Construction of Certain Small Representations for SO_{2m}

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0 Introduction

In recent years there is a growing interest in studying small representations and their application to the Langlands Conjectures. The most well known example of this type is the theta representation. One uses this representation to construct liftings from symplectic groups, or their double cover, to orthogonal groups and vice versa. This representation, which is defined on the double cover of the symplectic group, is associated with the minimal orbit. This means that this representation supports very "few" Fourier coefficients. This is in fact, the key ingredient which makes this and all other constructions work.

In the last few years there are many more examples of constructions of small representations. In [GRS1] the minimal representation for simply laced groups was constructed. Then, in [GRS4], these representations were used to construct a tower of liftings for the exceptional group G_2. A similar idea was used in [GRS5] to construct the descent map from GL_{2n} to the classical groups. A more recent example of such construction was studied in [BFG]. In that example the authors constructed a small representation for the double cover for odd orthogonal groups.

The constructions of many of these example is done by studying residues of Eisenstein series. One of the key steps is to understand the structure of the unramified constituent of these residues.

In this paper we construct some new examples of small representations for the group SO_{2m}. In fact we extend the construction done in [GRS1] for the minimal representation for these groups. As mentioned above, and as explained in [GRS2] for symplectic group, one can associate to a unipotent orbit a set of Fourier coefficients. In [GRS2] it is also explained how to associate to an automorphic representation a set of unipotent classes.

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Our main goal in this paper is a construction of small representations for SO_{2m} which are associated with the unipotent classes $(2^{2i}1^{2(m-2i)})$ for all $0 \leq i \leq m'$ where $m' = m/2-1$ if m is even and $m' = (m-1)/2-1$ if m is odd . This construction is done by studying the residues of the Siegel Eisenstein series of SO_{2m}. More precisely, let P_m denote one of the two standard parabolic subgroups of SO_{2m} whose Levi part is GL_m. Let $E_m(g,s)$ denote the Siegel Eisenstein series as defined in Section 2. In [KR] the poles of this Eisenstein series were determined. This is our starting point. In Section 2 we establish the fact that all these residues are square integrable and in case F is a totally real field we show that they fact are irreducible. We then study the unramified nature of each residue. We prove that at unramified places the residue representation is a constituent of two different maximal parabolic subgroups. In Section 3 we use this property to show that these residues supports "few" Fourier coefficients. Our main result in that Section is Theorem A. In that Theorem we prove that if $\mathcal{O}$ is any unipotent class which is greater or equal to (31^{2m-3}) then all the residues which we constructed has no nontrivial Fourier coefficient which corresponds to $\mathcal{O}$. We also prove some vanishing results corresponding to the unipotent classes $(2^{2j}1^{2(m-2j)})$ with $j > i$. In Section 4 we study some non-vanishing properties of these residues. In Theorem B we prove that these representations have nonzero Fourier coefficients corresponding to the unipotent class $(2^{2i}1^{2(m-2i)})$.

A first possible application to the construction of these representations, is to obtain more examples of liftings between split orthogonal groups. We hope to be able to use these liftings to construct new examples of CAP representations. However, it is possible that we will need to study other forms, not the split one, in order to achieve these liftings. As a second possible application we hope to use these residues to study the poles of the standard L-function of a cuspidal irreducible representation of $SO_{2m}(\mathbf{A})$ which is not necessary generic. We hope to relate the existence of a pole to the non-vanishing of a certain period, thus extending some of the results of [GRS6] to the non-generic case. In [GRS1], it is proved that the minimal representation, in our case when $i = 1$, is the theta lift of the identity representation from SL_2 to SO_{2m}. As indicated in [HL] it is expected that the representations we construct will be the theta lift of the identity function from Sp_{2i} to SO_{2m}. We hope to address these issues in the very near future.

After completing this paper it was brought to our attention that Theorem A can be also deduced from the local results of [We03].

1 Basic Definitions and Properties

Let SO_{2m} be the split orthogonal group. In term of matrices we will always consider this group with respect to the form represented by the matrix with ones on the other diagonal. Let F be a global field and $\mathbf{A}$ its ring of adeles. Let P_m be the maximal parabolic of SO_{2m} whose Levi part is GL_m. Let $E_m(g,s) = \sum_{SO_{2m}(F)/P_m(F)} f_s(\gamma g)$ be the Siegel Eisenstein series where $f_s \in Ind_{P_m(\mathbf{A})}^{SO_{2m}(\mathbf{A})} \delta_{P_m}^s$ is a standard section. It is well known that the series $E_m(g,s)$ converges on a right half plane. Moreover, it has simple poles at the points $s_{m,i} = (m-1-i)/(m-1)$ with $0 \le i \le m'$ where $m' = \frac{m}{2} - 1$ if m is even and $m' = \frac{(m-1)}{2} - 1$ if m is odd . Each of these possible poles can be realized by an appropriate choice of a section f. The proof of all this is contained in [KR].

Let $E_{m,i}(g)$ denote the residue of $E_m(g,s)$ at the point $s_{m,i}$. We shall denote by $E_{m,i}$ the corresponding representation. Here and elsewhere we will assume that $0 \le i \le m'$. When $i = 0$ we obtain the constant representation. In [GRS1] the residual representation $E_{m,1}$ is used to define the minimal automorphic representation of SO_{2m}. It is also shown there that the representation $E_{m,1}$ is square integrable. Our first result is to extend this property to the residues at the other points.

