

המחלקה למתמטיקה, בן-גוריון

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## אשנב למתמטיקה

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ביום שלישי, 8 בדצמבר, 2020

בשעה 16:10 – 17:30

במרשתת

ההרצאה

### העתקות פולינומיאליות ולוגיקה

תינתן על-ידי

מנחם קוג'מן

**תקציר:** באשנב נדבר על מאמר של Rudin, Walter בו הוא מציע הוכחה קצרה למשפט אקס: העתקה פולינומיאלית חז"ע מהמרוכבים בחזקת  $n$  למרוכבים בחזקת  $n$  היא על. נציג הוכחה המבוססת על משפט הקומפקטיות מלוגיקה ושלמות של מערכות אקסיומות. תוכלו להחליט בסוף האשנב איזו משתי ההוכחות קצרה יותר, או לא להחליט.



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Injective Polynomial Maps are Automorphisms

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Kronecker expansion is obtained

$$A = \mathbf{block}(A) = \sum_{k=1}^K s_k \mathbf{block}(\mathbf{v}_k \mathbf{u}_k^T) = \sum_{k=1}^K s_k U_k \otimes V_k.$$

By (B-2), it is straight-forward to verify the orthogonality of the  $U_k$ 's and the  $V_k$ 's.

*Remarks.* Since the SVD also determines optimal reduced rank approximations, best approximations using a fixed number of Kronecker products can be obtained from this Kronecker expansion. The motivation for this Kronecker expansion or "block" SVD arose in an image-processing application. Illuminating discussions of image processing and applications of the SVD, block matrix computations, and Kronecker products are found in Gonzalez and Wintz [6] or Jain [5]. An excellent treatment of the Kronecker product is found in Horn and Johnson [3]. Further generalizations of the Kronecker product and signal-processing applications are found in [4], [7], [2], [1].

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## Injective Polynomial Maps Are Automorphisms

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Walter Rudin

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This article presents a simple elementary proof of the following result.

**Theorem A.** *If  $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a polynomial map which is one-to-one, then*

- (a)  $F(\mathbb{C}^n) = \mathbb{C}^n$ , and
- (b)  $F^{-1}: \mathbb{C}^n \rightarrow \mathbb{C}^n$  is also a polynomial map.

Here  $n$  is a positive integer, and  $\mathbb{C}^n$  is the set of all  $z = (z_1, \dots, z_n)$ , each  $z_i$  lying in the complex field  $\mathbb{C}$ . In general, the notation  $\Phi: X \rightarrow Y$  indicates that  $\Phi$  is a map whose domain is  $X$  and whose range lies in  $Y$ . To say that  $F$  is a *polynomial* map means that  $F = (f_1, \dots, f_n)$  and each component  $f_i$  of  $F$  is a polynomial, mapping  $\mathbb{C}^n$  into  $\mathbb{C}$ .

Theorem A may be regarded as a small step toward a confirmation of the so-called Jacobian conjecture, which claims that if  $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a polynomial map whose Jacobian is a non-zero constant, then  $F$  is a polynomial automorphism of  $\mathbb{C}^n$ , i.e.,  $F$  is one-to-one and satisfies (a) and (b). This dates back to 1939 [5] but is still unproved (in June 1994), even for  $n = 2$ . Its history, many references, and some partial results, can be found in [2].

Theorem A shows that the Jacobian conjecture would be proved if one could show, for polynomial maps  $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$ , that “locally one-to-one” implies “globally one-to-one.” This formulation of the problem points to an interesting difference between  $\mathbb{C}^n$  and  $\mathbb{R}^n$ : Serguey Pinchuk [8] has (surprisingly!) constructed a polynomial map  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  whose Jacobian has no zero in  $\mathbb{R}^2$  but which is not one-to-one. The difference is, of course, that on  $\mathbb{R}^n$  there are nonconstant polynomials without zeros, whereas this cannot happen on  $\mathbb{C}^n$ .

