

UNIQUENESS AND DISJOINTNESS OF KLYACHKO MODELS

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ABSTRACT. We show the uniqueness and disjointness of Klyachko models for GL_n over a non-archimedean local field. This completes, in particular, the study of Klyachko models on the unitary dual. Our local results imply a global rigidity property for the discrete automorphic spectrum.

1. INTRODUCTION

In this work we show that over a local non-archimedean field, the mixed (symplectic-Whittaker) models introduced by Klyachko in [Kly84] are disjoint and that multiplicity one is satisfied. In [OS] we showed, over a p -adic field (a finite extension of \mathbb{Q}_p), the existence of Klyachko models for unitarizable representations. The upshot is then that for every irreducible, unitarizable representation of GL_n over a p -adic field there is a unique Klyachko model where it appears and it appears there with multiplicity one.

To formulate the main result more precisely we introduce some notation. Let F be a non-archimedean local field. For a positive integer r , denote by U_r the subgroup of upper triangular unipotent matrices in GL_r and let

$$Sp_{2k} = \{g \in GL_{2k} : {}^t g J_{2k} g = J_{2k}\}$$

where

$$(1) \quad J_{2k} = \begin{pmatrix} 0 & w_k \\ -w_k & 0 \end{pmatrix}$$

and $w_k \in GL_k(F)$ is the matrix with $(i, j)^{th}$ entry equal to $\delta_{i, n+1-j}$. Whenever $n = r + 2k$ we consider the subgroup $H_{r, 2k}$ of GL_n defined by

$$H_{r, 2k} = \left\{ \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} : u \in U_r, X \in M_{r \times 2k}, h \in Sp_{2k} \right\}.$$

Let ψ be a non trivial character of F . For $u = (u_{i,j}) \in U_r(F)$ we set

$$(2) \quad \psi_r(u) = \psi(u_{1,2} + \cdots + u_{r-1,r}).$$

Let $\psi_{r, 2k}$ be the character of $H_{r, 2k}(F)$ defined by

$$(3) \quad \psi_{r, 2k} \left(\begin{pmatrix} u & X \\ 0 & h \end{pmatrix} \right) = \psi_r(u).$$

When $n = r + 2k$ the space

$$\mathcal{M}_{r, 2k} = \text{Ind}_{H_{r, 2k}(F)}^{GL_n(F)}(\psi_r)$$

is called a mixed model. Here Ind denotes the functor of non-compact smooth induction. Representations of $GL_n(F)$ are always assumed to be smooth. When

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we say that the representation π of $GL_n(F)$ is unitary we really mean that π is a smooth representation that has a unitary structure. We say that a representation π of $GL_n(F)$ admits the mixed model $\mathcal{M}_{r,2k}$ if $\text{Hom}_{GL_n(F)}(\pi, \mathcal{M}_{r,2k}) \neq 0$. The space

$$\mathcal{M} = \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{M}_{n-2k,2k}$$

is referred to as the Klyachko model. The main result of this paper is the following.

Theorem 1. *Let F be a non-archimedean local field and let π be an irreducible representation of $GL_n(F)$ then*

$$(4) \quad m_\pi = \dim_{\mathbb{C}}(\text{Hom}_{GL_n(F)}(\pi, \mathcal{M})) \leq 1.$$

When F is a finite field, it is proved in [IS91] that $m_\pi = 1$ for every irreducible representation π of $GL_n(F)$. When F is a non-archimedean local field it is shown in [HR90] that there exists an irreducible representation π of $GL_3(F)$ so that $m_\pi = 0$. Thus, we cannot expect in general for the inequality (4) to be an equality. However, in [OS] we showed that if F is a p -adic field then $m_\pi \geq 1$ for every irreducible, unitary representation π of $GL_n(F)$. We therefore have the following.