We have

Theorem 1: *The residue representations $E_{m,i}$ are all square integrable.*

Proof: We argue by induction, the case $m = 3$ being the base of the induction for which the result is valid. Let $Q_{m,1}$ be the parabolic subgroup of SO_{2m} whose Levi part is $GL_1 \times SO_{2m-2}$. Let $U_{m,1}$ denote its unipotent radical. From [KR] we have the following inductive formula

$$C_{P_1}(E_m((a,h),s)) = |a|^{(m-1)s} E_{m-1}(h, \frac{m-1}{m-2}s) + |a|^{(m-1)(1-s)} E_{m-1}^a(h, \frac{m-1}{m-2}s - \frac{1}{m-2}) \quad (1)$$

Here $C_{P_1}(E_m(g,s))$ denotes the constant term along $U_{m,1}$ and $(a,h) \in GL_1 \times SO_{2m-2}$. Also, $E_m^a(g,s)$ is the Eisenstein Series corresponding to P_m^a, the parabolic subgroup associated to P_m. Since a similar formula is applicable for $E_m^a(g,s)$ it is readily verified by induction that this series has the same poles and the same exponents as $E_m(g,s)$.

Since our Eisenstein series is concentrated on the Borel, the criterion of square integrability of [MW] asserts that if η is an exponent of our representation then $\eta\delta_B^{-\frac{1}{2}}$ should be of the form $\sum_{\alpha\in\Delta} a_\alpha \alpha$ with $a_\alpha < 0$ for all simple roots α. To check that the set $P(m,i)$ of exponents of $E_{m,i}$ obey this, we note that an easy calculation shows that any

$$\chi(t(a_1,...,a_m)) = \Pi_{i=1}^m |a_i|^{y_i}$$

with $\forall 1 \le i \le m, y_i \ge 0; y_1 > 0; y_m = 0$ can be represented as a sum along the simple roots, $\chi = \sum_\alpha n_\alpha \alpha$ with n_α all positive.

Now let $P(m,i)$ denotes the set of vectors $\vec{v}=(v_1,...,v_m)$ in $\mathbb{Z}^m$ such that $\eta(t(a_1,...,a_m)) = \Pi_{i=1}^m |a_i|^{v_i}$ is an exponents of the automorphic form $E_{m,i}$. The inductive formula above implies the following algorithm to obtain $P(m,i)$ recursively.
For $j\in\mathbb{Z}, \vec{v}=(v_1,...,v_k)\in\mathbb{Z}^k$, let us denote by $j\times\vec{v}$ the vector $(j,v_1,...,v_k)\in\mathbb{Z}^{k+1}$.
Clearly, $P(m,0)$ consists of the zero vector $\vec{0}\in\mathbb{Z}^m$ and $P(m,i)$ is formed from $P(m-1,i-1)$ and $P(m-1,i)$ as follows: Assume $1\le i\le [\frac{m}{2}]-1$. If m is odd or if m is even and $i\neq[\frac{m}{2}]-1$ we have

$$P(m,i) = (m-i-1)\times P(m-1,i-1)\cup(i)\times P(m-1,i)$$

In case m is even and $i=[\frac{m}{2}]-1$ we have

$$P(m,i) = (m-i-1)\times P(m-1,i-1)$$

Note, that the three conditions above are to be checked for $\delta_B^{\frac{1}{2}}\eta^{-1}$ where η is an exponent of $E_{m,i}$. Since $\delta_B^{\frac{1}{2}}(t(a_1,...,a_m)) = a_1^{m-1}a_2^{m-2}...a_1^0$ we will check that for any $\vec{v}\in P(m,i)$ that the vector $\vec{\rho_m}-\vec{v}$, with $\vec{\rho_m}=(m-1,m-2,...,0)$ satisfy the three conditions mentioned above. Indeed, we may write $\vec{v}=(j,u_1,...,u_{m-1})$ with $\vec{u}=(u_1,...,u_{m-1})\in P(m-1,i-1)\cup P(m-1,i)$ and $j<m$. note that $\vec{\rho_m}-\vec{v}=(m-1-j)\times(\vec{\rho_{m-1}}-\vec{u})$ and the three conditions are verified by induction.

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In the next Theorem we extend the result of [GRS1] regarding the irreducibility of the residual representations. For the next theorem we assume that F is a totally real number field although we expect that the theorem, in fact the proof, is valid without this assumption. The problem being lack of information about the exact structure of degenerate principal series for the complex group $SO(2m,C)$. At any rate we have

Theorem 2: *Assume that F is totally real field. Then the representations $E_{m,i}$ are all irreducible.*

Proof: We use an argument from [La-Ra]. The automorphic representation $E_{m,i}$ generated by the residual Eisenstein series factors through $Ind_{P_m(\mathbf{A})}^{G(\mathbf{A})}\delta_P^{s_{m,i}} = \otimes_v I_v(s_{m,i})$ with $I_v(s_{m,i}) = Ind_{P_m(F_v)}^{G(F_v)}\delta_{P_v}^{s_{m,i}}$. Now, the local representation $I_v(s_{m,i})$ has a unique irreducible quotient. In the p-adic case this follows from [BJ], (proposition 5.2) while in the archimedean case this follows from [John] (cf. [Lee-Loke]). Thus $I_v(s_{m,i})$ has a unique semi-simple quotient, and a-fortiori a unique unitarizable quotient, and the latter is irreducible. Hence, the same is true for $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\delta_P^{s_{m,i}}$. On the other hand $E_{m,i}$ consists of square-integrable forms, and hence it is unitarizable. Thus $E_{m,i}$ is isomorphic to the unique unitarizable quotient of $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\delta_P^{s_{m,i}}$ and hence irreducible.

