} \downarrow \\ d^0 \delta^{-1} = \delta^0 d^{-1} \end{array}$$

$$d^0 (\delta^{-1} - \gamma^0 \circ d^{-1}) = \delta^0 d^{-1} - \delta^0 d^{-1} = 0 \quad : P^{-1} \rightarrow Q^0$$

so get

$$\begin{array}{ccc} & d^{-1} & P^{-1} \\ & \swarrow & \downarrow \delta^{-1} - \gamma^0 \circ d^{-1} \\ Q^{-2} & \xrightarrow{d^{-2}} & Z^{-1}(Q) \subset Q^{-1} \end{array} \quad \begin{array}{l} P^{-1} \text{ proj} \Rightarrow \exists \gamma^{-1} \\ \text{sols:} \end{array}$$

$$d^0 \gamma^{-1} = \delta^{-1} - \gamma^0 \circ d^{-1} \quad \text{is.}$$

$$\gamma^0 \circ d^{-1} + d^{-2} \circ \gamma^{-1} = \tilde{\varphi}^{i-1} - \tilde{\varphi}^{i-1}, \quad \text{eqn. (1) for } i=1.$$

$i \geq 2$: Exercise.

□

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