# Harmonic Functions

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Ben Gurion University

Eshnav, Jan 2017

classical notions

random walks

gambler's ruin

counting simple paths

# heat equation

Jean-Baptiste
Joseph Fourier
(1768–1830)

$u(x,t) =$ heat at time $t$, point $x \in [0, L]$

Fourier's law + conservation of energy:

$$\frac{\partial}{\partial t} u = \alpha \cdot \frac{\partial^2}{\partial x^2} u$$

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# heat equation

$$\frac{\partial}{\partial t}u = \Delta u \qquad \Delta = \frac{\partial^2}{\partial x^2}$$

$$f(x+\varepsilon) = f(x) + f'(x)\varepsilon + \tfrac{1}{2}f''(x)\varepsilon^2 + O(\varepsilon^3)$$
$$f(x-\varepsilon) = f(x) - f'(x)\varepsilon + \tfrac{1}{2}f''(x)\varepsilon^2 + O(\varepsilon^3)$$

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$$f(x+\varepsilon) + f(x-\varepsilon) = 2f(x) + f''(x)\varepsilon^2 + O(\varepsilon^3).$$

$$u(x, t+\delta) - u(x,t) = \delta \cdot \frac{2}{\varepsilon^2} \cdot \left( \frac{u(x+\varepsilon,t) + u(x-\varepsilon,t)}{2} - u(x,t) \right)$$

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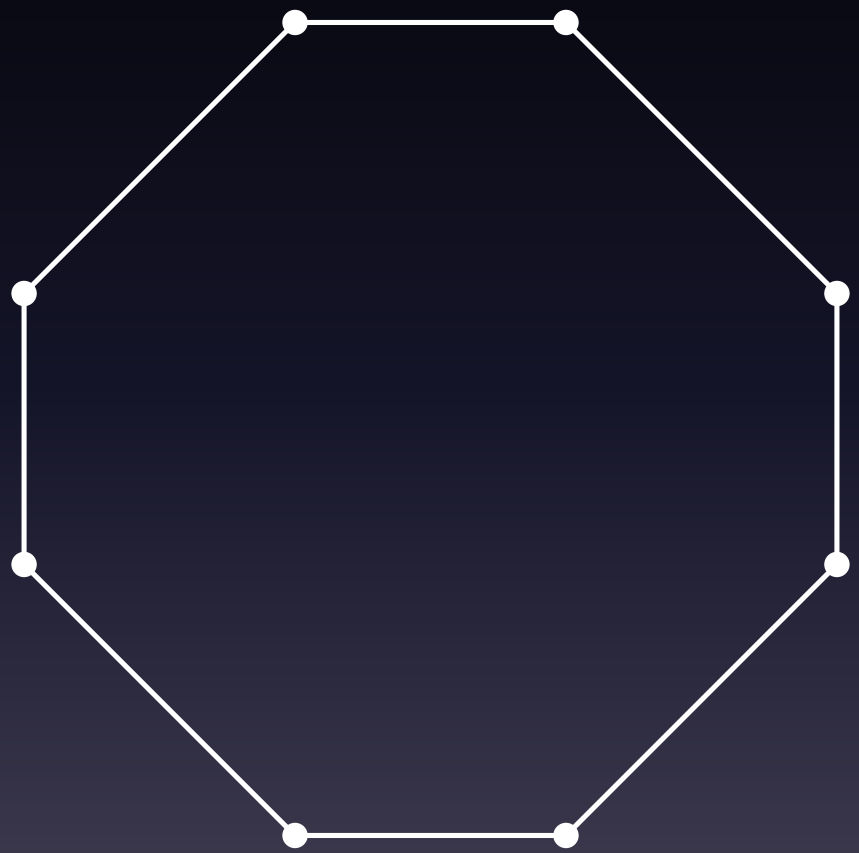
# discrete geometry - graphs

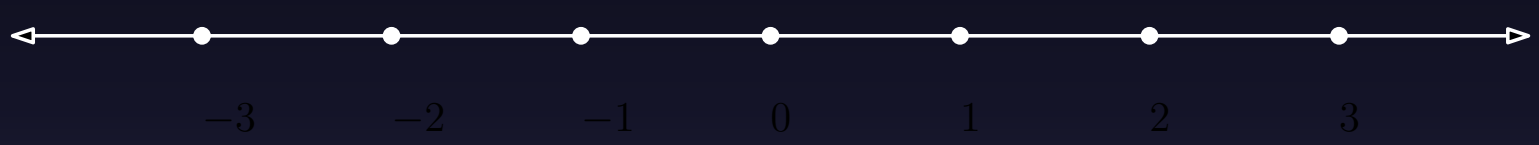
A graph is a collection of vertices (nodes) and edges (connections).