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We end this section with some local result. Let F be a local place where the representation $E_{m,i}$ is unramified. By abuse of notations we shall denote by $E_{m,i}$ this local representation. We shall also suppress the reference to F in the notations. Clearly this representation is a constituent of the induced representation $Ind_{P_m}^{SO_{2m}}\delta_{P_m}^{s_i}$. We will now show that in fact, this representation is a constituent of another induced representation. Let $Q_{m,i}$ denote the maximal parabolic subgroup of SO_{2m} whose Levi part is $GL_i \times SO_{2(m-i)}$. We now prove

Proposition 1: *The local representation $E_{m,i}$ is a constituent of $Ind_{Q_{m,i}}^{SO_{2m}}\chi_i\delta_{Q_i}^{1/2}$. Here χ_i is an unramified character of $Q_{m,i}$ defined on the GL_i component and extended trivially to $Q_{m,i}$.*

Proof: Let B_m be the Borel subgroup of SO_{2m}. We parameterize the maximal torus of SO_{2m} as $t = diag(a_1, a_2, ..., a_m, a_m^{-1}, ..., a_1^{-1})$. An easy computation shows that $\delta_{P_m}(t) = |a_1a_2...a_m|^{m-1}$ and $\delta_{B_m}(t) = |a_1|^{2m-2}|a_2|^{2m-4}...|a_m|^0$. We conclude that

$$\delta_{B_{GL_i}\times B_{SO_{2(m-i)}}} = |a_1|^{i-1}|a_2|^{i-3}\cdots|a_i|^{-i+1}|a_{i+1}|^{2m-2i-2}\cdots|a_m|^0$$

and thus

$$\delta^{1/2}_{B_{GL_i}\times B_{SO_{2(m-i)}}} = |a_1|^{(i-1)/2}|a_2|^{(i-3)/2}\cdots|a_i|^{(-i+1)/2}|a_{i+1}|^{m-i-1}\cdots|a_m|^0$$

Here, for a group G, we denote by B_G its Borel subgroup.

Let $\mu = \delta_{B_m}^{-1/2}\delta_{P_m}^{s_i}$. Thus $E_{m,i}$ is a sub-representation of $Ind_{B_m}^{SO_{2m}}\mu\delta_{B_m}^{1/2}$. We denote by $\nu_\alpha = |det|^\alpha\delta^{1/2}_{B_{GL_i}}\delta^{1/2}_{B_{SO_{2(m-i)}}}$ where $|det|^\alpha$ is defined as a character of GL_i. Define the following Weyl element $w_i = diag(J_i, J_{m-i}, J_{m-i}, J_i)$ in SO_{2m} where J_k denotes the $k \times k$ matrix with ones on the other diagonal, and zero elsewhere. A simple computation shows that twisting μ by w_i gives us the character $\nu_{-(i+1)/2}$. This means that $E_{m,i}$ is a constituent of $Ind_{B_m}^{SO_{2m}}(\nu_{-(i+1)/2}\delta^{1/2}_{B_{GL_i}}\delta^{1/2}_{B_{SO_{2(m-i)}}})\delta^{1/2}_{B_m}$. Induction by stages yields that $E_{m,i}$ is a constituent of $Ind_{Q_{m,i}}^{SO_{2m}}\nu_{-(i+1)/2}\delta^{1/2}_{Q_{m,i}}$. From this the proposition follows. ∎

2 Vanishing of Fourier Coefficients of $E_{m,i}$

In this section we will start our study of Fourier coefficients of the representations $E_{m,i}$. It will be convenient to do it using the language of unipotent classes. We will now recall some basic facts about these notions and their relations to Fourier coefficients.

In [GRS2] the connection between unipotent classes and Fourier coefficients of automorphic representations was studied in the case of the symplectic group. The correspondence between these two notions is defined in a similar way in the case of orthogonal groups. We now briefly explain this. Let $\mathcal{O} = (n_1 \cdots n_r)$ be a unipotent orbit for SO_{2m}. This means that

$n_1 \geq n_2 \cdots \geq n_r$ and $n_1 + \cdots + n_r = 2m$. As explained in [C-M] each even number occurs with even multiplicity. To such a unipotent orbit we associate a one dimensional torus element in SO_{2m} as follows. To each n_i we consider the set $\{t^{n_i-1}, t^{n_i-3}, \cdots, t^{-(n_i-3)}, t^{-(n_i-1)}\}$. Joining all these sets to one set and writing the elements in decreasing powers of t we obtain a torus element of SO_{2m} which we denote by $h_{\mathcal{O}}(t)$. Let V denote the maximal unipotent subgroup of SO_{2m} which consists of upper triangular matrices. The torus $h_{\mathcal{O}}(t)$ acts on V. On each one dimensional unipotent subgroup of V which corresponds to a positive root, the torus $h_{\mathcal{O}}(t)$ acts by multiplication by t^k for some $k \geq 0$. We let $V(\mathcal{O})$ denote the unipotent subgroup of V which consists of all one dimensional subgroups such that $h_{\mathcal{O}}(t)$ acts by multiplication of t^k with $k \geq 2$. As explained in [GRS2] to the group $V(\mathcal{O})$ we associate a family of characters $\psi_{V(\mathcal{O})}$. In this way to each unipotent class $\mathcal{O}$ we can associate a family of Fourier coefficients defined by

$$\int\limits_{V(\mathcal{O})(F)\backslash V(\mathcal{O})(\mathbf{A})} \varphi(vg)\psi_{V(\mathcal{O})}(v)dv \qquad (2)$$

Here φ is a vector in some automorphic representation of the group $SO_{2m}(\mathbf{A})$.

Finally, as explained in [C-M], there is a partial ordering which is defined on the set of all unipotent classes of the group SO_{2m}.

