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Prop Let $F : \underline{\mathrm{Mod}}\, A \to \mathrm{Mod}\, A$ be an A-lin funct.
TFAE: (i) F is exact
(ii) F preserves qu-isoms.

[will not prove this — not needed]

— skip

Example F not exact, $0 \to M^0 \xrightarrow{\varphi} M^1 \xrightarrow{\psi} M^2 \to 0$ and ex. seq. s.t. $0 \to F(M^0) \xrightarrow{F(\varphi)} F(M^1) \xrightarrow{F(\psi)} F(M^2) \to 0$ not exact, say at $F(M^i)$. Letting $d^0 := \varphi$ and $d^1 := \psi$, we get a complex (M, d), and $H(M) = 0$. Consider the zero complex $N = \bigoplus N^i$, $N^i = 0$. Then the zero hom. $\psi : N \to M$ is a qu-isom. Now $F(N)$ is also the zero complex, so $H(F(N)) = 0$. But $H^i(F(M)) \neq 0$, because $F(M)$ not exact at $F(M^i)$. Hence

$$H(F(\psi)) : H(F(N)) \to H(F(M))$$

is not a qu-isom.

However, additive functors do preserve hom. equ!

<u>Proposition</u> Let $F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A$ be an A-linear functor. If $\varphi : M \to N$ is a homotopy equivalence in $\underline{\mathcal{C}}(\underline{\mathrm{Mod}}\, A)$, then $F(\varphi) : F(M) \to F(N)$ is a homotopy equivalence. Therefore $F(\varphi)$ is a quasi-isomorphism.

<u>proof.</u> Say $\psi : N \to M$ is a homot-inverse of φ, and $\eta : \psi \circ \varphi \to \mathbb{1}_M$, $\zeta : \varphi \circ \psi \to \mathbb{1}_N$ are homotopies. Then $F(\eta) : F(M) \to F(M)$ is a degree -1 hom, and it satisfies

$$
\begin{aligned}
d_{F(M)} \circ F(\eta) + F(\eta) \circ d_{F(M)} &= F(d_M) \circ F(\eta) + F(\eta) \circ F(d_M) \\
&= F(d_M \circ \eta + \eta \circ d_M) = F(\psi \circ \varphi - \mathbb{1}_M) \\
&= F(\psi) \circ F(\varphi) - \mathbb{1}_{F(M)} .
\end{aligned}
$$

We see that $F(\eta) : F(\psi) \circ F(\varphi) \to \mathbb{1}_{F(M)}$ is a homotopy. Likewise $F(\zeta) : F(\varphi) \circ F(\psi) \to \mathbb{1}_{F(N)}$ is a homotopy.

$\square$

Projective Resolutions

A is a comm. ring.

Def Let $M \in \underline{\mathrm{Mod}}\, A$. A <u>projective resolution</u> of M is an exact sequence

$$\cdots \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{\eta} M \to 0 \to \cdots$$

of A-modules, such that all the modules P^i are <u>projective</u>.

Given a proj. res. as above, let P be the complex

$$P := \left(\cdots \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{d^{0}} 0 \to \cdots \right).$$

We also view M as a complex concentrated in degree 0: $\hat{\tilde{M}}$

$$\tilde{M} = \left(\cdots \to 0 \to M \to 0 \to \cdots \right).$$

The hom. $\eta : P^0 \to M$ becomes a hom. of complexes

$$\tilde{\eta} : P \to \tilde{M}.$$

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-2} \arrow[r, "d^{-2}"] \arrow[d, "0"'] & P^{-1} \arrow[r, "d^{-1}"] \arrow[d, "0"'] & P^{0} \arrow[r, "d^{0}"] \arrow[d, "\eta"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & M \arrow[r, "0"] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Indeed, $\eta \circ d^{-1} = 0 = 0 \circ 0$

Now all elements of P^0 are cocycles, and $H^0(P) = P^0 / d(P^{-1})$.

The hom.

$$H^0(\tilde{\eta}) : H^0(\underline{P}) \to H^0(\tilde{M}) = M$$ sends a coh. class $[p] \in H^0(\underline{P})$ to $\eta(p) \in M$. Since the seq.

$$\underline{P}^{-1} \xrightarrow{d^{-1}} \underline{P}^0 \xrightarrow{\eta} M \to 0$$

is exact, it follows that $H^0(\tilde{\eta})$ is an isom.