# graphs: cycle

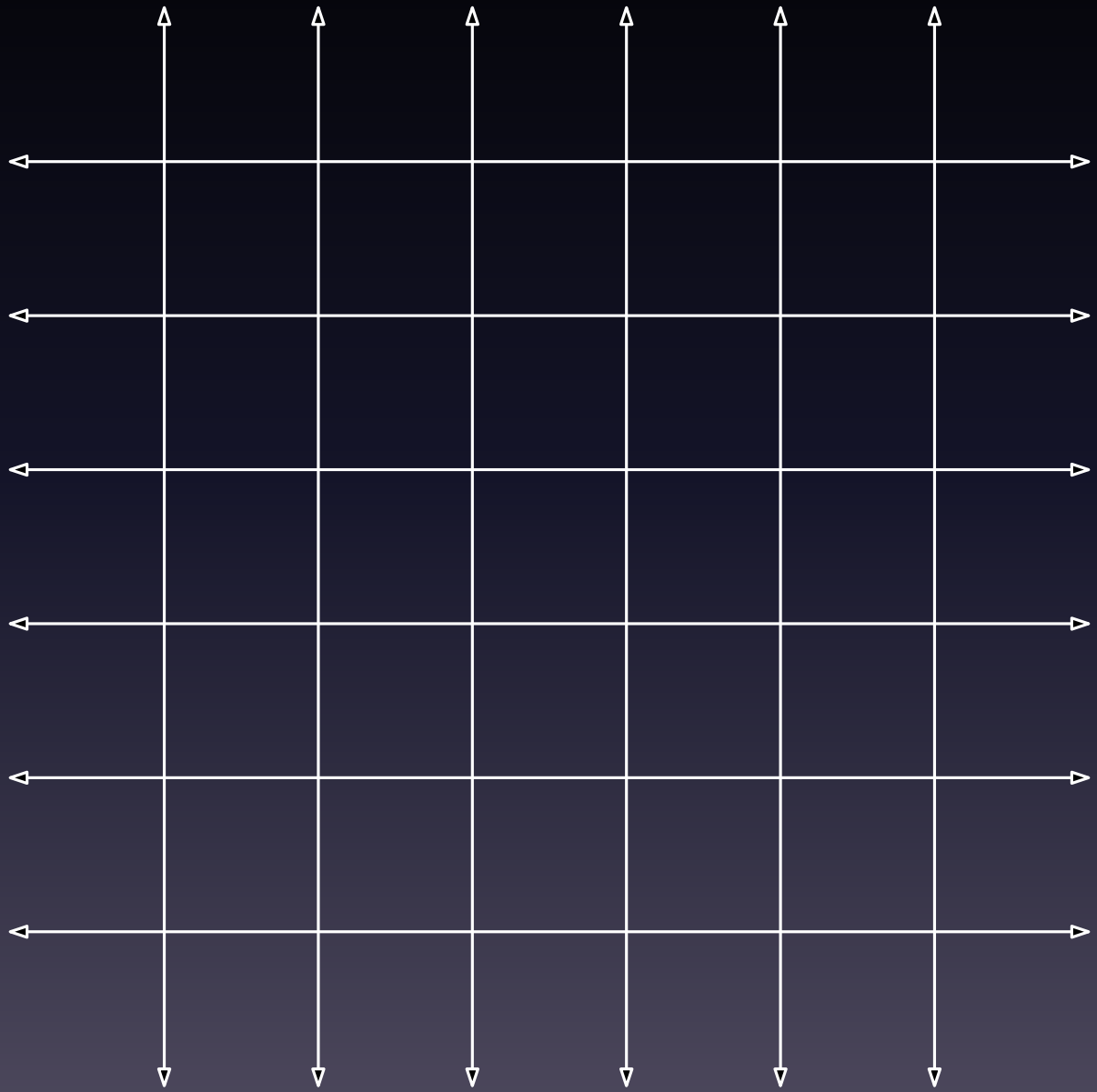


# graphs: finite line

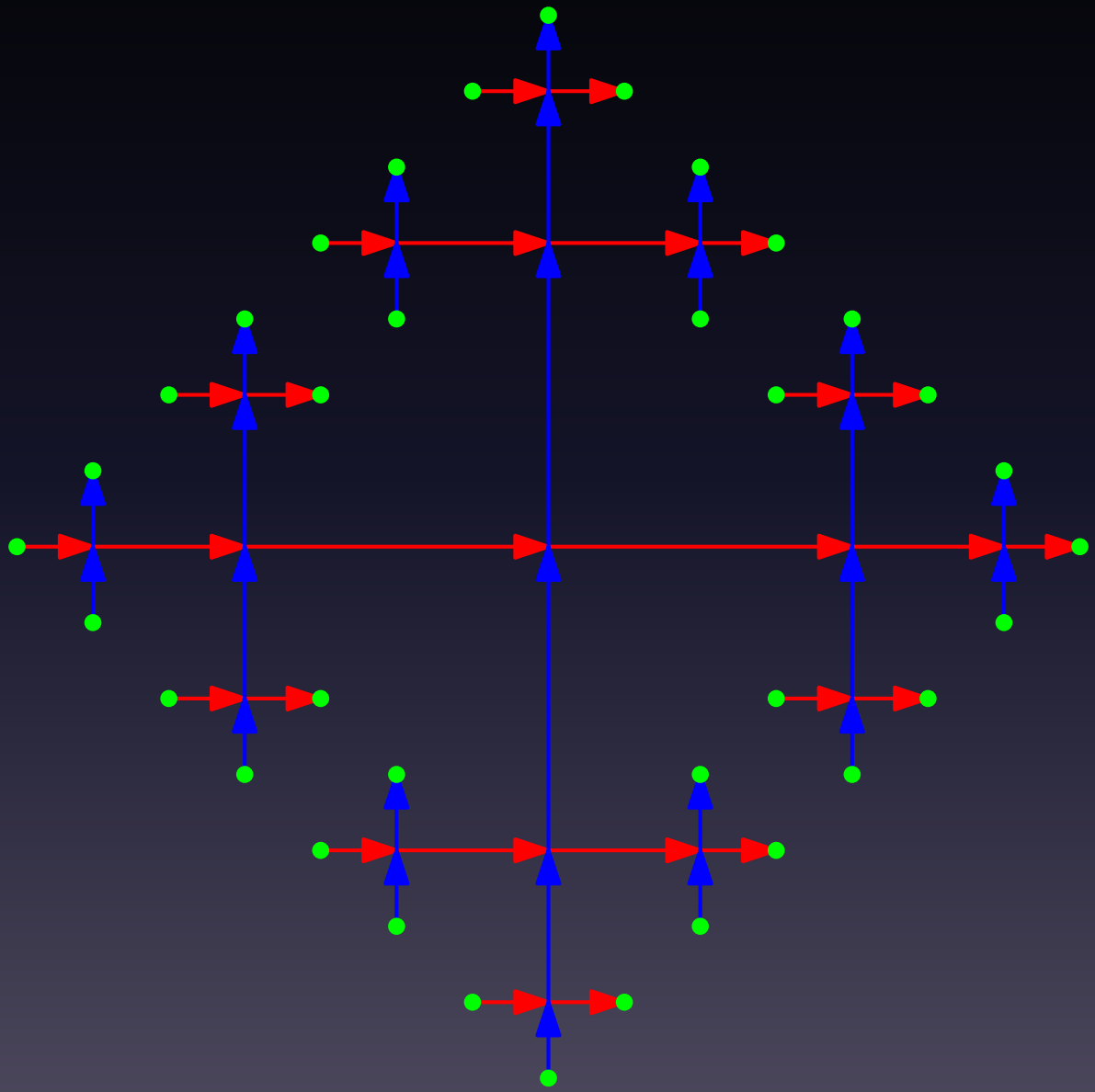
# graphs: $\mathbb{Z}$



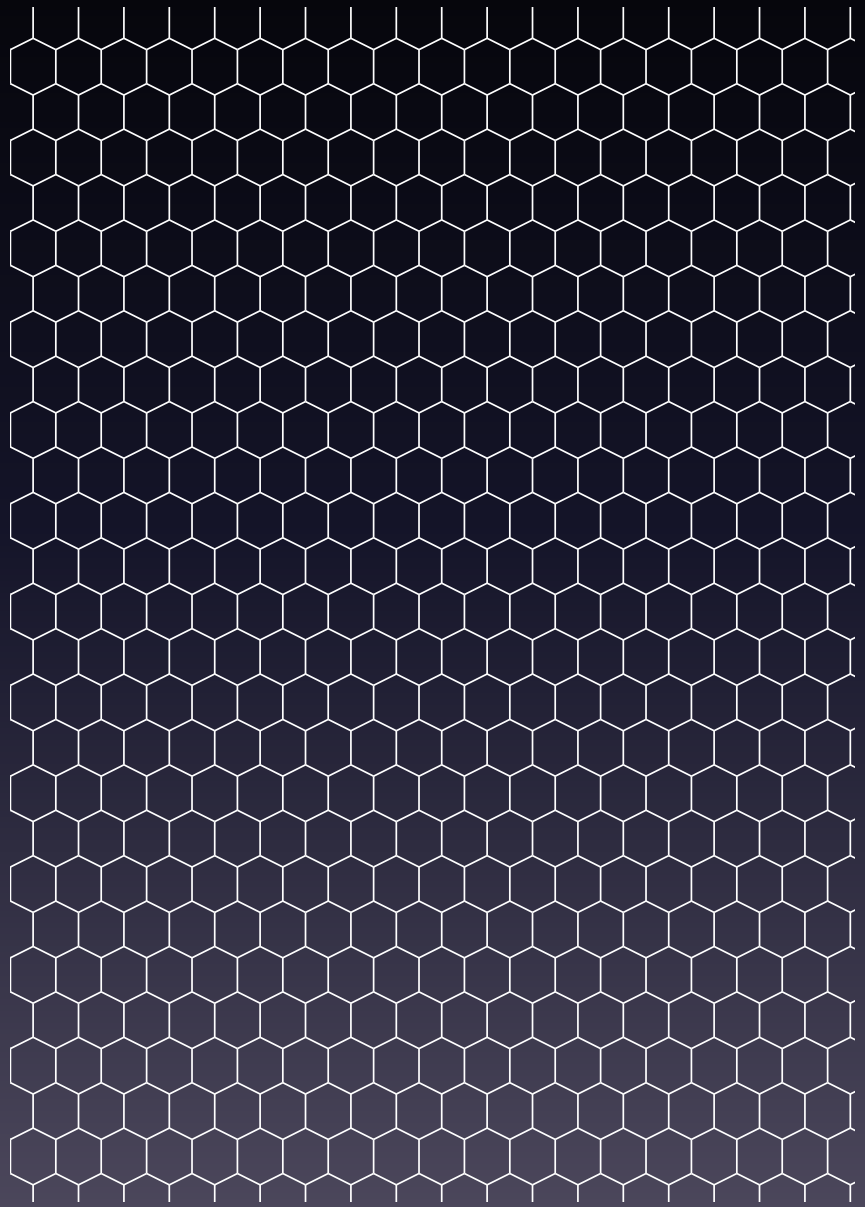
# graphs: $\mathbb{Z}^2$



# graphs: regular tree



# graphs: hexagonal lattice



# graphs: notation

- $x \sim y \iff x, y$ are neigbors
- $\deg(x) =$ the number of neighbors of $x$
- paths give a metric (geometry)
- boundaries

$$\partial D = \{x \notin D \ : \ \exists\, y \sim x \ , \ y \in D\}$$

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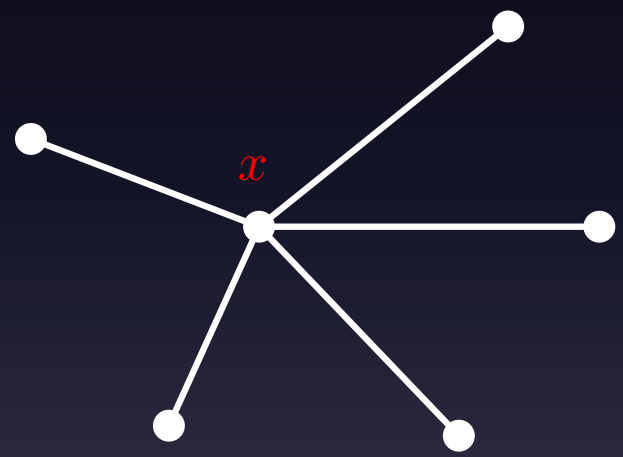

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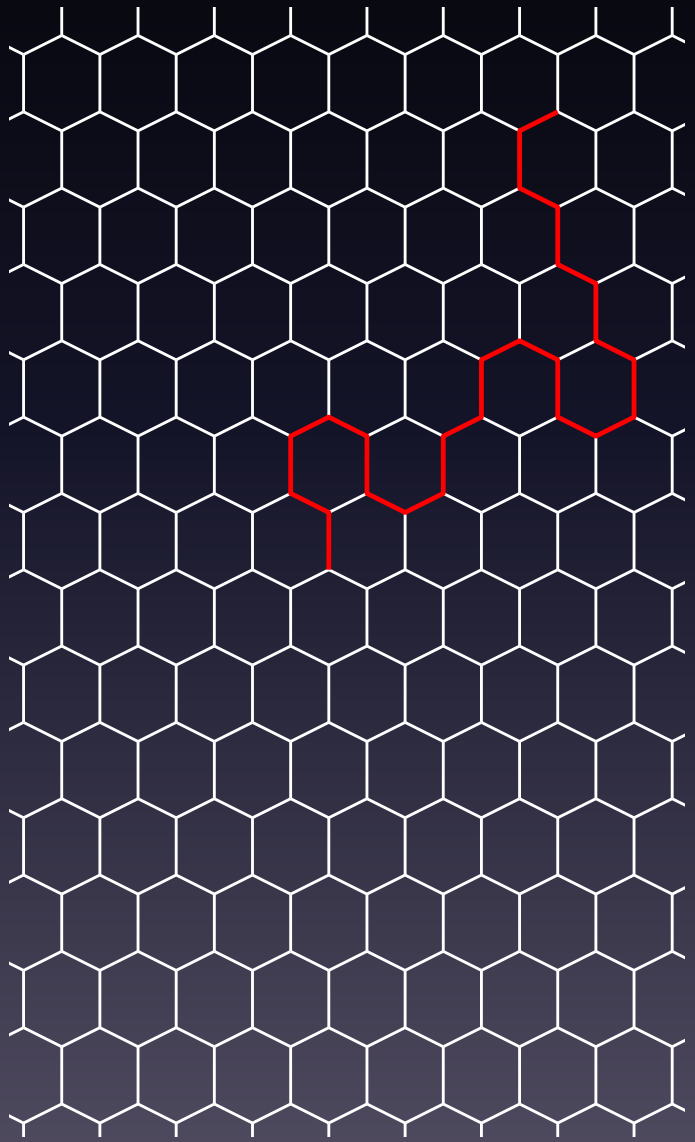