Our main result in this section is

Theorem A: *Let $\mathcal{O}$ be any unipotent classes which is either greater or equal to the unipotent class (31^{2m-3}) or greater then $(2^{2i}1^{2(m-2i)})$. Then the representation $E_{m,i}$ has non nonzero Fourier coefficient with respect to these unipotent orbits. In other words, the integrals (2) in which $\mathcal{O}$ is a unipotent classes as cited, vanish for any choice of $\varphi \in E_{m,i}$.*

We will prove Theorem A in several steps. We start with

Proposition 2: *Let $\mathcal{O}$ be one of the following unipotent classes:*
(a) $\mathcal{O} = (31^{2m-3})$.
(b) $\mathcal{O} = (2^{2j}1^{2(m-2j)})$ with $j > i$.
(c) $\mathcal{O} = ((2n_1)^{2l_1} \cdots (2n_k)^{2l_k}1^t)$ where $t \geq 0$ and $n_1 > 1$.
Then the representation $E_{m,i}$ has no nonzero Fourier coefficient corresponding to the unipotent orbit $\mathcal{O}$.

Proof: To prove the proposition, we will show that a local unramified component of the representation $E_{m,i}$ can not have a nonzero functional which corresponds to one of the above unipotent classes. To do this we follow the proof of Lemma 2 in [GRS3].

For this proof only, let F be a local field where the residue representation is unramified. As in the proof of Proposition 1 we will write $E_{m,i}$ for the local component. From that proposition it follows that $E_{m,i}$ is a constituent of two induced representations. From the exactness of the Jacquet Functor it is enough to show that at least one of these induced representations has no nonzero local functional which corresponds to one of the above unipotent orbits. This is done by the Bruhat theory. Thus we have to prove the following. Suppose that $E_{m,i}$ is a constituent of $Ind_R^{SO_{2m}}\chi\delta_R^{1/2}$ where R is one of the parabolic subgroups P_m or $Q_{m,i}$ as defined in section one. Then we have to prove that the space of double cosets $R\backslash SO_{2m}/V(\mathcal{O})$ has no admissible elements. In our context, a double coset is not admissible if for some representative γ there exists $v\in V(\mathcal{O})$ such that $\psi_{V(\mathcal{O})}(v)\neq 1$ and $\gamma v\gamma^{-1}\in R$.

We start with case (a). Let $e_{i,j}$ be the standard basis for $Mat_{2m,2m}$. We let $U_1 = \{I_{2m}+r_{1,2}e'_{1,2}+\cdots+r_{1,2m-1}e'_{1,2m-1}+ze_{1,2m}\}$ where $e'_{p,j}=e_{p,j}-e_{2m-j+1,2m-p+1}$ and z depends on the variables $r_{1,j}$. Thus U_1 is the unipotent radical of the parabolic subgroup whose Levi part is $GL_1\times SO_{2(m-1)}$. We define a character $\psi_{U_1}^a(u)=\psi(u_{1,m}+au_{1,m+1})$ where $a\in F^*$. It follows from the above description of the connection between unipotent orbits and Fourier coefficients that the integrals

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E_{m,i}(ug)\psi_{U_1}^a(u)du \tag{3}$$

describe the integrals (2) for the case $\mathcal{O}=(31^{2m-3})$. Thus we have to show that (3) are zero for all choice of data. We use the above local argument where we choose $R=P_m$. That is, R is the parabolic subgroup whose Levi part is GL_m. Clearly every representative of the double cosets $P_m\backslash SO_{2m}/U_1$ is of the form wv where w is a Weyl element and $v\in V$. Denote $u_1(x)=I_{2m}+xe'_{1,m}$ and $u_2(y)=I_{2m}+ye'_{1,m+1}$. Then from the definition $u_1(x),u_2(y)\in U_1$ and $\psi_{U_1}^a(u_1(x))=\psi(x)$ and $\psi_{U_1}^a(u_2(y))=\psi(ay)$. Assume first that the representative is a Weyl element. A simple computations shows that if $w=(w_{k,j})$ is in SO_{2m} then $w_{k,j}=w_{2m+1-k,2m+1-j}$. A matrix multiplication implies that either $wu_1(x)w^{-1}\in P_m$ for all $x\in F$ or $wu_2(y)w^{-1}\in P_m$ for all $y\in F$. From this it follows that w cannot be an admissible representative. Let U_1^{m+1} be the subgroup of U_1 such that $r_{1,2}=\cdots=r_{1,m+1}=0$. Clearly, the restriction of $\psi_{U_1}^a$ to U_1^{m+1} is trivial. Let wv be a double coset representative. It is an easy matrix multiplication to show that given $v\in V$, there is a $u'\in U_1^{m+1}$ such that $vu_1(x)u_2(y)u'v^{-1}=u_1(x)u_2(y)$. Using this we are basically reduced to the case of a Weyl element representative which we showed that it is not admissible. Thus wv is not admissible and we are done.

Next consider case (b). We first describe the group $V(\mathcal{O})$ in this case. For any $j>1$ define the unipotent group

$$L_j=\left\{\begin{pmatrix} I_{2j} & & X\\ & I_{2(m-2j)} & \\ & & I_{2j}\end{pmatrix}\ :\ JX^t+XJ=0\right\}$$

We define a character of L_j by $\psi_j(l)=\psi(tr'x)$ where $tr'x=x_{1,1}+\cdots+x_{j,j}$. Thus integral (2) becomes in this case

$$\int\limits_{L_j(F)\backslash L_j(\mathbf{A})} E_{m,i}(lg)\psi_j(l)dl \tag{4}$$