Corollary 1. *Let F be a p -adic field and let π be an irreducible, unitary representation of $GL_n(F)$ then $m_\pi = 1$.*

By Frobenius reciprocity [BZ76, §2.28] for a representation π of $GL_n(F)$ we have

$$(5) \quad \text{Hom}_{GL_n(F)}(\pi, \mathcal{M}_{r,2k}) = \text{Hom}_{H_{r,2k}(F)}(\pi, \psi_r).$$

It follows that for an irreducible, unitary representation π of $GL_n(F)$ there is a unique integer $0 \leq \kappa(\pi) \leq \lfloor \frac{n}{2} \rfloor$ such that

$$\text{Hom}_{H_{n-2\kappa(\pi),2\kappa(\pi)}(F)}(\pi, \psi_{n-2\kappa(\pi),2\kappa(\pi)}) \cong \mathbb{C},$$

i.e. such that π is $(H_{n-2\kappa(\pi),2\kappa(\pi)}, \psi_{n-2\kappa(\pi),2\kappa(\pi)})$ -distinguished and that the space of such functionals is one dimensional. Moreover, $\kappa(\pi)$ is the explicit value assigned in [OS, Theorem 8] in terms of Tadic's classification of the unitary dual.

The *if* direction of the following corollary was proved in [OS07, Theorem 1]. The other implication is straightforward from Theorem 1. Since it will not serve us further in this work the corollary is formulated using the notation of [OS07] without recalling it.

Corollary 2. *Let F be a p -adic field and let π be an irreducible, unitary representation of $GL_{2n}(F)$. Then π is distinguished by $Sp_{2n}(F)$ if and only if*

$$\pi \cong U(\delta_1, 2n_1) \times \cdots \times U(\delta_r, 2n_r) \times \pi(U(\delta'_1, 2n'_1), \alpha_1) \times \cdots \times \pi(U(\delta'_s, 2n'_s), \alpha_s)$$

for some discrete series representations $\delta_1, \dots, \delta_r, \delta'_1, \dots, \delta'_s$, some positive integers $n_1, \dots, n_r, n'_1, \dots, n'_s$ and some real numbers $\alpha_1, \dots, \alpha_s$ such that $-\frac{1}{2} < \alpha_i < \frac{1}{2}$.

In [OS] we also studied globally over a number field, the mixed (symplectic-Whittaker) periods on the discrete automorphic spectrum of GL_n . Let F be a number field and let ψ be a non-trivial character of $F \backslash \mathbb{A}_F$. We use (2) to view ψ_r as a character of $U_r(\mathbb{A}_F)$ and (3) to view $\psi_{r,2k}$ as a character of $H_{r,2k}(\mathbb{A}_F)$. For an automorphic form ϕ in the discrete automorphic spectrum of $GL_n(\mathbb{A}_F)$ and a decomposition $n = r + 2k$ we consider the mixed period integral

$$(6) \quad P_{r,2k}(\phi) = \int_{H_{r,2k}(F) \backslash H_{r,2k}(\mathbb{A}_F)} \phi(h) \psi_{r,2k}(h) dh.$$

We say that an irreducible, discrete spectrum automorphic representation π of $GL_n(\mathbb{A}_F)$ is $(H_{r,2k}, \psi_{r,2k})$ -distinguished if $P_{r,2k}$ is not identically zero on the space of π . In [OS] we provided an explicit integer $0 \leq \kappa(\pi) \leq \lfloor \frac{n}{2} \rfloor$ such that π is distinguished by $(H_{n-2\kappa(\pi), 2\kappa(\pi)}, \psi_{n-2\kappa(\pi), 2\kappa(\pi)})$. Furthermore, we showed that this period integral is factorizable. Corollary 1 (particularly, the disjointness of Klyachko models) then shows that there is a unique such integer. Furthermore, it implies an interesting rigidity property of the discrete automorphic spectrum of GL_n .

Corollary 3. *Let F be a number field and let $\pi = \otimes_v \pi_v$ be an irreducible, discrete spectrum automorphic representation of $G(\mathbb{A}_F)$. Then there exists a unique integer $k = k(\pi)$ such that π is $(H_{n-2k, 2k}, \psi_{n-2k})$ -distinguished. Moreover the following are equivalent:*

- (1) π is $(H_{r,2k}, \psi_{r,2k})$ -distinguished;
- (2) π_v is $(H_{r,2k}, \psi_{r,2k})$ -distinguished for all places v of F ;
- (3) π_{v_0} is $(H_{r,2k}, \psi_{r,2k})$ -distinguished for some finite place v_0 of F .