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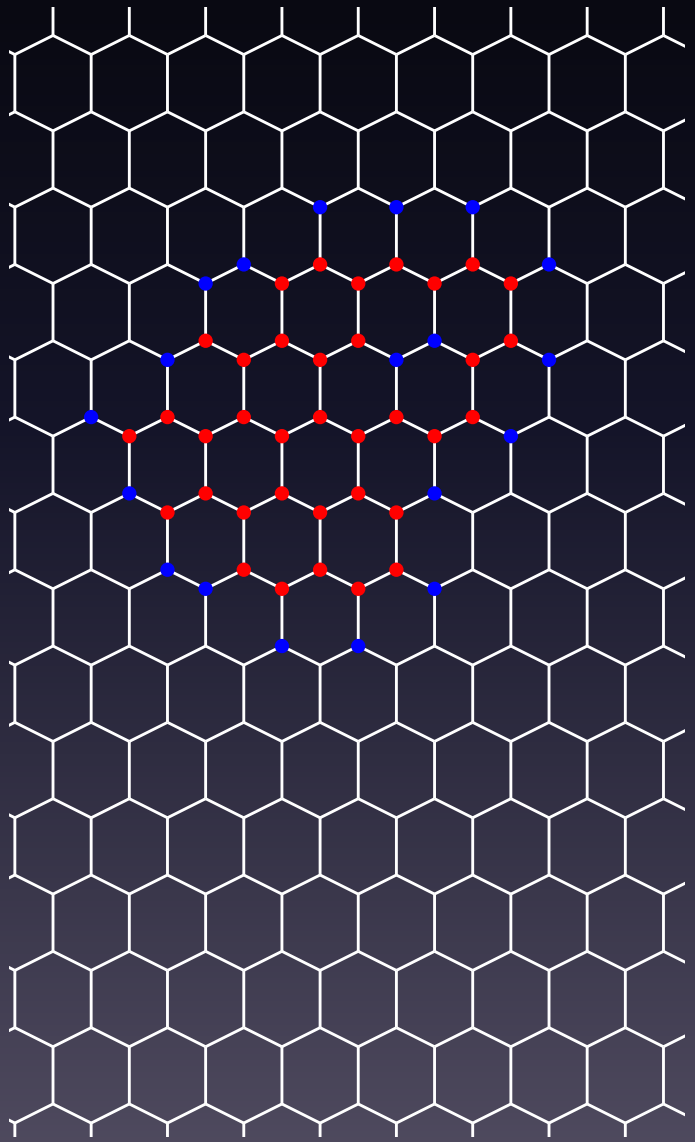
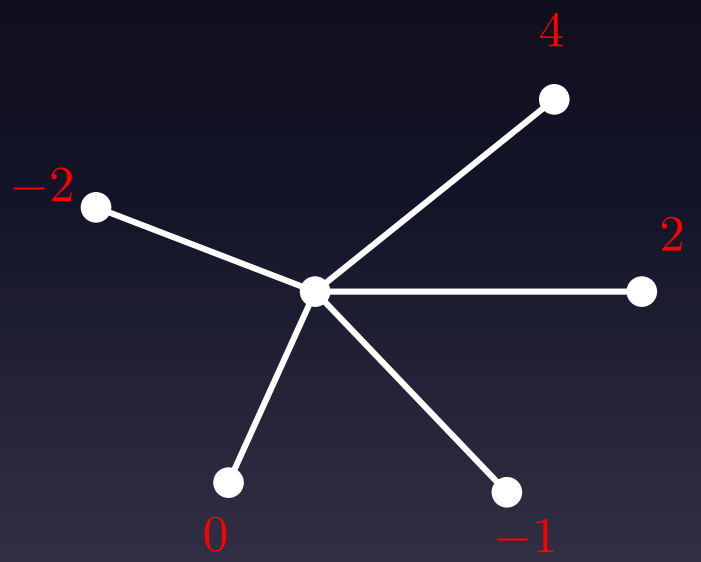


# harmonic function

Definition

A function is harmonic at $x$ if

$$f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y)$$



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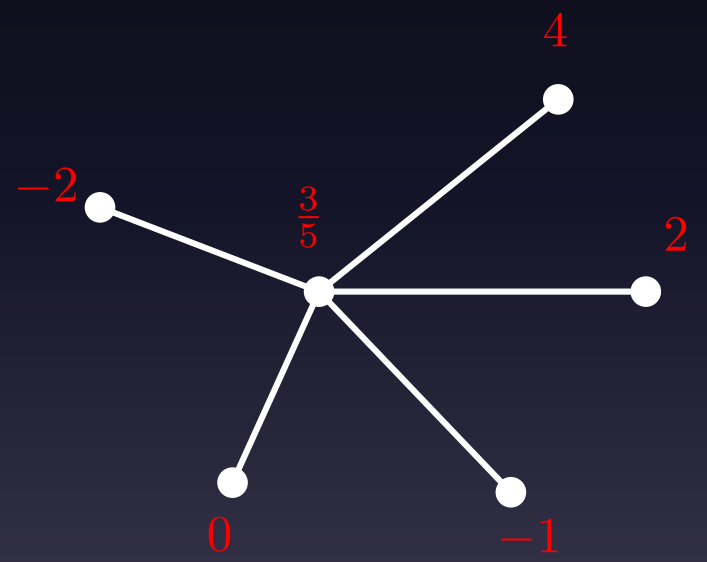

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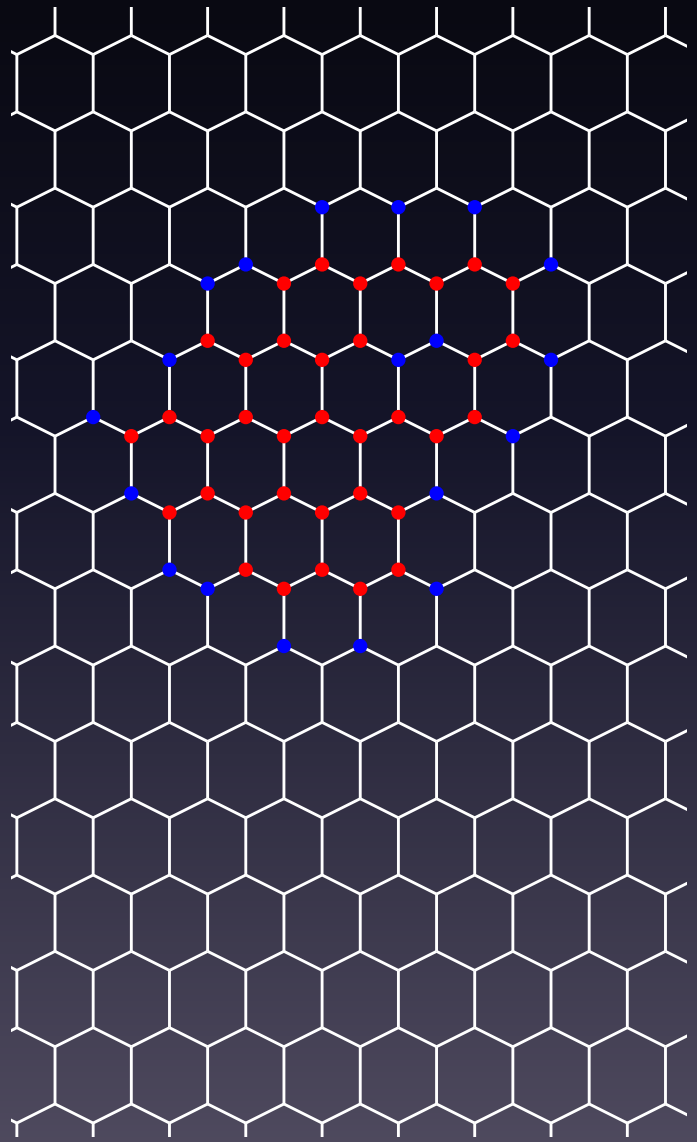
A function is harmonic at $x$ if

$$\Delta f(x) := \frac{1}{\deg(x)} \sum_{y \sim x} (f(y) - f(x)) = 0$$

# harmonic functions

- maximum (minimum) principle
- boundary conditions uniquely define harmonic functions
- Liouville's Theorem: A bounded function harmonic on all of $\mathbb{Z}^d$ is constant.



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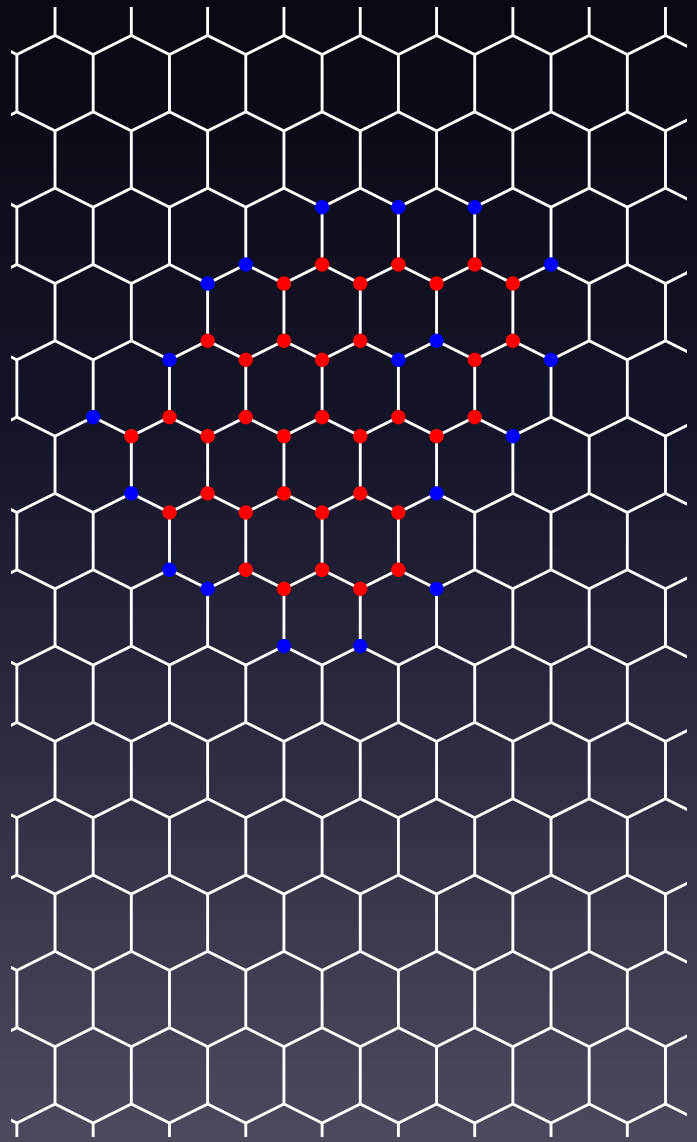

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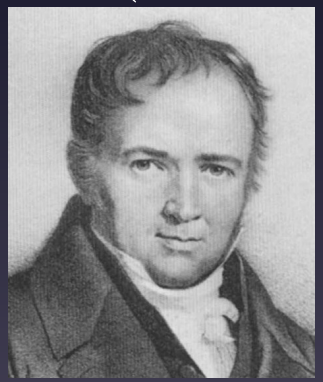
Peter Gustav Lejeune Dirichlet (1805–1859)



Siméon Denis Poisson (1781–1840)

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Joseph Liouville
(1809–1882)

# heat evolution

$$u(x,t+\delta) - u(x,t) = \delta \cdot \frac{2}{\varepsilon^2} \cdot \left( \frac{u(x+\varepsilon,t)+u(x-\varepsilon,t)}{2} - u(x,t) \right)$$