We have proved half of:

<u>Prop</u>. Let $M \in \underline{\mathrm{Mod}}\, A$. Let $\underline{P} \in \underline{C}(\underline{\mathrm{Mod}}\, A)$ be a complex s.t. $\underline{P}^i$ are projective for all i, and $\underline{P}^i = 0$ for $i > 0$. Let $\eta : \underline{P}^0 \to M$ be an A-lin. hom. TFAE:

(i) The seq.

$$\cdots \to \underline{P}^{-1} \xrightarrow{d_{\underline{P}}^{-1}} \underline{P}^0 \xrightarrow{\eta} M \to 0 \to \cdots$$

is a proj. res. of M.

(ii) Let $\tilde{M} := (\cdots \to 0 \to M \to 0 \to \cdots)$ and $\tilde{\eta} := \{\eta^i\}$, $\eta^0 = \eta$, $\eta^i = 0$ for $i \neq 0$. Then

$$\tilde{\eta} : \underline{P} \to \tilde{M}$$

is a quasi-isom. of complexes.

<u>Exercise</u>: Finish proof; i.e. (ii) $\Rightarrow$ (i).

<u>Prop.</u> Let M be an A-module. There exists a projective resolution $\underline{P} \to M$.

<u>Pf</u>. Actually we will produce a <u>free</u> resolution. Choose a surjection $\eta: P^0 \to M$ from some free A-mod. P^0. Then choose a surj.

$$d^{-1}: P^{-1} \to \mathrm{Ker}(\eta) \text{ from some free } A\text{-mod. } P^{-1}.$$

Proceed recursively. $\square$

<u>Theorem.</u>

Let $\varphi: M \to N$ be a hom. in $\underline{\mathrm{Mod}}\, A$, and let $\eta: \underline{P} \to M$, $\zeta: \underline{Q} \to N$ be projective resolutions.

(1) There is a hom. of complexes

$$\tilde{\varphi}: \underline{P} \to \underline{Q}$$

s.t.

$$\zeta \circ \tilde{\varphi} = \varphi \circ \eta.$$

(2) Suppose $\tilde{\varphi}': \underline{P} \to \underline{Q}$ is another hom. of complexes s.t. $\zeta \circ \tilde{\varphi}' = \varphi \circ \eta$.
Then $\tilde{\varphi}'$ and φ are homotopic.

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$$\begin{tikzcd} P \arrow[r, "\tilde{\varphi}"] \arrow[d, "\eta"'] & Q \arrow[d, "\zeta"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, bend left, "\tilde{\varphi}'"] \arrow[r, bend right, "\tilde{\varphi}"'] \arrow[r, phantom, "\sim_h"] & Q \end{tikzcd}$$

<u>proof</u>. (1) We define hom's $\varphi^i : P^i \to Q^i$ recursively

s.t. diag's commute:

$$\begin{tikzcd} P^{-i} \arrow[r, "d"] \arrow[d, "\varphi^i"'] & \cdots \arrow[r] & P^0 \arrow[r, "\eta"] \arrow[d, "\varphi^0"'] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\ Q^{-i} \arrow[r, "d"] & \cdots \arrow[r] & Q^0 \arrow[r, "\zeta"] & N \arrow[r] & 0 \end{tikzcd}$$

$i=0$:

$$\begin{tikzcd} P^0 \arrow[r, "\eta"] \arrow[d, dashed, "\varphi^0"'] \arrow[dr] & M \arrow[d, "\varphi"] \\ Q^0 \arrow[r, two heads, "\zeta"] & N \end{tikzcd}$$

$\varphi \circ \eta : P^0 \to N$, P^0 proj.

$\zeta : Q^0 \twoheadrightarrow N$,

$\Downarrow$

$\exists\ \varphi^0 : P^0 \to Q^0$

$i=1$:

$$\begin{tikzcd} P^{-1} \arrow[r, "d"] \arrow[d, dashed, "\varphi^1"'] \arrow[dr, "\varphi^0 \circ d"] & \mathrm{Ker}(\eta) \subset P^0 \arrow[d] \arrow[d, shift left=8, "\varphi^0"] \\ Q^{-1} \arrow[r, two heads, "d"] & \mathrm{Ker}(\zeta) \subset Q^0 \end{tikzcd}$$

$i \geq 2$:

$$\begin{tikzcd} P^{-i} \arrow[r, "d"] \arrow[d, dashed, "\varphi^i"'] \arrow[dr] & Z^{-i+1}(P) \subset P^{-i+1} \arrow[d] \arrow[d, shift left=10, "\varphi^{-i+1}"] \\ Q^{-i} \arrow[r, two heads] & Z^{-i+1}(Q) \subset P^{-i+1} \end{tikzcd}$$