Remark 1. This rigidity property is best understood when Klyachko models are read off the Arthur type defined in [Clo04]. This interpretation will be the subject of a forthcoming note.

The rest of this work is devoted to the proof of Theorem 1. It is organized as follows. After setting up the notation in §2, in §3-§4 we reduce Theorem 1 to a statement about invariant distributions on orbits. This statement is made more explicit in §5 and is then proved by induction in §6.

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2. NOTATION

Let F be a non-archimedean local field and for any positive integer r let $G_r = GL_r(F)$. We denote by I_r the identity matrix in G_r . We also set $G_0 = \{1\}$. Throughout, we fix a positive integer n and let $G = G_n$. For a partition (n_1, \dots, n_t) of n we denote by $P_{(n_1, \dots, n_t)}$ the standard parabolic subgroup of G of type (n_1, \dots, n_t) . It consists of matrices in upper triangular block form. If $P = P_{(n_1, \dots, n_t)}$ we denote by \bar{P} the parabolic opposite to P . It consists of matrices in lower triangular block form. When we say that $P = MU$ is the standard Levi decomposition of P we mean that U is its unipotent radical, and $M = P \cap \bar{P} = \{\text{diag}(g_1, \dots, g_t) : g_i \in G_{n_i}\}$. We then denote by \bar{U} the unipotent radical of \bar{P} . We denote by $a^{(r)}$ the r -tuple (a, \dots, a) , thus for example $P_{(1)^n}$ is the subgroup of upper triangular matrices in G . For any standard Levi subgroup M of G denote by W_M the Weyl group of M and let $W = W_G$. If M' is another standard Levi subgroup then any double coset in $W_M \backslash W / W_{M'}$ has a unique element of minimal length which we refer to as a left W_M and right $W_{M'}$ reduced Weyl element. We denote by ${}_M W_{M'}$ the set of all left W_M and right $W_{M'}$ reduced Weyl elements. For integers a and b we

set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$. For any subset $A \subseteq [1, n]$ we denote by SA the permutation group in the elements of A . It will be convenient to identify W with $S[1, n]$. If $P = MU$ and $P' = M'U'$ are standard parabolic subgroups of G with their standard Levi decompositions, the Bruhat decomposition of G gives the disjoint union

$$(7) \quad G = \bigsqcup_{w \in {}_M W_{M'}} Pw\overline{P'}.$$

For any matrix X let tX denote the transpose matrix. For a skew-symmetric matrix $\mathcal{I} = -{}^t\mathcal{I} \in G_{2k}$ let

$$Sp(\mathcal{I}) = \{g \in G_{2k} : {}^tg\mathcal{I}g = \mathcal{I}\}$$

and let

$$J_{2k} = \begin{pmatrix} 0 & w_k \\ -w_k & 0 \end{pmatrix}$$

where $w_k \in G_k$ is the matrix with ij^{th} entry $\delta_{i, n+1-j}$. Denote by U_r the subgroup of upper triangular unipotent matrices and by \overline{U}_r the subgroup of lower triangular unipotent matrices in G_r . For non-negative integers r and k let

$$H_{r, 2k} = \left\{ \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} : u \in U_r, X \in M_{r \times 2k}(F), h \in Sp(J_{2k}) \right\}$$

and let

$$\overline{H}_{r, 2k} = \left\{ \begin{pmatrix} u & 0 \\ X & h \end{pmatrix} : u \in \overline{U}_r, X \in M_{2k \times r}(F), h \in Sp(J_{2k}) \right\}.$$

Note that $\overline{H}_{r, 2k}$ is the image of $H_{r, 2k}$ under transpose. For $g \in G$ let

$$g^\tau = {}^tg^{-1}.$$

The restriction to $H_{r, 2k}$ of the involution $\tau : G \rightarrow G$ defines a group isomorphism from $H_{r, 2k}$ to $\overline{H}_{r, 2k}$. Let $n = r + 2k = r' + 2k'$ and let $\mathcal{H}^{r, r'} = \mathcal{H}_n^{r, r'} = H_{r, 2k} \times \overline{H}_{r', 2k'}$. Thus

$$\mathcal{H}^{r, r'} = \{(h_1, h_2^\tau) : h_1 \in H_{r, 2k}, h_2 \in H_{r', 2k'}\}.$$