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harmonic functions are stable under heat evolution

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# random walk

Given a graph $G$, walk randomly on a graph: at every time step, independently of the past, choose a uniform neighbor and move to it

# Dirichlet problem solution

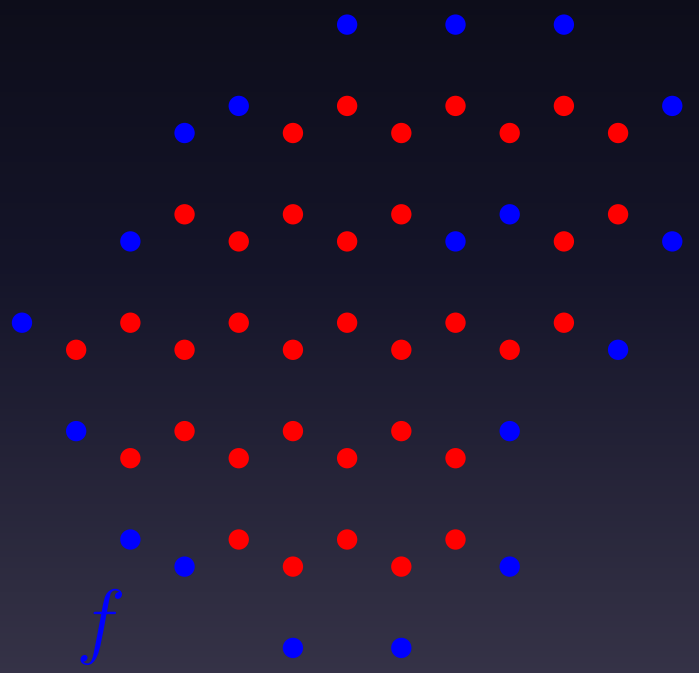
- in a graph $G$ let $D$ be some finite domain with a boundary

  $\partial D = \{x \notin D \ : \ \exists\, y \sim x \ , \ y \in D\}$

- $f : \partial D \to \mathbb{R}$ boundary conditions
- the function

  $$u(x) = \mathbb{E}[f(X_T) \mid X_0 = x]$$

  is harmonic in $D$ and coincides



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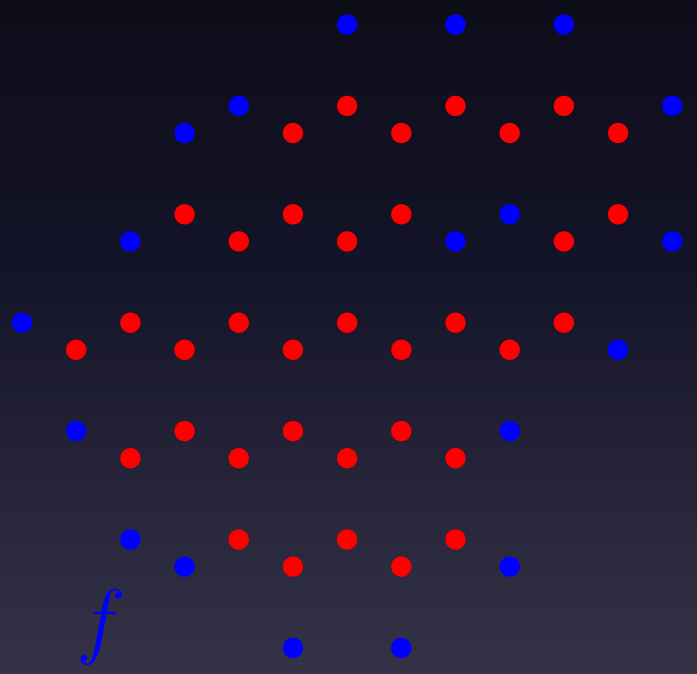

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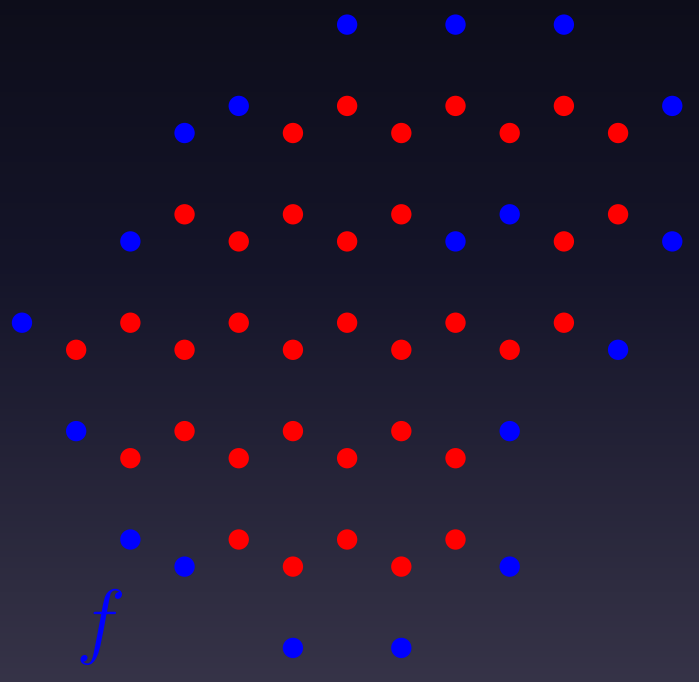

# heat equation solution

- In a graph $G$ let $D$ be some finite domain with a boundary

$$\partial D = \{x \notin D \ : \ \exists\, y \sim x \ , \ y \in D\}.$$

- Let $f : \partial D \cup D \to \mathbb{R}$ be some initial / boundary conditions.
- The function

$$u(x,t) = \mathbb{E}[f(X_{T \wedge t}) \mid X_0 = x]$$

is harmonic in $D$ and coincides with the initial conditions $f$, and solves the heat equation

$$u(x,t+1) - u(x,t) = \Delta u(x,t).$$

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# fair game

- play the following game:
  at each step, a coin is tossed
  you gain one coin, or lose one coin, each with probability $\frac{1}{2}$
- let $X_0, X_1, \ldots,$ be the number of coins at step $t$
- question: starting with $x$ coins, what is the probability to win $N$ coins without going bankrupt ?

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# gambler's ruin

- $f(x) =$ probability to reach $N$ before 0 started at $x$
- $f(0) = 0, f(N) = 1$
- $0 < x < N \quad \Rightarrow \quad f(x) = \frac{1}{2} f(x+1) + \frac{1}{2} f(x-1)$
- $f$ is harmonic in $(0, N)$
- solution: $f(x) = \frac{x}{N}$
  (unique by maximum principle)

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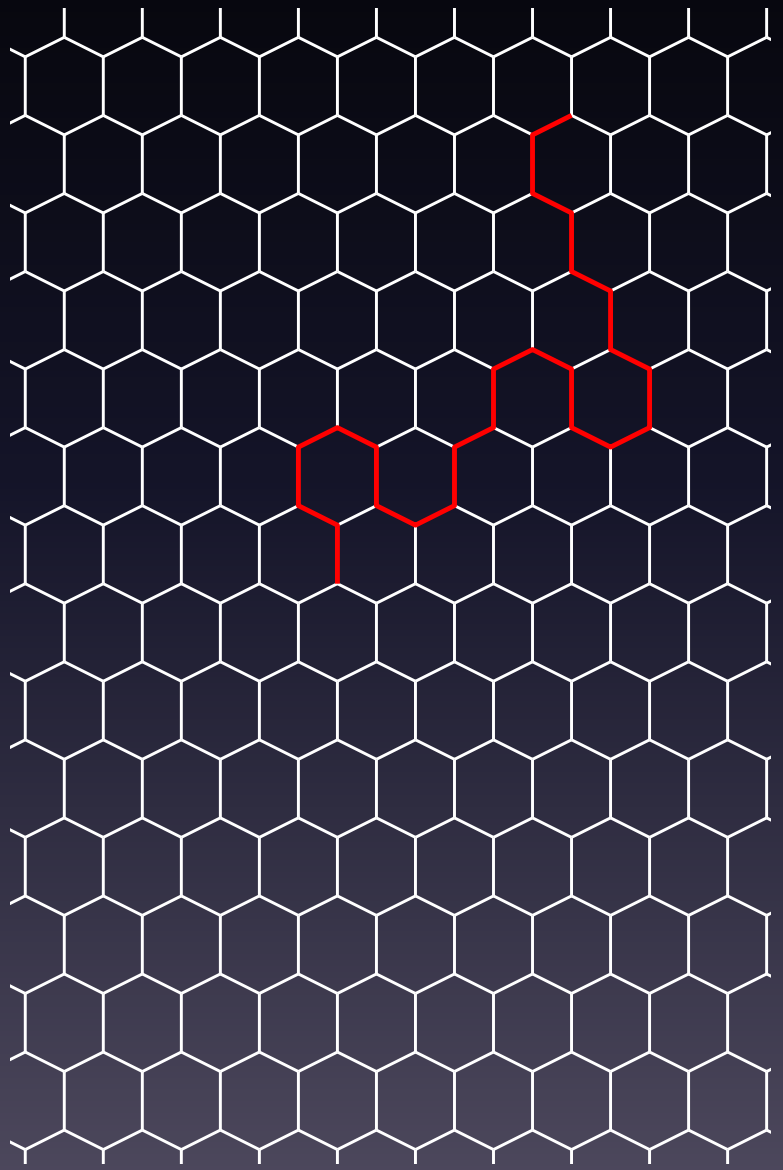
# SAW

- in a graph $G$ fix some root vertex $o$
- let $\mathsf{SAW}_n$ be the set of all simple paths of length $n$ started at $o$
- Problem: count $|\mathsf{SAW}_n| = ?$