In other words this is integral (2) for the unipotent class $\mathcal{O}=(2^{2j}1^{2(m-2j)})$. To handle this case we use $R=Q_{m,i}$. As above, we can choose the representatives for $Q_{m,i}\backslash SO_{2m}/L_j$ to be wv where w is a Weyl element and $v\in V$. For all $1\le k\le j$ we define $u_k(y_k)=I_{2m}+y_ke'_{k,2(m-j)+k}$. We also define L'_j to be the subgroup of L_j such that $x_{p,q}=0$ for all $p\ge q$. Clearly, the restriction of ψ_j to L'_j is trivial, and $\psi_j(u_k(y_k))=\psi(y_k)$. Denote by $U_{m,i}$ the unipotent radical of $Q_{m,i}$ and let $U^-_{m,i}$ denote the opposite unipotent group to $U_{m,i}$. Assume first that the representative is a Weyl element w. Then it is easy to see that if $wu_k(y_k)w^{-1}\notin U^-_{m,i}$ then $wu_k(y_k)w^{-1}\in Q_{m,i}$. Thus we have to prove that there is $1\le k\le j$ such that $wu_k(y_k)w^{-1}\notin U^-_{m,i}$. To see this notice that as we vary k all variables y_k and $-y_k$ occur in different rows and columns. In other words, if $k_1\ne k_2$ then the variables y_{k_i} in the matrices $u_{k_i}(y_{k_i})$ are in different rows and columns. Since conjugation by w permutes the rows and columns it follows that the same thing happens for all matrices $wu_k(y_k)w^{-1}$ with $1\le k\le j$. However, from the definition of $U^-_{m,i}$ it follows that at most i of the matrices $wu_k(y_k)w^{-1}$ can be in $U^-_{m,i}$. Since $j>i$ it follows that at least for one k we have $wu_k(y_k)w^{-1}\in Q_{m,i}$. Hence w is not admissible. Next we consider the general case, that is when the representative is wv. This is done as in case (a). More precisely, a matrix multiplication shows that given v there is an $l\in L'_j$ such that $vu_1(y_1)\cdots u_j(y_j)lv^{-1}=u_1(y_1)\cdots u_j(y_j)$. Since we saw that w cannot be admissible, it follows that wv is not admissible.

Finally we consider case (c). Since this case is similar to case (a) we just indicate the group $V(\mathcal{O})$ and the relevant characters in this case. Recall that $\mathcal{O}=((2n_1)^{2l_1}\cdots(2n_k)^{2l_k}1^t)$. Since this is a unipotent orbit for SO_{2m} the sum has to be even and hence t is even . Following the general description as explained in [GRS2] we consider the parabolic subgroup of SO_{2m} whose Levi part is given by

$$GL_{2l_1}^{n_1-n_2}\times GL_{2(l_1+l_2)}^{n_2-n_3}\times\cdots\times GL_{2(l_1+\cdots+l_{k-1})}^{n_{k-1}-n_k}\times GL_{2(l_1+\cdots+l_k)}^{n_k}\times SO_t$$

The actual size and the number of times that each GL_j occurs in the above group, is not important to us. All that matters is that this Levi group has the form $GL_{r_1}\times\cdots\times GL_{r_l}\times SO_t$ for some $l>1$. The reason that $l>1$ follows from the fact that $n_1>1$.

Let $\bar r=(r_1,\cdots,r_l,t)$. Let $Q_{\bar r}$ denote the parabolic subgroup of SO_{2m} whose Levi part is $GL_{r_1}\times\cdots\times GL_{r_l}\times SO_t$. We denote its unipotent radical by $U^0_{\bar r}$. We define a subgroup $U_{\bar r}$ of $U^0_{\bar r}$ as follows. If $t=0$ we define $U_{\bar r}=U^0_{\bar r}$. In the case when $t\ne 0$ we first notice that $U^0_{\bar{r}}/[U^0_{\bar{r}},U^0_{\bar{r}}]$ can be identified with $Y = Mat_{r_1\times r_2} \oplus \cdots \oplus Mat_{r_{l-1}\times r_l} \oplus Mat_{l_k\times t}$. For $y = (y_1,\cdots,y_{l-1},y_l) \in Y$ we let Y^0 denote the subgroup of all $y \in Y$ such that $y_l = 0$. We now define $U_{\bar{r}}$ to be the subgroup of $U^0_{\bar{r}}$ generated by the image of Y^0 in $U^0_{\bar{r}}$ and by $[U^0_{\bar{r}},U^0_{\bar{r}}]$. In both cases we have $V(\mathcal{O}) = U_{\bar{r}}$

Next we define a character $\psi_{U_{\bar{r}}}$ of $U_{\bar{r}}$. To do that we first identify $U_{\bar{r}}/[U_{\bar{r}},U_{\bar{r}}]$ with $X = Mat_{r_1\times r_2} \oplus\cdots\oplus Mat_{r_{l-1}\times r_l} \oplus Mat^0_{r_k}$ where $Mat^0_{r_k} = \{A \in Mat_{r_k} \ : \ A^tJ + JA^t = 0\}$ and J is the matrix with ones on the other diagonal and zero elsewhere. To define the character $\psi_{U_{\bar{r}}}$ we define it on $U_{\bar{r}}/[U_{\bar{r}},U_{\bar{r}}]$ and then extend it trivially to $U_{\bar{r}}$. Let $x = (x_1,\cdots,x_l,x^0_l)$ be an element in X. We define $\psi_{U_{\bar{r}}}(u) = \psi(tr(x_1 + \cdots + x_l) + tr'x^0_l)$ where for $x^0_l = (x^0_l[i,j])$ we set $tr'x^0_l = x^0_k[1,1] + \cdots + x^0_k[r_l/2, r_l/2]$. (Recall that r_l is even). These characters agree with the definition of $\psi_{V(\mathcal{O})}$ in this case.