We denote by $e_{\mathcal{H}^{r, r'}}$ the identity element of $\mathcal{H}^{r, r'}$. It will also be useful to consider the map $\xi : \mathcal{H}^{r, r'} \rightarrow \mathcal{H}^{r', r}$ defined by

$$\xi(h_1, h_2^\tau) = (h_2, h_1^\tau).$$

The group $\mathcal{H}^{r, r'}$ acts on G by

$$h \cdot g = h_1 g {}^t h_2, \quad h = (h_1, h_2^\tau) \in \mathcal{H}^{r, r'}, \quad g \in G.$$

We observe that

$$(8) \quad {}^t(h \cdot g) = \xi(h) \cdot {}^t g, \quad h \in \mathcal{H}^{r, r'}, \quad g \in G.$$

When $r = r'$ the map ξ is an involution of $\mathcal{H}^{r, r}$. The formula (8) allows us then to define the semi direct product

$$\widetilde{\mathcal{H}}^{r, r} = \mathcal{H}^{r, r} \rtimes \{\pm 1\}$$

with multiplication rule

$$(h, \epsilon)(h', \epsilon') = (h \xi_\epsilon(h'), \epsilon \epsilon') \quad \text{where} \quad \xi_\epsilon(h) = \begin{cases} h & \epsilon = 1 \\ \xi(h) & \epsilon = -1. \end{cases}$$

Here $h, h' \in \mathcal{H}^{r,r}$, $\epsilon, \epsilon' \in \{\pm 1\}$. The group $\widetilde{\mathcal{H}}^{r,r}$ acts on G by

$$(h, \epsilon) \cdot g = h \cdot T_\epsilon(g) \text{ where } T_\epsilon(g) = \begin{cases} g & \epsilon = 1 \\ {}^t g & \epsilon = -1. \end{cases}$$

In order to unify notation, when $r \neq r'$ we shall set $\widetilde{\mathcal{H}}^{r,r'} = \mathcal{H}^{r,r'} \times \{1\}$.

For a non-trivial character ψ of F we define as in §1 the generic character ψ_r of U_r by (2) and the character $\psi_{r,2k}$ of $H_{r,2k}$ by (3). Let $\theta^{r,r'}$ be the character of $\mathcal{H}^{r,r'}$ defined by

$$\theta^{r,r'}(h_1, h_2^\tau) = \psi_{r,2k}(h_1)\psi_{r',2k'}(h_2).$$

We also extend $\theta^{r,r'}$ to the character $\widetilde{\theta}^{r,r'}$ of $\widetilde{\mathcal{H}}^{r,r'}$ defined by

$$\widetilde{\theta}^{r,r'}(h, \epsilon) = \epsilon \theta^{r,r'}(h).$$

3. REDUCTION TO INVARIANT DISTRIBUTIONS

Let $n = r + 2k = r' + 2k'$ be 2 decompositions of n . Let $\mathcal{H} = \mathcal{H}^{r,r'}$ and $\theta = \theta^{r,r'}$. The action of $\widetilde{\mathcal{H}}$ on G defines an action on $C_c^\infty(G)$ and on the space $\mathfrak{D}(G) = C_c^\infty(G)^*$ of distributions on G by

$$(h \cdot \phi)(g) = \phi(h^{-1} \cdot g) \text{ and } (h \cdot D)(\phi) = D(h^{-1} \cdot \phi)$$

for $h \in \widetilde{\mathcal{H}}$, $g \in G$, $\phi \in C_c^\infty(G)$ and $D \in \mathfrak{D}(G)$. In this section we show that Theorem 1 reduces to the following.