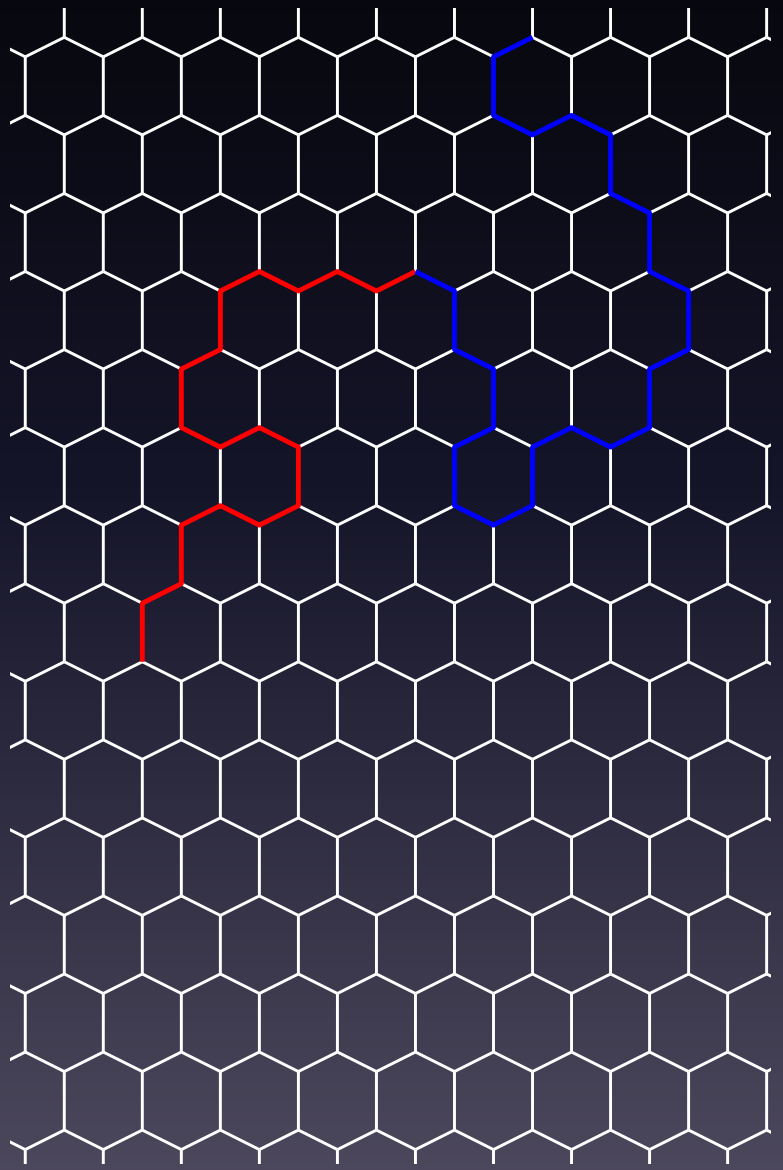


# SAW

- easy using Fekete's Lemma: the limit exists:

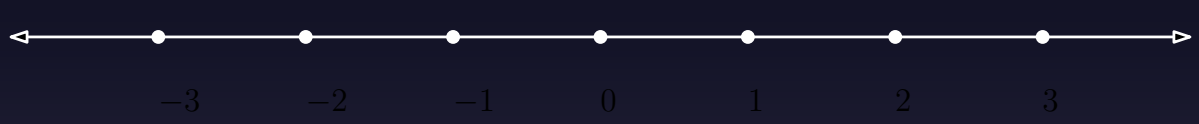
$$\mu = \mu(G) := \lim_{n\to\infty} |\mathsf{SAW}_n|^{1/n}$$

- $\mu$ is called the connective constant



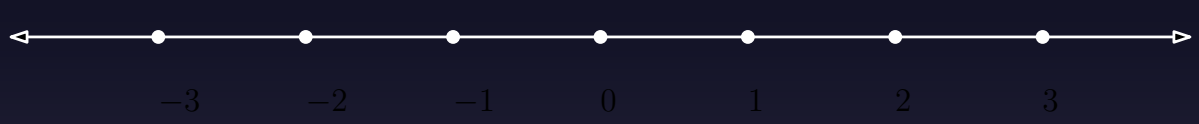
# connective constant

Example: $\mu(\mathbb{Z}) =$



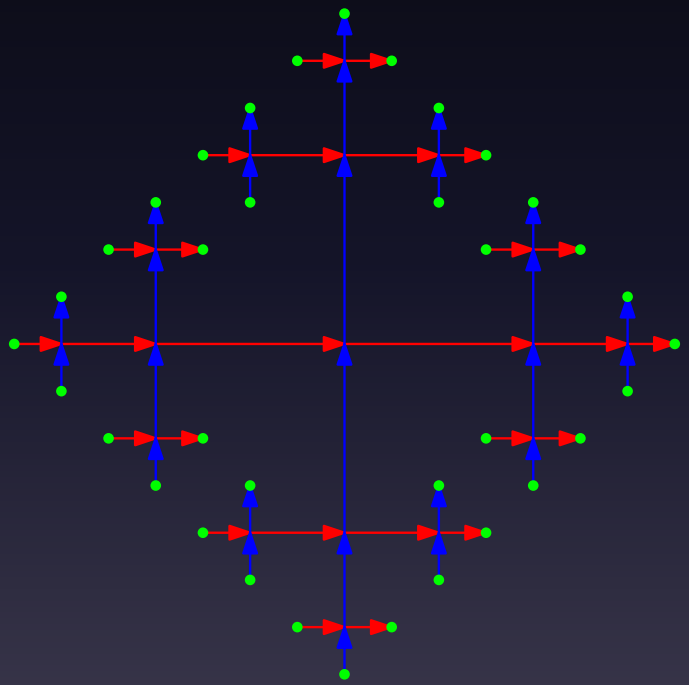
# connective constant

Example: $\mu(\mathbb{Z}) = 1$



# connective constant

Example: regular tree, $\mu(\mathbb{T}_d) =$



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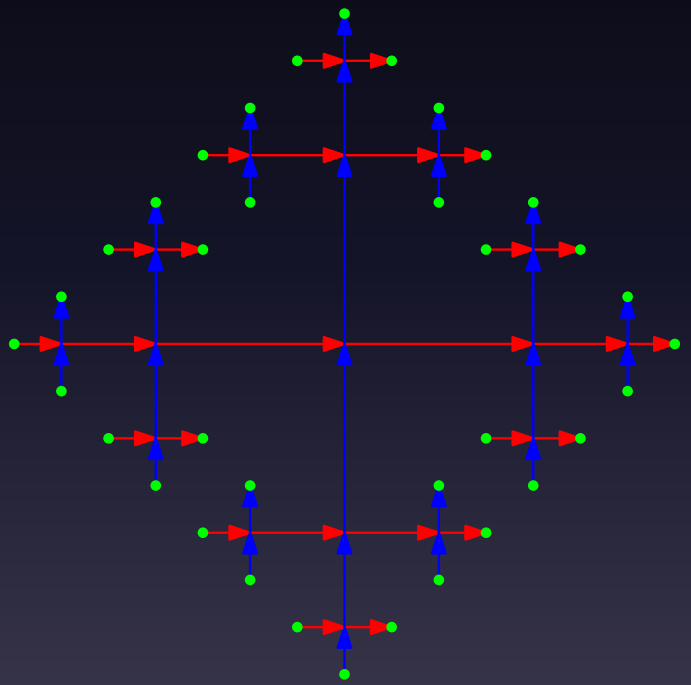
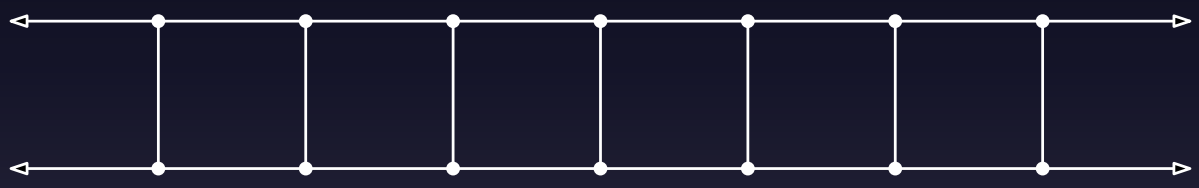


# connective constant

Example: the ladder $\mu(L) =$



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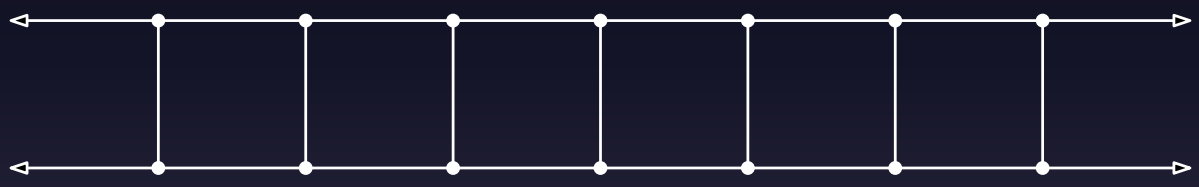
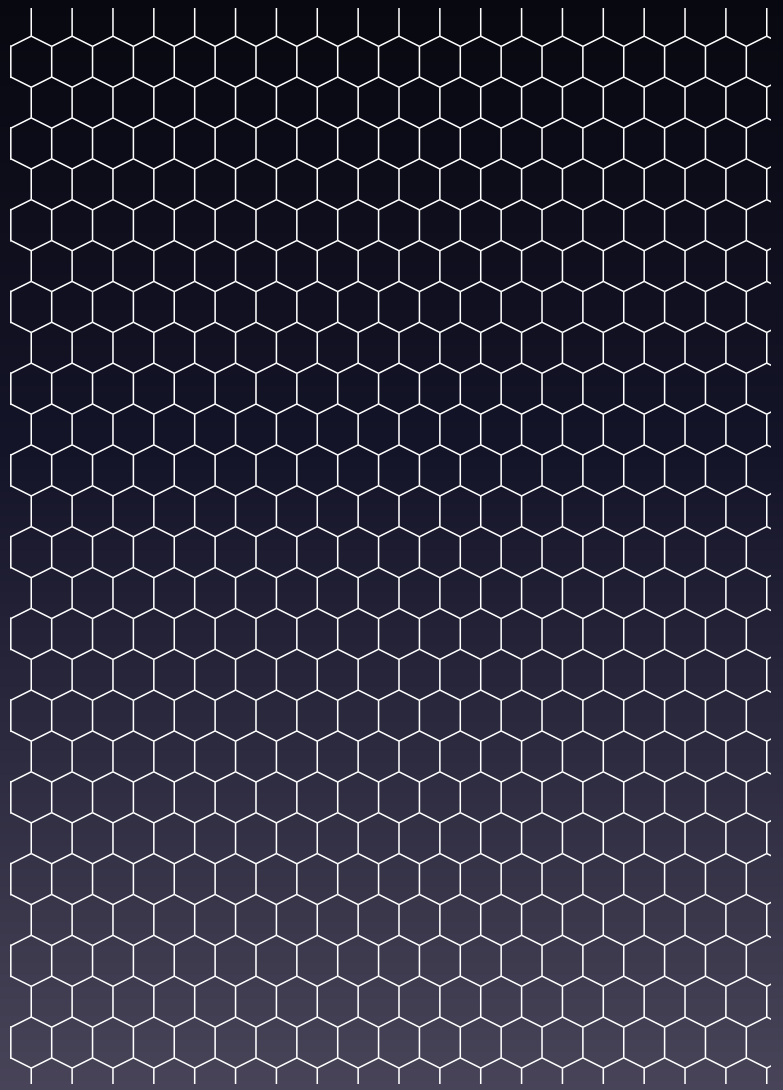
Example: the ladder $\mu(L) = \frac{1+\sqrt{5}}{2}$