(2) Now we have $\tilde{\varphi}, \tilde{\varphi}' : \underline{P} \to Q$ that lift $\varphi : M \to N$. We have to construct a homotopy $\gamma : \tilde{\varphi} \to \tilde{\varphi}'$! We construct $\gamma^{-i} : \underline{P}^{-i} \to Q^{-i-1}$ recursively on $i \geq 0$, s.t.

$$\begin{tikzcd}
& \underline{P}^{-i} \arrow[r, "d"] \arrow[dl, "\gamma^{-i}"'] \arrow[d, shift left] \arrow[d, shift right] & \underline{P}^{-i+1} \arrow[dl] \arrow[d, shift left] \arrow[d, shift right] & \cdots & \underline{P}^{0} \arrow[dl, "\gamma^0"'] \arrow[d, shift left, "\tilde{\varphi}'^0,\ \tilde{\varphi}^0"] \arrow[d, shift right] \\
Q^{-i-1} \arrow[r, "d"] & Q^{-i} \arrow[r, "d"] & Q^{-i+1} & Q^{-1} \arrow[r, "d"'] & Q^{0}
\end{tikzcd}$$

(∗) $\boxed{\tilde{\varphi}'^{-j} - \tilde{\varphi}^{-j} = d \circ \gamma^{-j} + \gamma^{-j+1} \circ d : \underline{P}^{-j} \to Q^{-j}}$

for all $\boxed{j \leq i.}$ Of course $\gamma^{-j} = 0$ for $j < 0$ (since $\underline{P}^{-j} = 0$). Triv. holds for $j \leq -1$.

$i = 0$:

$$\begin{tikzcd}
\underline{P}^0 \arrow[r, "\eta"] \arrow[d, "\tilde{\varphi}'^0 - \tilde{\varphi}^0"] & M \arrow[d, "\varphi"] \\
Q^0 \arrow[r, "\varepsilon"] & N
\end{tikzcd}$$

$\varepsilon \circ (\tilde{\varphi}'^0 - \tilde{\varphi}^0) = (\varphi - \varphi) \circ \eta = 0$ so $(\tilde{\varphi}'^0 - \tilde{\varphi}^0)(\underline{P}^0) \subset \operatorname{Ker}(\varepsilon)$.

$$\begin{tikzcd}
& \underline{P}^0 \arrow[dl, dashed, "\gamma^0"'] \arrow[d, "\tilde{\varphi}'^0 - \tilde{\varphi}^0"] \\
Q^{-1} \arrow[r, two heads] & \operatorname{Ker}(\varepsilon) \subset Q^0
\end{tikzcd}$$

$\underline{P}^0$ proj $\Rightarrow \exists\, \gamma^0$. $\gamma^{-1} = 0$, so (∗) holds for $j = 0$.

Now do $i=1$: $\qquad \delta^i := \tilde{\varphi}'^i - \tilde{\varphi}^i$

$$\begin{tikzcd}
P^{-1} \arrow[r, "d^{-1}"] \arrow[d, "\delta^{-1}"'] & P^0 \arrow[r] \arrow[dl, "\gamma^0"'] \arrow[d, "\delta^0"] & 0 \arrow[d, "0"] \\
Q^{-1} \arrow[r, "d^{-1}"] & Q^0 \arrow[r] & 0
\end{tikzcd}$$

$\delta^0 = d^{-1}\circ\gamma^0$

$d^{-1}\circ\delta^{-1} = \delta^0\circ d^{-1}$

$$d^{-1}\circ\left(\delta^{-1} - \gamma^0\circ d^{-1}\right) = \delta^0\circ d^{-1} - \delta^0\circ d^{-1} = 0 \qquad : P^{-1}\to Q^0$$

so get

$$\begin{tikzcd}
 & P^{-1} \arrow[dl, dashed, "\gamma^{-1}"'] \arrow[d, "\delta^{-1} - \gamma^0\circ d^{-1}"] \\
Q^{-2} \arrow[r, two heads, "d^{-2}"'] & Z^{-1}(Q) \subset Q^{-1}
\end{tikzcd}$$

P^{-1} proj $\Rightarrow \exists\, \gamma^{-1}$ sats:

$$d^{-2}\circ\gamma^{-1} = \delta^{-1} - \gamma^0\circ d^{-1} \quad \text{ie.}$$

$$\gamma^0\circ d^{-1} + d^{-2}\circ\gamma^{-1} = \tilde{\varphi}'^{-1} - \tilde{\varphi}^{-1},$$ equ. (†) for $j=1$.

$i\geq 2$: Exercise.

$\square$

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