Proposition 1. *If $D \in \mathfrak{D}(G)$ is such that $h \cdot D = \widetilde{\theta}(h)D$ for all $h \in \widetilde{\mathcal{H}}$ then $D = 0$, i.e.*

$$(9) \quad \text{Hom}_{\widetilde{\mathcal{H}}}(C_c^\infty(G), \widetilde{\theta}) = 0.$$

3.1. Proposition 1 implies Theorem 1. Let π be an irreducible representation of G . Set $H = H_{r,2k}$, $H' = H_{r',2k'}$, $\psi = \psi_{r,2k}$ (forgive the abuse of notation) and $\psi' = \psi_{r',2k'}$. Denote by \overline{H} (resp. \overline{H}') the image of H (resp. H') under τ . Let $\ell \in \text{Hom}_H(\pi, \psi)$ and $\ell' \in \text{Hom}_{H'}(\pi, \psi')$. The representation $\pi^\tau(g) = \pi(g^\tau)$ realizes the contragradient representation $\widetilde{\pi}$ on the space V_π of π [GK75] (see also [BZ76, Theorem 7.3]). Note that $\ell' \in \text{Hom}_{\overline{H}'}(\pi^\tau, (\psi')^\tau)$ defines a functional $\widetilde{\ell}'$ on the space $V_{\widetilde{\pi}}$ of $\widetilde{\pi}$ and that $\widetilde{\ell}' \in \text{Hom}_{\overline{H}}(\widetilde{\pi}, (\psi)^\tau)$. Note further that $\ell \circ \pi(\phi)$ is a smooth vector in V_π . Define the distribution D on G by

$$(10) \quad D(\phi) = \widetilde{\ell}'(\ell \circ \pi(\phi)), \phi \in C_c^\infty(G).$$

For $h \in H$ and $h' \in H'$ we have $\pi((h^{-1}, {}^t h') \cdot \phi) = \pi(h) \circ \pi(\phi) \circ \pi({}^t h')$ and therefore

$$((h, (h')^\tau) \cdot D)(\phi) = \widetilde{\ell}'(\ell \circ \pi(h) \circ \pi(\phi) \circ \pi({}^t h')).$$

By our assumption on ℓ and ℓ' we have, $\ell \circ \pi(h) = \psi(h)\ell$ and $\widetilde{\ell}' \circ \widetilde{\pi}((h')^\tau) = \psi'(h')\widetilde{\ell}'$, $h \in H$, $h' \in H'$. Also note that for any $\tilde{v} \in V_{\widetilde{\pi}}$ viewed as a smooth functional on π the composition $\tilde{v} \circ \pi(g)$ is again a smooth functional on π and in fact

$$(\tilde{v} \circ \pi(g))(v) = \tilde{v}(\pi(g)v) = (\widetilde{\pi}(g^{-1})\tilde{v})(v)$$

i.e.,

$$\tilde{v} \circ \pi(g) = \widetilde{\pi}(g^{-1})\tilde{v}.$$

Applying this to $\tilde{v} = \ell \circ \pi(\phi)$ and $g = {}^t h'$ we get that

$$\begin{aligned} ((h, (h')^\tau) \cdot D)(\phi) &= \psi(h) \tilde{\ell}'((\ell \circ \pi(\phi)) \circ \pi({}^t h')) \\ &= \psi(h) \tilde{\ell}'(\tilde{\pi}((h')^\tau)(\ell \circ \pi(\phi))) = \theta(h, (h')^\tau) D(\phi). \end{aligned}$$

We see that D is (\mathcal{H}, θ) -equivariant. If $r \neq r'$ it follows from Proposition 1 that $D = 0$. If we assume further that ℓ is non-zero then the vectors $\ell \circ \pi(\phi)$, $\phi \in C_c^\infty(G)$ span $V_{\tilde{\pi}}$. We conclude that $\tilde{\ell}'$ must vanish identically on $V_{\tilde{\pi}}$ and hence also $\ell' = 0$. This shows that

$$(11) \quad \dim_{\mathbb{C}}(\text{Hom}_{H_{r,2k}}(\pi, \psi_{r,2k})) \dim_{\mathbb{C}}(\text{Hom}_{H_{r',2k'}}(\pi, \psi_{r',2k'})) = 0 \text{ whenever } r \neq r'.$$