# connective constant

Theorem (Duminil-Copin & Smirnov 2010)

$\mu(\mathbb{H}) =$



# connective constant

Theorem (Duminil-Copin & Smirnov 2010)

$$\mu(\mathbb{H}) = \sqrt{2+\sqrt{2}}$$

Hugo Duminil-Copin

Stas Smirnov

# calculating $\mu(\mathbb{H})$

- $\frac{1}{\mu}$ is the radius of convergence for the generating function

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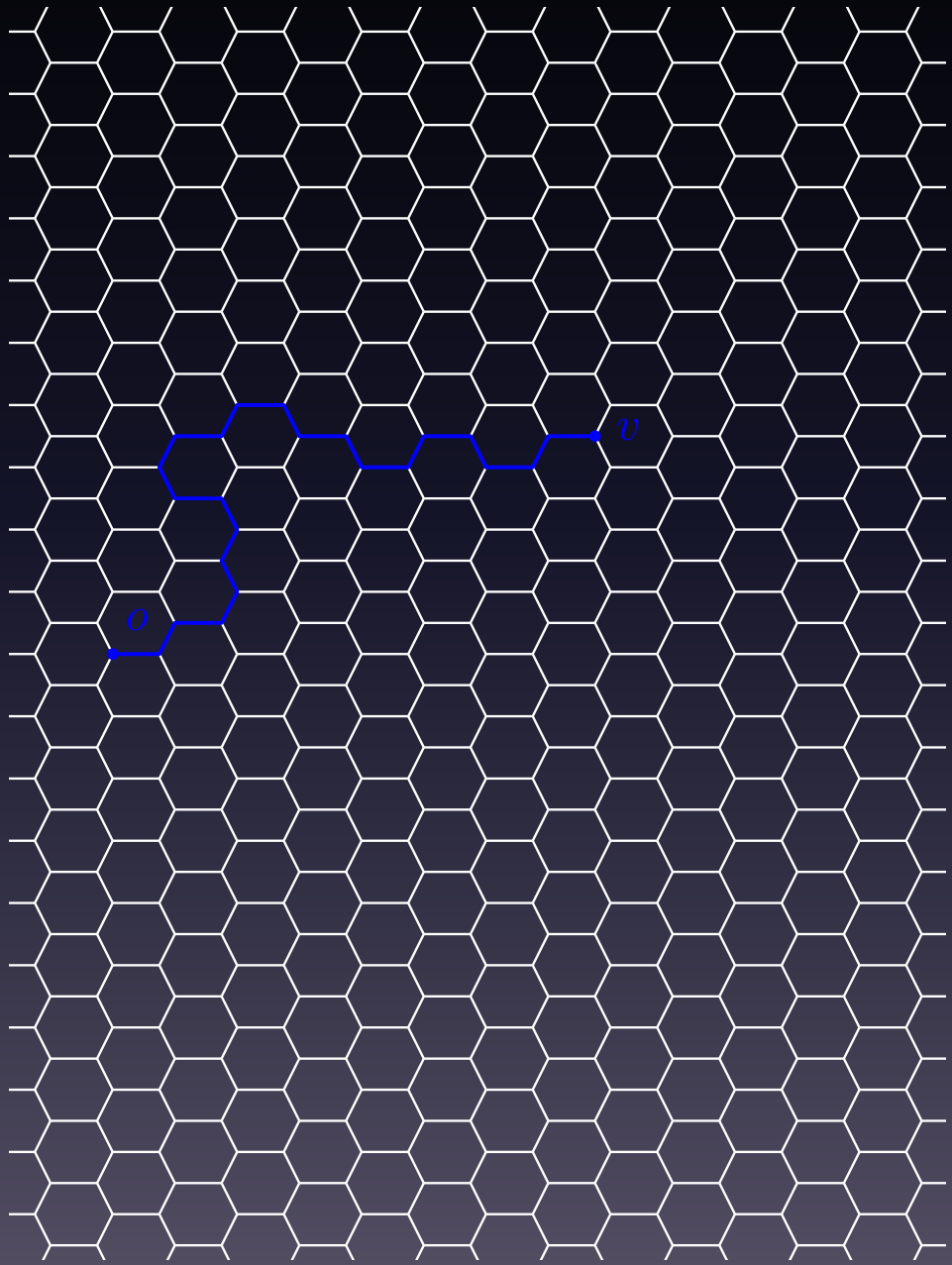

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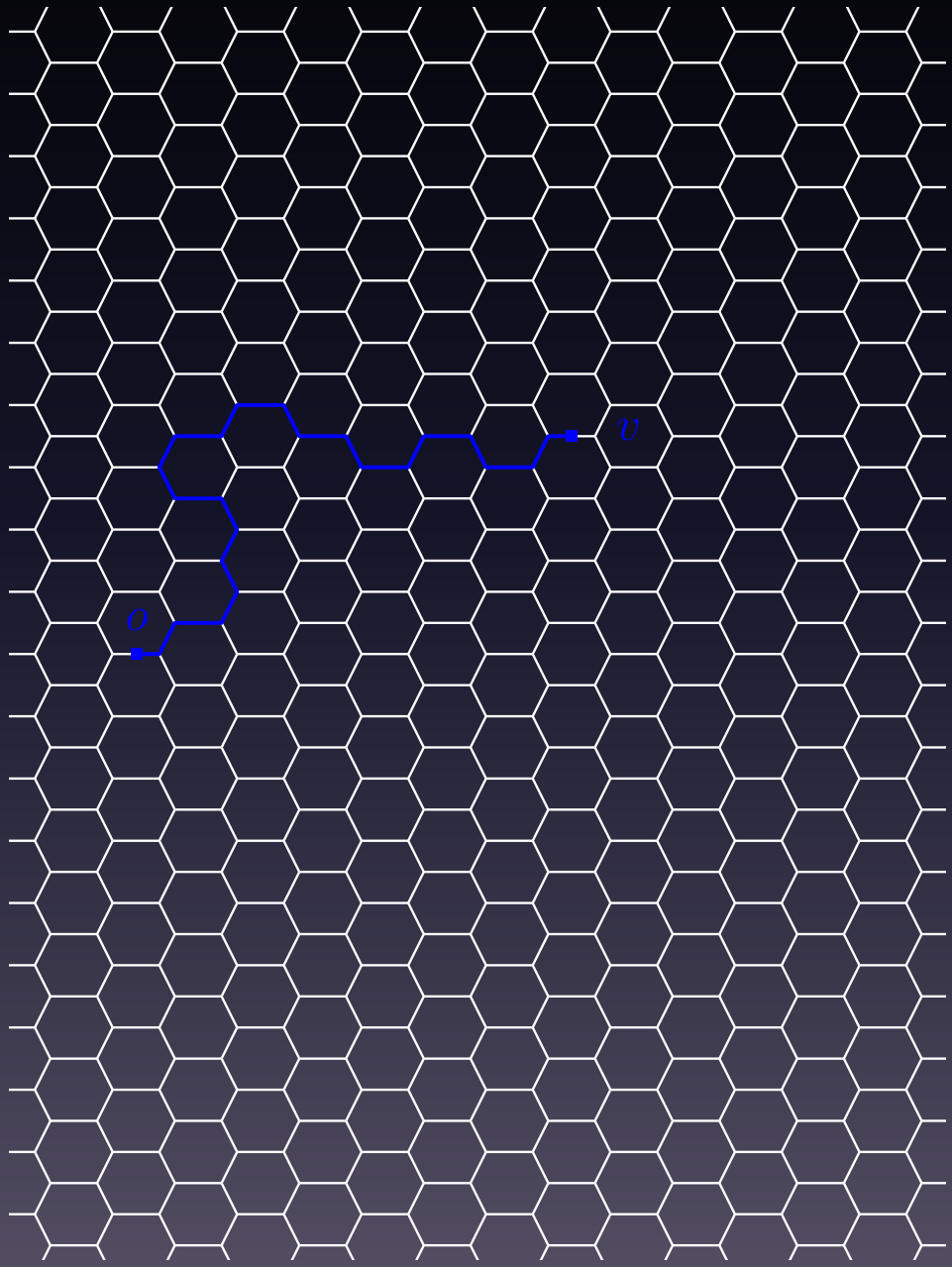
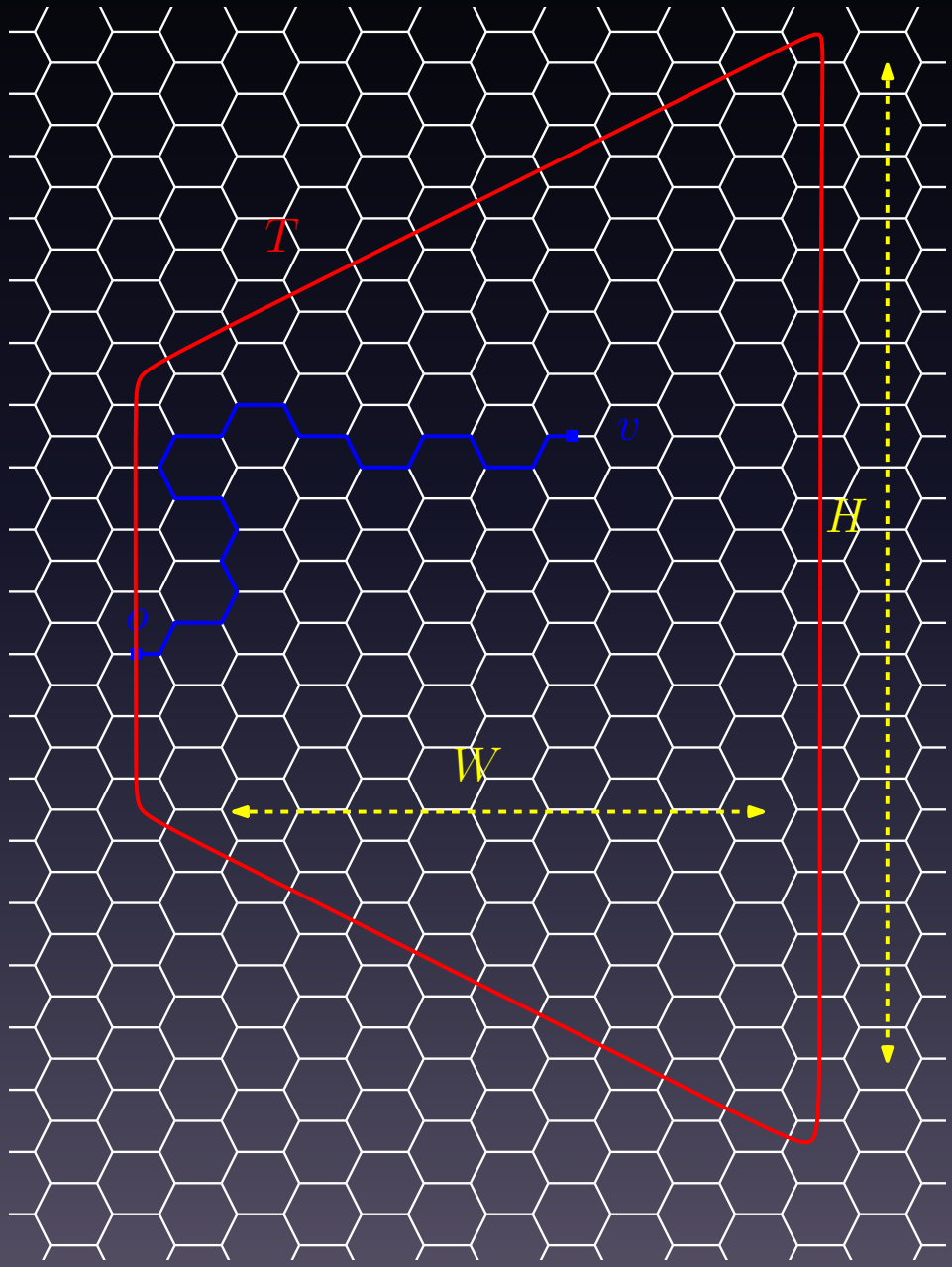