To prove that (2) are zero for all choice of data, we use the same type of argument as in case (a) where we use $R = P_m$. Once again, any double coset representative is of the form wv. Let us just mention that the fact that $n_1 > 1$ implies that w is not admissible. Indeed, from the definition of $\psi_{V(\mathcal{O})}$ in this case we can find two one dimensional unipotent subgroups such that $\psi_{V(\mathcal{O})}$ restricted to these groups is not trivial and these groups are not commutative. Hence at least one of them must be conjugated into P_m. This follows from the fact that the unipotent radical subgroup of P_m, and hence its opposite group, are abelian. The general case is done in a similar way as in case (a) and will be omitted. This completes the proof of the proposition. ∎

Next we define certain subgroups of the group U_1 which was defined in the proof of Proposition 2 part (a). For all $2 \le k$ we define $U_1^k = \{u = (u_{i,j}) \in U_1 \ : \ u_{1,i} = 0, \quad 2 \le i \le k\}$. We prove

Lemma 1: *For all $2 \le k \le m-1$ the integral*

$$\int\limits_{U_1^k(F)\backslash U_1^k(\mathbf{A})} E_{m,i}(ug)\psi^a_{U_1}(u)du \tag{5}$$

is zero for all choice of data. Here $\psi^a_{U_1}$ is viewed as a character of U_1^k by restriction.

Proof: For $2 \le j \le k$ we expand integral (5) along the unipotent groups $x_j(r_j) = I_{2m} + r_je'_{1,j}$. We thus obtain

$$\int\limits_{U_1^k(F)\backslash U_1^k(\mathbf{A})} \sum_{\alpha_j\in F^*} \int\limits_{(F\backslash\mathbf{A})^{k-1}} E_{m,i}(x_2(r_2)\cdots x_k(r_k)ug)\psi^a_{U_1}(u)\psi(\alpha_2r_2 + \cdots + \alpha_kr_k)dr_jdu \tag{6}$$

Conjugating by a suitable discrete matrices $\gamma(\vec{\alpha_j})$ in $SO_{2m-2}(F)$, integral (6) equals

$$\sum_{\gamma} \int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E_{m,i}(u\gamma g)\psi^a_{U_1}(u)du \tag{7}$$

However, from Proposition 2 part (a) this last integral is zero for all choice of data. ■

Proof of Theorem A: Let $\mathcal{O} = (n_1 \cdots n_s)$ be a unipotent orbit. Because of Proposition 2 we may assume that $\mathcal{O}$ is greater than (31^{2m-3}). Then $n_1 \geq 3$. Assume first that at least one of the n_i is odd and greater then one. In this case the torus $h_{\mathcal{O}}(t)$ contains the factors t and $t^0 = 1$ in it. Assume that t^1 occurs p_1 times in $h_{\mathcal{O}}(t)$ and that one occurs p_2 times. Then from the definition of $h_{\mathcal{O}}(t)$ we have $p_1 \leq 2p_2$. As in [GRS2], to determine the character $\psi_{V(\mathcal{O})}$ we need to consider the various orbits of $GL_{p_1}(F) \times SO_{2p_2}(F)$ which acts on $Mat_{p_1 \times 2p_2}(F)$ whose stabilizer is $SO_{p_1}(F) \times SO_{2p_2-p_1}(F)$. It is not hard to check that a representative of such an orbit can be chosen so that one of the rows will be a vector of nonzero length. This means that $V(\mathcal{O})$ contains a subgroup which is conjugated to U_1^k for some $k \leq m-1$ and that $\psi_{V(\mathcal{O})}$ induces a character ψ_{U_1} on U_1^k. Here U_1^k is defined before Lemma 1 and $\psi_{U_1}^a$ is defined in the proof of Proposition 2 part (a) . It follows from Lemma 1 that this integral is zero. Thus we may assume that if $\mathcal{O} = (n_1 \cdots n_s)$ then either all the n_j's are even numbers or the number one. But these cases were covered in Proposition 2 part (c). This completes the proof of the Theorem. ■

For Theorem B, stated below, we need one more result on the vanishing properties of the Fourier coefficients. Recall that V is the maximal unipotent subgroup of SO_{2m} which consists of upper unipotent matrices. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$ where $\epsilon_j \in F$. We define a character ψ_V^ϵ on V as follows. For $v = v_{k,j} \in V$ we define $\psi_V^\epsilon(v) = \psi(\epsilon_1 v_{1,2} + \cdots + \epsilon_{m-1} v_{m-1,m} + \epsilon_m v_{m-1,m+1})$.

Lemma 2: *Suppose that for some $j \leq m-2$, the numbers ϵ_j and ϵ_{j+1} are both nonzero or that ϵ_{m-2} and ϵ_m are both nonzero. Then the integral*

$$\int_{V(F)\backslash V(\mathbf{A})} E_{m,i}(vg)\psi_V^\epsilon(v)dv \tag{8}$$

is zero for all choice of data.

Proof: The proof of this lemma is similar to the proof of Proposition 2. All we have to verify is that the space of double coset $P_m \backslash SO_{2m}/V$ has no admissible representatives. Clearly, all representatives can be chosen to be Weyl elements. A double coset will be admissible if and only if its representative w conjugates into L^- all simple roots of SO_{2m} for which ψ_V^ϵ is not trivial. Here L is the unipotent radical of P_m and $L^- = L^t$. Clearly, L is abelian. From our assumption on the ϵ_j's, the character ψ_V^ϵ is not trivial on the matrices $\{I_{2m} + v_{j,j+1}e'_{j,j+1}\}$ and $\{I_{2m} + v_{j+1,j+2}e'_{j+1,j+2}\}$ or on $\{I_{2m} + v_{m-2,m-1}e'_{m-2,m-1}\}$ and $\{I_{2m} + v_{m-1,m+1}e'_{m-1,m+1}\}$. If we consider each pair separately we see that the two matrices are not commutative. Thus it is impossible to find a Weyl element which will conjugate them simultaneously into L^- which is abelian. ■

3 On Non-vanishing Properties of $E_{m,i}$

In this section we will determine the Fourier coefficient which the representation $E_{m,i}$ supports. More precisely, we will prove

Theorem B: *The representation $E_{m,i}$ has a nonzero Fourier coefficient which corresponds to the unipotent orbit* $(2^{2i}1^{2(m-2i)})$.

