אוניברסיטת
בן-גוריון בנגב

The Department of Mathematics

2016–17–B term

Course Name Derived categories IV

Course Number 201.2.0364

Course web page
https://www.math.bgu.ac.il/~amyekut/teaching/2016-17/der-cats-IV/course_page.html

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Office Hours https://math.bgu.ac.il/en/teaching/hours

Abstract

Requirements and grading[1]

[1] Information may change during the first two weeks of the term. Please consult the webpage for updates

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Derived Categories IV

Course in 2nd Semester 2016-17

Course description. The course is a continuation of *Derived Categories I, II and III*, that were given in the previous semesters. In the new course we will concentrate on more specialized topics (see below), most notably *Grothendieck duality for schemes.*

Audience. This is an advanced course, aimed at M.Sc. and Ph.D. students, post-docs and researchers. *Participants from outside the BGU community are welcome.* The lectures will be in English.

Prerequisites. It is expected that participants shall have a reasonable knowledge of the material covered in the earlier courses (which includes all the standard material on derived categories – see below).

A working knowledge of commutative algebra, ring theory and algebraic geometry is also required for this course.

Organization. The course will meet once a week for a 2 hour lecture.
Time: Wednesday 12:00-14:00
Location: building 58 room -101
First Meeting: 15 March 2017

Potential participants are urged to get in touch with the lecturer in advance. The course notes will be published on the arxiv, and very likely also as a book. Here is a link to the course web page:
http://www.math.bgu.ac.il/~amyekut/teaching/2016-17/der-cats-IV/course_page.html

On the subject. See the prevoius announcements on the course web page.

Content of the first three courses. We started with a review of categories (in general and abelian). After that we made an in-depth study of DG (differential graded) algebra: DG rings, DG modules, DG categories and DG functors. We introduced the DG category $\mathsf{C}(A, \mathsf{M})$ of DG A-modules in M, where A is a DG ring and M is an abelian category. This new framework includes in it both the category of unbounded complexes in M and the category of DG A-modules.

followed the original approach of Grothendieck in [RD = "Residue and Duality"], but with many more details, providing full proofs, and connecting everything to the DG background. We proved that the homotopy category $\mathsf{K}(A, \mathsf{M})$ of $\mathsf{C}(A, \mathsf{M})$ is pretriangulated. We also proved that for any DG functor $F : \mathsf{C}(A, \mathsf{M}) \to \mathsf{C}(B, \mathsf{N})$, the induced functor $F : \mathsf{K}(A, \mathsf{M}) \to \mathsf{K}(B, \mathsf{N})$ is triangulated.

We then made a thorough study of localization of categories. The localization of $\mathsf{K}(A, \mathsf{M})$ with respect to the quasi-isomorphisms is the derived category $\mathsf{D}(A, \mathsf{M})$. The category $\mathsf{D}(A, \mathsf{M})$ is pretriangulated, and the localization functor $\mathrm{Q} : \mathsf{K}(A, \mathsf{M}) \to \mathsf{D}(A, \mathsf{M})$ is both a triangulated functor and an Ore localization.

Next we talked about derived functors. To enable a precise treatment of this concept, we introduced some 2-categorical language. We proved that any triangulated functor $F : \mathsf{K}(A, \mathsf{M}) \to \mathsf{E}$, with target an arbitrary pretriangulated category E, has unique left and right derived functors $\mathrm{L}F, \mathrm{R}F : \mathsf{D}(A, \mathsf{M}) \to \mathsf{E}$.

Existence of derived functors relies on the availability of resolutions. We defined K-projective, K-injective and K-flat resolutions in $\mathsf{K}(A, \mathsf{M})$. Then we proved existence of these resolutions in several important algebraic situations.

Derived bifunctors were the next topic. We gave sufficient conditions for their existence. There was a detailed treatment of the important derived bifunctors $\mathrm{RHom}(-, -)$ and $(- \otimes^{\mathrm{L}} -)$.

The last general topic was cohomological dimension of triangulated functors and the way-out yoga of Grothendieck, expanding the ideas from [RD].

We then started studying the first of the more specialized topics: dualizing complexes over noetherian commutative rings. After definining this concept, we proved existence of dualizing complexes over a commutative ring A that is essentially finite type over a regular ring. We proved the weak uniqueness of dualizing complexes (following [RD]). After that we studied residue complexes over commutative rings. We ended the semester with an initial discussion of rigid complexes (in the sense of Van den Bergh).

The notes from parts I-II of the course are available as the first half of "Derived Categories – a Textbook" the arxiv:
`http://arxiv.org/abs/1610.09640v1`.
The notes from part III of the course (unedited) are available on the course web page:
`http://www.math.bgu.ac.il/~amyekut/teaching/2016-17/der-cats-III/course_page.html`

Topics for the fourth course. Here is a tentative list of topics. This material is all taken from research papers.

(1) **Rigidity, residues and duality over commutative rings.** We will study rigid residue complexes. We will prove their uniqueness and existence, the trace and localization functoriality, and the ind-rigid trace homomorphism.
(2) **Derived categories in geometry.** This topic concerns geometry in the wide sense. We will prove existence of K-flat and K-injective resolutions, and talk about derived direct and inverse image functors.
(3) **Rigidity, residues and duality over schemes.** The goal is to present an accessible approach to global Grothendieck duality for proper maps of schemes. This approach is based on rigid residue complexes and the ind-rigid trace. We will indicate an extension of this approach to DM stacks.
(4) **Derived categories in noncommutative ring theory.** Subtopics: dualizing complexes, tilting complexes, the derived Picard group, derived Morita theory, survey of noncommutative and derived algebraic geometry.

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Course topics

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