# reductions

$$P(z) = \sum_{v \in T} \sum_{\omega : o \to v} z^{|\omega|}$$



- define a function on mid-edges in $T$:

$$F(p) = \sum_{\omega: o \to p} z^{|\omega|} e^{i\alpha\theta(\omega)}$$

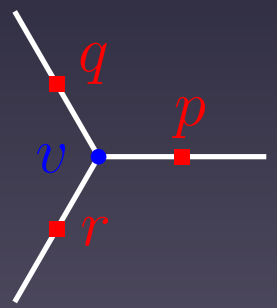
  where $\theta(\omega)$ is the winding of $\omega$

- extend $F$ to the vertices by averaging $F$ on the mid edges around each vertex:

$$F(v) = (p - v)F(p) + (q - v)F(q) + (r - v)F(r)$$

  where $p, q, r$ are the mid-edges adjacent to $v$

  Question: can we find $z, \alpha$ so that $F$ is "holomorphic" ? that is, so that $F$ is 0 on each vertex?



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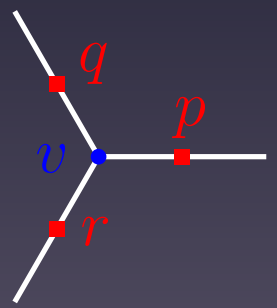

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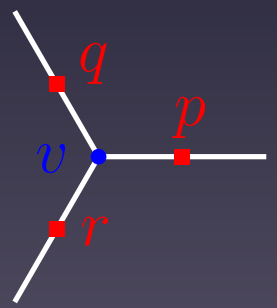
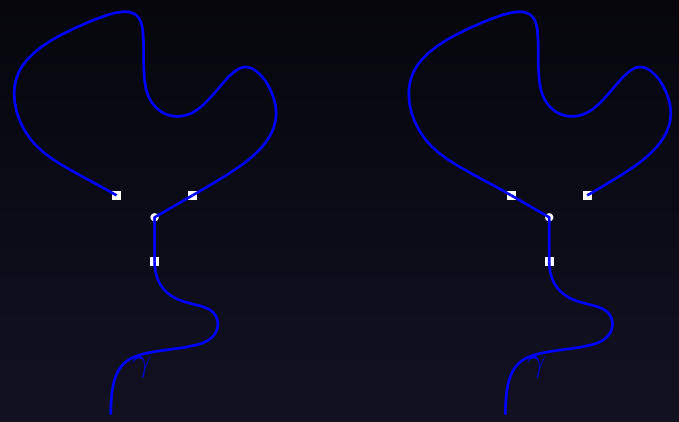
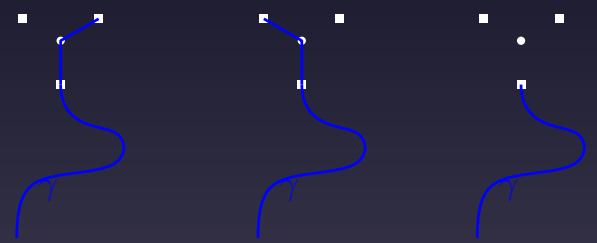


$$0 = e^{i\frac{4}{3}\pi} \cdot z^k e^{i\alpha\frac{4}{3}\pi} + e^{-i\frac{4}{3}\pi} \cdot z^k e^{-i\alpha\frac{4}{3}\pi}$$
$$+ e^{-i\frac{4}{3}\pi} \cdot z e^{-i\alpha\frac{1}{3}\pi} + e^{i\frac{4}{3}\pi} \cdot z e^{i\alpha\frac{1}{3}\pi} + 1 \cdot 1$$




$z^k \exp(i\alpha\frac{4}{3}\pi)$ $z^k \exp(-i\alpha\frac{4}{3}\pi)$

end-mid= $\frac{4}{3}\pi$ end-mid= $-\frac{4}{3}\pi$




$z \exp(-i\alpha\frac{1}{3}\pi)$ $z \exp(i\alpha\frac{1}{3}\pi)$ 1

end-mid= $-\frac{4}{3}\pi$ end-mid= $\frac{4}{3}\pi$ end-mid= 0

Equations:

$$0 = \cos\left((\alpha+1)\tfrac{4}{3}\pi\right)$$
$$-1 = 2z\cos\left((\alpha+4)\tfrac{1}{3}\pi\right)$$

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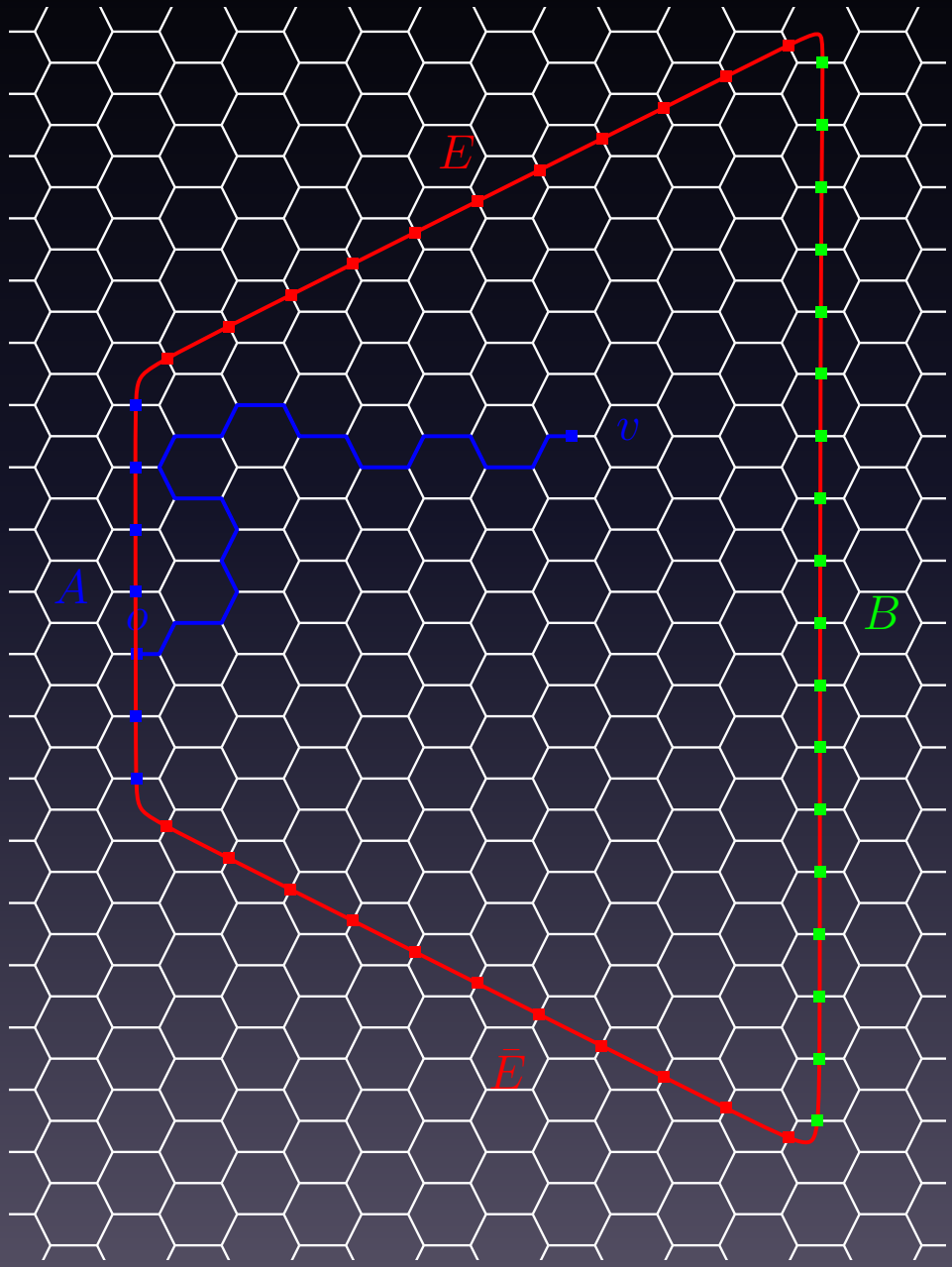
*e.g.* $k = 0$ and $\alpha = -\frac{5}{8}$ and