Theorem A is not new. In [7] Don Newman proved (a) with  $\mathbb{R}^2$  in place of  $\mathbb{C}^n$ . In [3] this was extended to  $\mathbb{R}^n$ , for arbitrary  $n$ , with the aid of a good dose of homology theory; that paper also contains a brief sketch of the analogous result for maps from  $k^n$  to  $k^n$ , for arbitrary algebraically closed fields  $k$ . Ax [1; Th. 2] extended this to morphisms of algebraic varieties, using nonprincipal ultraproducts of fields. Theorem (2.1) on p. 294 of [2] lists eight (mostly algebraic) conditions on polynomial maps  $F$  that are equivalent; Theorem A is one of those equivalences:  $F$  is one-to-one if and only if  $F$  is an automorphism.

I believe that the proof given here is much simpler than any of the above. (Proof: I have no trouble understanding it.) It uses two facts from complex analysis:

**Fact 1.** *If (i)  $u, v: \mathbb{C}^n \rightarrow \mathbb{C}$  are polynomials with no common factor of positive degree,*

*(ii)  $\Omega$  is an open subset of  $\mathbb{C}^n$ , and*

*(iii)  $v(p_0) = 0$  at some point  $p_0$  in  $\Omega$ ,*

*then  $\Omega$  contains points  $p$  at which  $v(p) = 0$  but  $u(p) \neq 0$ .*

This must be prehistoric. A proof can be found on pp. 14, 15 of [11]. Note that it fails on  $\mathbb{R}^n$ .

*Example:*  $u(x, y) = x^2 + y^2$ ,  $v(x, y) = x^2 + (y - x)^2$ .

**Fact 2.** *If  $F$  satisfies the hypothesis of Theorem A, then the Jacobian of  $F$  is  $\neq 0$  at every point of  $\mathbb{C}^n$ .*

This is in fact true for holomorphic maps from open sets in  $\mathbb{C}^n$  into  $\mathbb{C}^n$  that are locally one-to-one, and it used to be a fairly difficult theorem (see, for instance, [6; pp. 86–88]) until Jean-Pierre Rosay published a truly simple proof [9].

Combined with the inverse function theorem (Th. 9.24 in [10]), Fact 2 implies what will actually be used, namely:

*The range  $F(\mathbb{C}^n)$  of  $F$  is an open subset of  $\mathbb{C}^n$ .*

(Remark: That  $F(\mathbb{C}^n)$  is open is also an immediate consequence of Brouwer's "Invariance of Domain" theorem, concerning continuous one-to-one maps from  $\mathbb{R}^N$  into  $\mathbb{R}^N$  [4; p. 95] but that theorem is much more difficult than the route via Fact 2.)

We now start the proof.

Let  $f_1, \dots, f_n$  be the components of  $F$ , and let  $k$  be the subfield of  $\mathbb{C}$  generated by the coefficients of the polynomials  $f_i$ . Since  $k$  is countable, there are only countably many polynomials with coefficients in  $k$ . The union of their zero-sets (ignoring the zero-polynomial) is thus a countable union of closed sets without interior, hence cannot cover the complete metric space  $\mathbb{C}^n$ . It follows that there is a point  $\xi$  in  $\mathbb{C}^n$ , fixed from now on, with the following property:

(\*) *If  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  is a polynomial with coefficients in  $k$ , and  $f(\xi) = 0$ , then  $f(z) = 0$  for every  $z$  in  $\mathbb{C}^n$ .*

Put  $\eta = F(\xi)$ .

**Claim.** *The extension fields*

$$k(\eta) = k(\eta_1, \dots, \eta_n)$$

and

$$k(\eta, \xi) = k(\eta_1, \dots, \eta_n, \xi_1, \dots, \xi_n)$$

are equal.

Here  $k(\eta)$  is the smallest subfield of  $\mathbb{C}$  that contains  $k$  and  $\eta_1, \dots, \eta_n$ , and similarly for  $k(\eta, \xi)$ .

If the claim is false, there is an isomorphism  $\varphi$  of  $k(\eta, \xi)$  into  $\mathbb{C}$  that fixes every element of  $k(\eta)$  but moves some  $\xi_i$ . (See the lemma at the end of the paper.) Put

$$\omega = (\varphi(\xi_1), \dots, \varphi(\xi_n))$$

and note that  $\omega \neq \xi$ .

Since  $f_j(\xi) = \eta_j$  is in  $k(\eta)$  and the coefficients of  $f_j$  are in  $k$ , we have, for  $1 \leq j \leq n$ ,

$$f_j(\xi) = \varphi(f_j(\xi_1, \dots, \xi_n)) = f_j(\varphi(\xi_1), \dots, \varphi(\xi_n)) = f_j(\omega).$$

Hence  $F(\xi) = F(\omega)$ , which contradicts the assumption that  $F$  is one-to-one. This proves the claim.