Assume now that $r = r'$. Recall that $e_{\mathcal{H}}$ is the unit element of \mathcal{H} . Note that $(e_{\mathcal{H}}, -1) \cdot \phi = {}^t \phi$ where ${}^t \phi(g) = \phi({}^t g)$, $\phi \in C_c^\infty(G)$, $g \in G$. Note further that for every $h \in \mathcal{H}$ we have

$$(h, 1)(e_{\mathcal{H}}, -1) = (e_{\mathcal{H}}, -1)(\xi(h), 1)$$

and that $\theta(\xi(h)) = \theta(h)$. Since $D \in \text{Hom}_{\mathcal{H}}(C_c^\infty(G), \theta)$, it also follows that

$$D_1 = D - (e_{\mathcal{H}}, -1) \cdot D \in \text{Hom}_{\mathcal{H}}(C_c^\infty(G), \theta).$$

Furthermore, since $\tilde{\theta}(e_{\mathcal{H}}, -1) = -1$ and $(e_{\mathcal{H}}, -1) \cdot D_1 = -D_1$ we conclude that $D_1 \in \text{Hom}_{\tilde{\mathcal{H}}}(C_c^\infty(G), \tilde{\theta})$. Proposition 1 now implies that

$$(12) \quad D = (e_{\mathcal{H}}, -1) \cdot D.$$

Let $B : C_c^\infty(G) \times C_c^\infty(G) \rightarrow \mathbb{C}$ be the bilinear form defined by

$$(13) \quad B(\phi_1, \phi_2) = D(\phi_1 * \phi_2)$$

where

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(x) \phi_2(x^{-1}g) dx.$$

Note that

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1) \circ \pi(\phi_2) \text{ and } {}^t(\phi_1 * \phi_2) = {}^t \phi_2 * {}^t \phi_1, \phi_1, \phi_2 \in C_c^\infty(G).$$

Thus, (12) implies that

$$B(\phi_1, \phi_2) = B((e_{\mathcal{H}}, -1) \cdot \phi_2, (e_{\mathcal{H}}, -1) \cdot \phi_1).$$

This implies that $R_B = (e_{\mathcal{H}}, -1) \cdot L_B$ where

$$L_B = \{\phi \in C_c^\infty(G) : B(\phi, \cdot) \equiv 0\} \text{ and } R_B = \{\phi \in C_c^\infty(G) : B(\cdot, \phi) \equiv 0\}$$

are respectively the left and right kernels of B . In other words

$$(14) \quad R_B = \{{}^t \phi : \phi \in L_B\}.$$

For a functional λ on V_{π} let

$$\mathfrak{K}(\lambda, \pi) = \{\phi \in C_c^\infty(G) : \lambda \circ \pi(\phi) = 0\}.$$

Note that

$$B(\phi_1, \phi_2) = (\tilde{\ell}' \circ \pi(\phi_2^\vee))(\ell \circ \pi(\phi_1))$$

where

$$\phi^\vee(g) = \phi(g^{-1})$$

and therefore

$$L_B = \mathfrak{K}(\ell, \pi) \text{ and } R_B = \{\phi^\vee : \phi \in \mathfrak{K}(\tilde{\ell}', \tilde{\pi})\}.$$

By our definitions we have

$$\mathfrak{K}(\tilde{\ell}', \tilde{\pi}) = \mathfrak{K}(\ell', \pi^\tau) = \{({}^t\phi)^\vee : \phi \in \mathfrak{K}(\ell', \pi)\}$$

and therefore

$$R_B = \{{}^t\phi : \phi \in \mathfrak{K}(\ell', \pi)\}.$$

It now follows from (14) that

$$\mathfrak{K}(\ell, \pi) = \mathfrak{K}(\ell', \pi).$$

Since π is irreducible we get that $\ker \ell = \ker \ell'$ and therefore that ℓ and ℓ' are proportional. We therefore proved that

$$(15) \quad \dim_{\mathbb{C}}(\mathrm{Hom}_{H_{r,2k}}(\pi, \psi_{r,2k})) \leq 1 \text{ for all } 0 \leq k \leq \lfloor \frac{n}{2} \rfloor.$$

Theorem 1 is now a straightforward consequence of (5), (11) and (15).