$$z^{-1} = 2\cos\tfrac{\pi}{8} = \sqrt{2+\sqrt{2}}$$

# summary so far

$$P(z) = \sum_{v \in T} \sum_{\omega : o \to v} z^{|\omega|}$$



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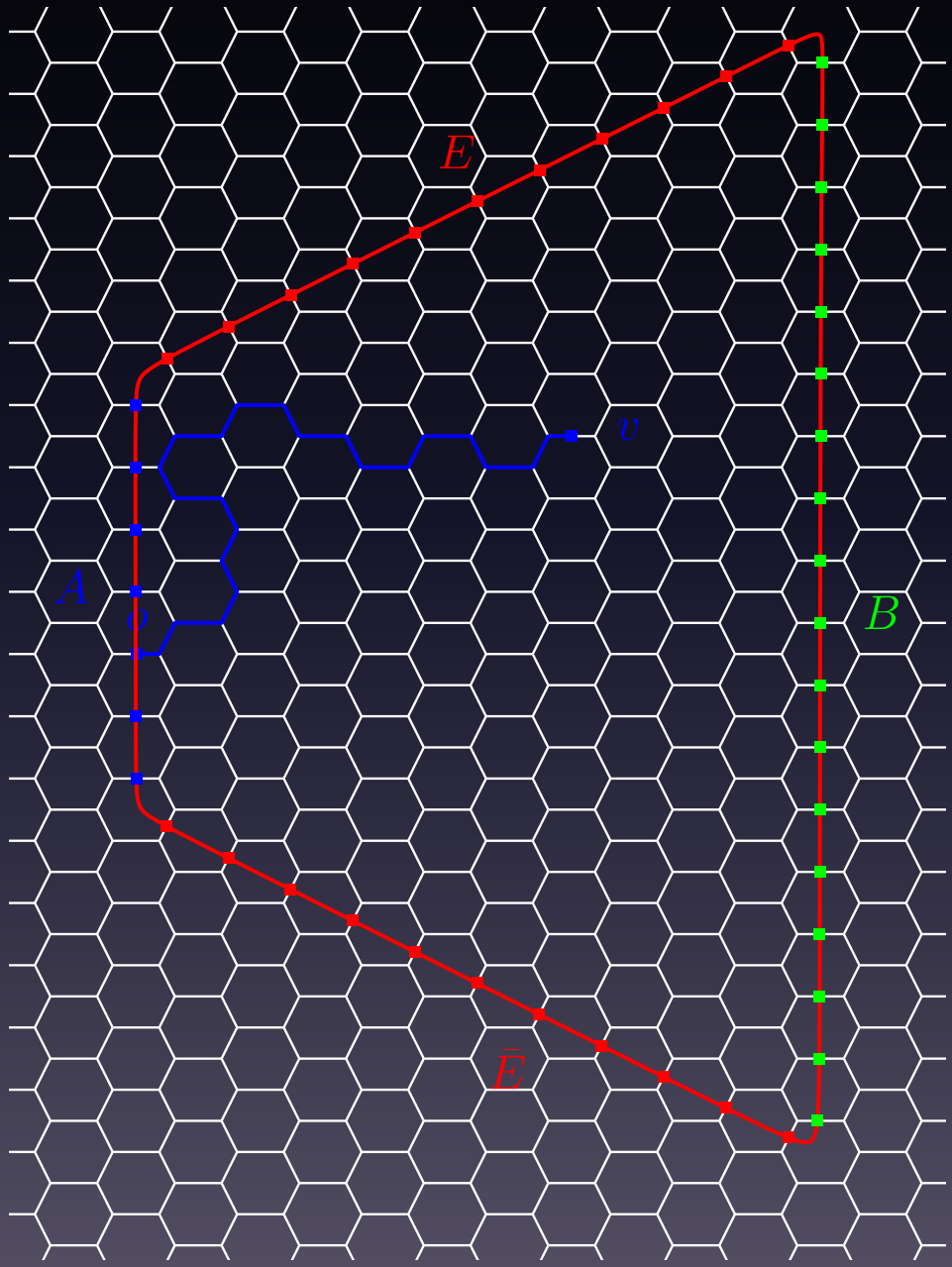

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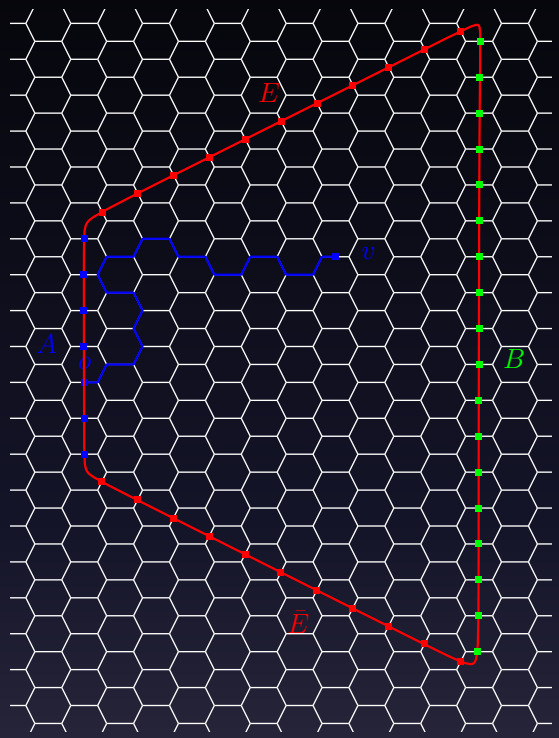


$$0 = \sum_{v \in T} F(v) = -F(A) + F(B) + \lambda F(E) + \bar{\lambda} F(\bar{E})$$

$$(\lambda = e^{i\alpha\pi})$$

the winding is constant on $A, B, E$ and $\bar{E}$!

$$0 = -1 - e^{i\alpha\pi} P(A^+) - e^{-i\alpha\pi} P(A^-) + P(B) + \lambda^2 P(E) + \lambda^{-2} P(\bar{E})$$

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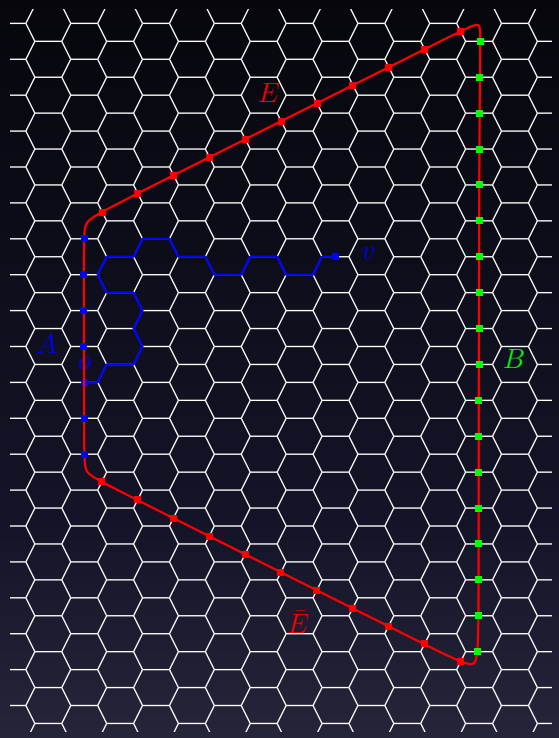


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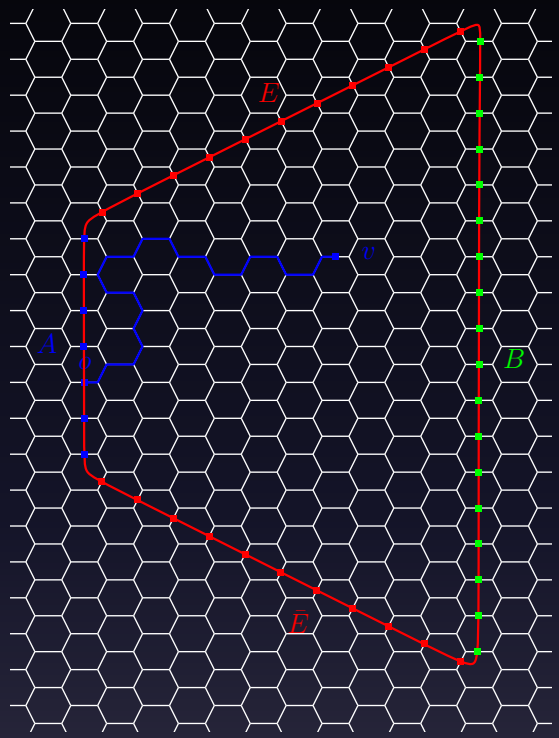


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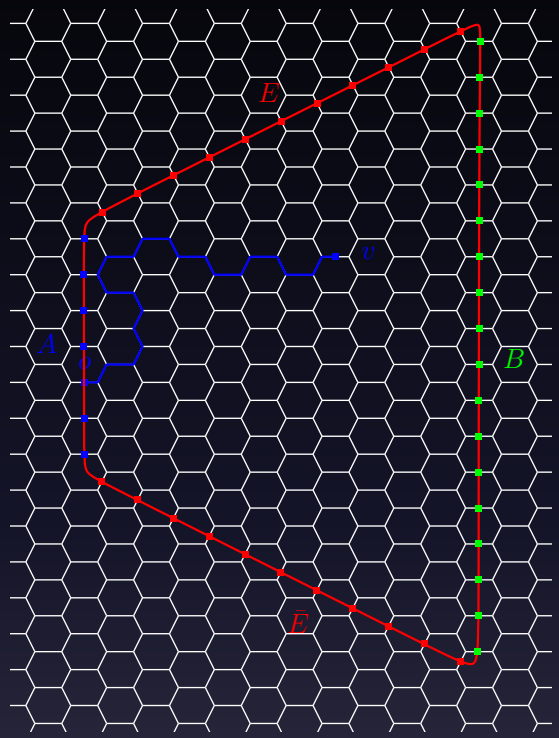


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taking $H \to \infty$ (height)

$$1 = -\cos(\alpha\pi) \sum_{\omega: o \to A} z^{|\omega|} + \sum_{\omega: o \to B} z^{|\omega|}$$

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