Let us first recall the Fourier coefficient which corresponds to this orbit. This was done in Proposition 2 case (b). In that part we defined the group L_i and the character ψ_i defined on this group. Thus the Fourier coefficient which corresponds to the above unipotent orbit is given by integral (4). To prove Theorem B we need to show that there is a choice of data such that integral (4) is not zero. To do this we will use induction.

For $1 \le j \le m-1$ let U_j denote the unipotent subgroup of SO_{2m} defined by all matrices of the form $U_j = \{I_{2m} + r_{j,j+1}e'_{j,j+1} + \cdots + r_{j,2m-j}e'_{j,2m-j} + ze_{j,2m-j+1}\}$ where $e'_{k,t} = e_{k,t} - e_{2m-t+1,2m-k+1}$ and z depends on the $r_{j,p}$. In matrices we have

$$U_1 = \{\begin{pmatrix} 1 & r & z \\ & I_{2m-2} & r^* \\ & & 1 \end{pmatrix} : r \in Mat_{1\times 2m-2}\} \quad U_2 = \{\begin{pmatrix} 1 & & & & \\ & 1 & r & z & \\ & & I_{2m-4} & r^* & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} : r \in Mat_{1\times 2m-4}\}$$

and so on. We clearly have that $V = U_1U_2\ldots U_{m-1}$ where V is the maximal unipotent subgroup of SO_{2m} which consists of upper triangular matrices. Define a character ψ_1^0 of U_1 as follows. For $u = (u_{k,j}) \in U_1$ define $\psi_1^0(u) = \psi(u_{1,2})$. We first claim that the integral

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})} E_{m,i}(ug)\psi_1^0(u)du \tag{9}$$

is not zero for some choice of data. Indeed, consider the Fourier expansion of $E_{m,i}(g)$ along U_1. From Proposition 2 case (a) we know that the contribution to this expansion from all vectors of nonzero length is zero. If integral (9) is zero for all choice of data, it follows that $E_{m,i}(g)$ will equal its constant term along U_1. This is impossible unless $i = 0$ in which case, $E_{m,0}$ is the trivial representation.

Next we expand (9) along the group U_2. In this case we claim that only the constant term contributes to the expansion. Indeed, the contribution from vectors of nonzero length to this expansion is zero because the group U_2 can be conjugated to the group U_1^2 which was defined right before Lemma 1. If the character on U_2 corresponds to to a vector of nonzero length, it then follows from Lemma 1 that it is zero. Next we consider the contribution to the expansion from the nonzero vectors of length zero. Let ψ_2^0 denote the character of U_2

defined as follows. For $u=(u_{k,j})\in U_2$ define $\psi_2^0(u)=\psi(u_{2,3})$. Thus the contributions from these vectors will be a sum of integrals of the type

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})}\int\limits_{U_2(F)\backslash U_2(\mathbf{A})} E_{m,i}(u_1u_2g)\psi_1^0(u_1)\psi_2^0(u_2)du_2du_1 \tag{10}$$

We claim that this integral is zero. Indeed, continuing and expanding this integral along U_3, then U_4 and so on we obtain, that (10) is a sum of integrals of the type (8). Thus, in the notations of Lemma 2, we have that $\epsilon_1=\epsilon_2=1$. Hence by that Lemma each such term is zero, and hence (10) is zero for all choice of data.

Thus, (9) equals

$$\int\limits_{U_1(F)\backslash U_1(\mathbf{A})}\int\limits_{U_2(F)\backslash U_2(\mathbf{A})} E_{m,i}(u_1u_2g)\psi_1^0(u_1)du_2du_1 \tag{11}$$

This integral defines a nonzero automorphic representation for the group $SO_{2m-4}(\mathbf{A})$. To prove Theorem B we will show that this representation is $E_{m-2,i-1}$. Thus we will be able to apply induction.

Let $Q=Q_{m,2}$ denote the maximal parabolic subgroup of SO_{2m} whose Levi part is $GL_2\times SO_{2m-4}$ and such that its unipotent radical, denoted $U(Q)$ is upper triangular. To analyze integral (11) we consider the integral

$$I(h,g)=\int\limits_{F\backslash\mathbf{A}}\int\limits_{U(Q)(F)\backslash U(Q)(\mathbf{A})} E_m(u(x(r)h,g),s)\psi(r)dudr \tag{12}$$

Here $(h,g)\in GL_2\times SO_{2m-4}$ and $x(r)$ is the upper unipotent subgroup of GL_2. Clearly, $I(h,g)$ is an automorphic function in the variable g, that is on the group $SO_{2(m-2)}$. We unfold this integral and notice that $P_m\backslash SO_{2m}/Q$ has three double cosets. First, we have the identity $w_1=e$ and the other two representatives can be chosen to be

$$w_2=\begin{pmatrix}1&&&\\&&&1\\&&I_{2m-4}&\\&1&&\\&&&&1\end{pmatrix}\qquad w_3=\begin{pmatrix}&&I_2\\&I_{2m-4}&\\I_2&&\end{pmatrix}$$

Thus (12) is equal to the sum of the three integrals

$$I_j=\int\limits_{F\backslash\mathbf{A}}\int\limits_{U(Q)(F)\backslash U(Q)(\mathbf{A})}\sum_{\gamma\in(Q\cap w_j^{-1}P_mw_j)(F)\backslash Q(F)} f_s(w_j\gamma u(x(r)h,g))\psi(r)dudr \tag{13}$$

