(103) # Zariski Topology

Let A be a (comm.) ring. The set $\operatorname{Spec} A$ has on it a topology. The closed sets are the sets

$$Z(\mathfrak{a}) := \{ \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subset \mathfrak{p} \},$$

where $\mathfrak{a}$ is any ideal of A. This is the "set of zeroes" of the "functions" $f \in \mathfrak{a}$. It is quite easy to show that the collection $\{Z(\mathfrak{a})\}$ satisfies the axioms of closed sets of a topology. Note that

$$Z(\mathfrak{a}) = \bigcap_{f \in \mathfrak{a}} Z(f), \quad \text{where } Z(f) := \{\mathfrak{p} \mid f \in \mathfrak{p}\}.$$

The complement of $Z(f)$ is

$$D(f) := \{ \mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \}.$$ "principal open set"

Thus any open set U is

$$U = \operatorname{Spec} A - Z(\mathfrak{a}) = \bigcup_{f \in \mathfrak{a}} D(f).$$

We see that the principal open sets are basis of the Zariski topology.

Zeroes in what sense? Any point $\mathfrak{p} \in \operatorname{Spec} A$ has a field associated to it; the residue field $\underline{k}(\mathfrak{p}) = A_{\mathfrak{p}} / \mathfrak{p}_{\mathfrak{p}}$. For $f \in A$ let $f(\mathfrak{p}) \in \underline{k}(\mathfrak{p})$ be the image of f under the can. hom. $A \to \underline{k}(\mathfrak{p})$. Then

$$\mathfrak{p} \in Z(f) \Leftrightarrow f \in \mathfrak{p} \Leftrightarrow f(\mathfrak{p}) = 0.$$

Consider a collection $\{s_i\}_{i\in I}$ of elements of A. It is not hard to show that

$$\bigcup_i D(s_i) = \operatorname{Spec} A \quad \text{iff} \quad \sum_i A\cdot s_i = A.$$

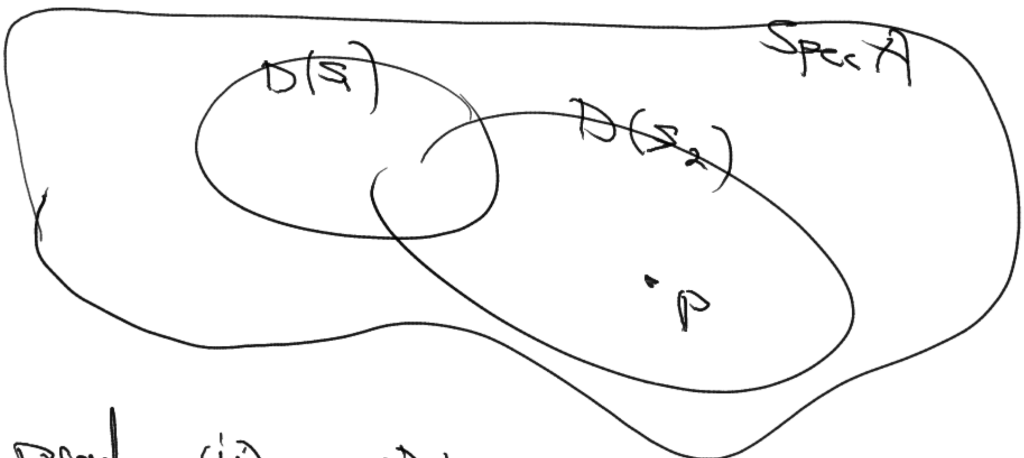
This implies that $\exists\, I_0 \subset I$ finite s.t.

$$\bigcup_{i\in I_0} D(s_i) = \operatorname{Spec} A. \quad \text{(Wait with it)}$$

__Theorem__ Let A be a noetherian ring, and P a fin. gen. A-module. TFAE:

(i) $P_{\mathfrak p}$ is a free $A_{\mathfrak p}$-mod. for all $\mathfrak p \in \operatorname{Spec} A$.

(ii) There is a collection $\{s_i\}$ of elts. of A s.t. $\bigcup D(s_i) = \operatorname{Spec} A$, and P_{s_i} is a free A_{s_i}-mod. for all i.



__Proof__. (ii) $\Rightarrow$ (i):
Take $\mathfrak p \in \operatorname{Spec} A$. $\exists\, i$ s.t. $\mathfrak p \in D(s_i)$. So $s_i \notin \mathfrak p$ and so the hom. $A \to A_{\mathfrak p}$ factors

through A_{s_i}. Therefore

$$P_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A P \cong A_{\mathfrak{p}} \otimes_{A_{s_i}} \underbrace{(A_{s_i} \otimes_A P)}_{\text{free}}$$

is free.

(i) $\Rightarrow$ (ii): For any $\mathfrak{p}$ we will find s s.t. $\mathfrak{p} \in D(s)$ and $A_s \otimes_A P$ is free.

Since $P_{\mathfrak{p}}$ is free, it has a basis $\left(\frac{p_1}{s_1}, \dots \frac{p_n}{s_n}\right)$ over $A_{\mathfrak{p}}$. (Here $s_i \in A - \mathfrak{p}$). But then $\left(\frac{p_1}{1}, \dots, \frac{p_n}{1}\right)$ is also a basis of $P_{\mathfrak{p}}$. We have an ex. seq. of A-mods

$$0 \to L \to A^{\oplus n} \xrightarrow{\varphi} P \to M \to 0$$

where $\varphi(a_1, \dots) = \sum a_i p_i$, $L := \operatorname{Ker}(\varphi)$ and $M := \operatorname{Coker}(\varphi)$.

After applying the exact functor $A_{\mathfrak{p}} \otimes_A -$ we get an ex. seq.

$$0 \to L_{\mathfrak{p}} \to A_{\mathfrak{p}}^{\oplus n} \xrightarrow{\varphi_{\mathfrak{p}}} P_{\mathfrak{p}} \to M_{\mathfrak{p}} \to 0.$$

Now $\varphi_{\mathfrak{p}}$ is bijective, so $L_{\mathfrak{p}} = 0 = M_{\mathfrak{p}}$.

By the next lemma, $\exists\, s \in A - \mathfrak{p}$, s.t. $L_s = 0 = M_s$. Hence

$$\varphi_s : A_s^{\oplus n} \to P_s$$

is bijective, so P_s is free. $\triangle$

(100) Lemma. Let L be a fin. gen. A-mod. If $L_{\mathfrak{p}} = 0$ for some prime $\mathfrak{p}$, then $\exists$ $s \in A - \mathfrak{p}$ s.t. $L_s = 0$.

pf. Let $l_1, \dots, l_n$ be generators of the module L. Since $\frac{l_i}{1} = 0$ in $L_{\mathfrak{p}}$, it follows that $\exists$ $s_i \in A - \mathfrak{p}$ s.t. $s_i \cdot l_i = 0$ in L. Let $s := s_1 \cdots s_n$. Then $s \in A - \mathfrak{p}$, $s \cdot l_i = 0$, and in the module L_s we have $\frac{l_i}{1} = 0$. But $\frac{l_1}{1}, \dots, \frac{l_n}{1}$ generate L_s. $\square$