In particular, each  $\xi_j$  is in  $k(\eta)$ . This means that there are polynomials  $u_j, v_j$ , with coefficients in  $k$ , and without common factors of positive degree, such that  $v_j(\eta) \neq 0$  and

$$\xi_j = u_j(\eta) / v_j(\eta) \quad (1 \leq j \leq n). \quad (1)$$

Thus  $\xi_j v_j(F(\xi)) - u_j(F(\xi)) = 0$ . Property (\*) implies now that

$$z_j v_j(F(z)) = u_j(F(z)) \quad (1 \leq j \leq n, z \in \mathbb{C}^n). \quad (2)$$

Put  $\Omega = F(\mathbb{C}^n)$ . We saw, as a consequence of Fact 2, that  $\Omega$  is open. If  $v_j$  had a zero in  $\Omega$ , Fact 1 would imply that there is a point in  $\Omega$  where  $v_j = 0$  but  $u_j \neq 0$ , contradicting (2).

Hence  $v_j \circ F : \mathbb{C}^n \rightarrow \mathbb{C}$  is a polynomial without zeros, hence is constant, hence each  $v_j$  is constant. Without loss of generality,  $v_j = 1$ . Putting

$$G = (u_1, \dots, u_n), \quad (3)$$

(2) becomes

$$G(F(z)) = z \quad \text{for all } z \text{ in } \mathbb{C}^n. \quad (4)$$

Hence  $F(G(F(z))) = F(z)$ . This says that  $F \circ G$  is the identity map on  $\Omega$ . If two polynomials agree on  $\Omega$ , they agree on  $\mathbb{C}^n$ . Thus

$$F(G(w)) = w \quad \text{for all } w \text{ in } \mathbb{C}^n. \quad (5)$$

The theorem follows from (4) and (5), with  $F^{-1} = G$ .

**Lemma.** Suppose that  $\mathcal{F}$  is a subfield of  $\mathbb{C}$ ,  $\xi_1, \dots, \xi_m$  are in  $\mathbb{C}$ , and  $\mathcal{F}_1 = \mathcal{F}(\xi_1, \dots, \xi_m)$ . Then either  $\mathcal{F}_1 = \mathcal{F}$ , or there is an isomorphism  $\varphi$  of  $\mathcal{F}_1$  into  $\mathbb{C}$  that fixes every element of  $\mathcal{F}$  but moves at least one  $\xi_i$ .

*Proof:* Assume  $\mathcal{F}_1 \neq \mathcal{F}$ . Then there is a nonempty subset of  $\{\xi_1, \dots, \xi_m\}$ , say  $(\xi_1, \dots, \xi_j)$  (after reordering) that is minimal with respect to the property

$$\mathcal{F}_1 = \mathcal{F}(\xi_1, \dots, \xi_j).$$

Put  $\mathcal{F}_2 = \mathcal{F}(\xi_1, \dots, \xi_{j-1})$ . (This is  $\mathcal{F}$  when  $j = 1$ .) Then

$$\mathcal{F} \subset \mathcal{F}_2 \subsetneq \mathcal{F}_2(\xi_j) = \mathcal{F}_1.$$

Let  $\varphi$  fix every element of  $\mathcal{F}_2$  and choose  $\varphi(\xi_j)$  as follows:

If  $\xi_j$  is transcendental over  $\mathcal{F}_2$ , let  $\varphi(\xi_j)$  be any complex number  $\neq \xi_j$  that is also transcendental over  $\mathcal{F}_2$  (such as  $1 + \xi_j$ ).

If  $\xi_j$  is algebraic over  $\mathcal{F}_2$ , with minimal polynomial  $p(x)$ , let  $\varphi(\xi_j)$  be another root of  $p(x)$ .

To every  $w$  in  $\mathcal{F}_1$  corresponds a rational function  $r$ , with coefficients in  $\mathcal{F}_2$ , such that  $w = r(\xi_j)$ . Setting  $\varphi(w) = r(\varphi(\xi_j))$  gives the desired isomorphism.

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