We will show that $I_1 = I_3 = 0$ by studying the discrete sum in each case. For w_1 we obtain the sum over $P_{m-2}(F)\backslash SO_{2(m-2)}(F)$. This means that the sum does not depends on the h variable and hence, conjugating $x(r)$ to the left we obtain zero because of the additive character. Case w_3 is similar. In this case the discrete sum is over $(P_{m-2}(F)\backslash SO_{2(m-2)}(F))U(Q)(F)$. Thus we get zero contribution in this case. We are left with w_2. Write $U(Q) = U'(Q)U''(Q)$ where

$$U'(Q) = \{u = \begin{pmatrix} 1 & & & & & \\ & 1 & r & & & \\ & & I_{m-2} & & & \\ & & & I_{m-2} & r^* & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\} \quad U''(Q) = \{u = \begin{pmatrix} 1 & & r_1 & r_2 & z & * \\ & 1 & 0 & r_3 & * & z^* \\ & & I_{m-2} & & r_3^* & r_2^* \\ & & & I_{m-2} & 0 & r_1^* \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\}$$

Here, all variables $r \in Mat_{1\times(m-2)}$. For w_2 the discrete sum equals

$$(B_2(F)\backslash GL_2(F))(P_{m-2}(F)\backslash SO_{2(m-2)}(F))U''(Q)(F)$$

From this, factoring the integration over $U(Q)$, integral (12) equals

$$\sum_{\delta\in B_2(F)\backslash GL_2(F)} \int_{F\backslash \mathbf{A}} \sum_{\gamma\in P_{m-2}(F)\backslash SO_{2(m-2)}(F)} \int_{U'(Q)(\mathbf{A})} f_s(w_2u(x(r)\delta h, \gamma g))\psi(r)dudr \tag{14}$$

Consider the integral over $U'(Q)(\mathbf{A})$ only. An easy computation shows that this integral defines a section in the induced space $Ind_{P_{m-2}(\mathbf{A})}^{SO_{2(m-2)}(\mathbf{A})}\delta_{P_{m-2}}^{((m-1)s-1)/(m-3)}$. This means that as a function of $g \in SO_{2(m-2)}(\mathbf{A})$, integral (14) can be realized in the space of the Eisenstein series $E_{m-2}(g, ((m-1)s-1)/(m-3))$. We know that integral (11) is the residue of integral (12) at the point $s_{m,i} = (m-1-i)/(m-1)$. Since integral (11) is nonzero it follows that the residue of integral (14) at $s_{m,i} = (m-1-i)/(m-1)$ is nonzero and that it is realized in the space of the residue representation $E_{m-2,i-1}$ of the group $SO_{2(m-2)}(\mathbf{A})$. If $i > 1$ then the representation $E_{m-2,i-1}$ is not the trivial representation. Hence, when we expand (11) along U_3 we obtain a non trivial contribution from the nonzero length vectors. In other words, if $i > 1$ then the integral

$$\int_{U_1(F)\backslash U_1(\mathbf{A})} \int_{U_2(F)\backslash U_2(\mathbf{A})} \int_{U_3(F)\backslash U_3(\mathbf{A})} E_{m,i}(u_1u_2u_3g)\psi_1^0(u_1)\psi_3^0(u_3)du_3du_2du_1 \tag{15}$$

is nonzero for some choice of data. Here $\psi_3^0(u_3)$ is defined as follows. If $u_3 = (u_{i,j}) \in U_3$ then $\psi_3^0(u_3) = \psi(u_{3,4})$. Arguing as in integrals (9) and (10) we deduce that (15) equals

$$\int_{V_2(F)\backslash V_2(\mathbf{A})} E_{m,i}(vg)\psi_{V_2}(v)dv \tag{16}$$

where for each j we define $V_j = U_1U_2\dots U_{2j}$ and for $v = (v_{l,k}) \in V_j$ we define $\psi_{V_j}(v) = \psi(v_{1,2} + v_{3,4} + \cdots + v_{2j-1,2j})$. The integral (16) defines an automorphic representation on SO_{2m-8}. Arguing as above we deduce that this representation is realized in the space of $E_{m-4,i-2}$. Hence, if $i > 2$ it is not the identity representation and we can proceed by induction to deduce that

$$\int\limits_{V_i(F)\backslash V_i(\mathbf{A})} E_{m,i}(vg)\psi_{V_i}(v)dv \tag{17}$$

is not zero for some choice of data. Recall, from the description in proposition 2, that we need to show that the integral

$$\int\limits_{L_i(F)\backslash L_i(\mathbf{A})} E_{m,i}(lg)\psi_i(l)dl \tag{18}$$

is nonzero for some choice of data. To show that we will prove that the non-vanishing of (17) implies the non-vanishing of (18). To do that define the following Weyl element w of SO_{2m}. Let $w[k,r]$ denote the $(k,r)-th$ entry of w. Recall that to define a Weyl element of SO_{2m} it is enough to specify it in the first m rows. For all $1 \le j \le i$ let $w[j,2j-1] = w[i+j,2(m-i+j)-1] = 1$. Also, for all $2i+1 \le j \le m$ let $w[j,j] = 1$. All other entries of the first m rows are zeros. A simple conjugation shows that the integral

$$\int\limits_{L_i^0(F)\backslash L_i^0(\mathbf{A})} E_{m,i}(lg)\psi_i(l)dl \tag{19}$$

is obtained as an inner integration to integral (17). Here $L_i^0 = \{l = (l_{k,r}) \in L_i \ : \ l_{k,r} = 0 \ \text{ if } \ k > r\}$.

Next we expand integral (19) along the abelian group $L_i^0\backslash L_i$ with points in $F\backslash\mathbf{A}$. Using a certain discrete matrix we obtain that this expansion is given as a sum of integrals where each summand is integral (18). Thus integral (18) is nonzero. This completes the proof of Theorem B. ■

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