4. REDUCTION TO \mathcal{H} -ORBITS

We keep the notation introduced in §3. For every $g \in G$ we denote by \mathcal{H}_g the stabilizer of g in \mathcal{H} and by $\tilde{\mathcal{H}}_g$ the stabilizer of g in $\tilde{\mathcal{H}}$. The purpose of this section is to reduce Proposition 1 to the following.

Proposition 2. *For every $g \in G$ the character $\tilde{\theta}$ is non-trivial on $\tilde{\mathcal{H}}_g$.*

Remark 2. The objects involved and the statement of Proposition 2 makes sense over any field F and in fact, our proof is valid in this generality. In particular, using Mackey theory, it gives an alternative proof of the uniqueness and disjointness of Klyachko models over a finite field.

4.1. Proposition 2 implies Proposition 1. Assume now that Proposition 2 holds. We deduce that Proposition 1 also holds. Let $\mathbf{1}_{\tilde{\mathcal{H}}_g}$ denote the trivial character of $\tilde{\mathcal{H}}_g$. Note that $h \cdot g \mapsto \tilde{\mathcal{H}}_g h^{-1}$ is a homeomorphism of $\tilde{\mathcal{H}}$ -spaces $\tilde{\mathcal{H}} \cdot g \simeq \tilde{\mathcal{H}}_g \backslash \tilde{\mathcal{H}}$ that induces an $\tilde{\mathcal{H}}$ -isomorphism

$$(16) \quad C_c^\infty(\tilde{\mathcal{H}} \cdot g) \simeq \mathrm{ind}_{\tilde{\mathcal{H}}_g}^{\tilde{\mathcal{H}}}(\mathbf{1}_{\tilde{\mathcal{H}}_g})$$

where ind denotes smooth induction with compact support. Therefore, by Frobenius reciprocity [BZ76, §2.29]

$$(17) \quad \mathrm{Hom}_{\tilde{\mathcal{H}}}(C_c^\infty(\tilde{\mathcal{H}} \cdot g), \tilde{\theta}) = \mathrm{Hom}_{\tilde{\mathcal{H}}_g}(\delta_{\tilde{\mathcal{H}}_g}, \theta|_{\tilde{\mathcal{H}}_g})$$

where $\delta_{\tilde{\mathcal{H}}_g}$ is the modulus function of $\tilde{\mathcal{H}}_g$. Since the image of $\tilde{\theta}$ lies in the unit circle (in fact, the image of θ lies in the group of p -powered roots of unity where p is the residual characteristic of F) and since $\delta_{\tilde{\mathcal{H}}_g}$ is positive, we get that whenever $\theta|_{\tilde{\mathcal{H}}_g}$ is non-trivial we also have

$$(18) \quad \tilde{\theta}|_{\tilde{\mathcal{H}}_g} \neq \delta_{\tilde{\mathcal{H}}_g}.$$

It follows from Proposition 2 that (18) holds for every $g \in G$ and therefore by (16) that

$$(19) \quad \mathrm{Hom}_{\tilde{\mathcal{H}}}(C_c^\infty(\tilde{\mathcal{H}} \cdot g), \tilde{\theta}) = 0, \quad g \in G.$$

Proposition 1 follows from (19) using the theory of Gelfand-Kazhdan [GK75]. Indeed, we apply [BZ76, Theorem 6.9] to the following setting. We view $C_c^\infty(G)$ as a

module over itself by convolution. By [BZ76, Proposition 1.14] it uniquely defines a sheaf \mathcal{F} over the l -space G . We let $\tilde{\mathcal{H}}$ act on $C_c^\infty(G)$ by

$$h \cdot_{\tilde{\theta}} \phi = \tilde{\theta}(h)h \cdot \phi.$$

This defines an action of $\tilde{\mathcal{H}}$ on the sheaf \mathcal{F} . The space of $\tilde{\mathcal{H}}$ -invariant distributions on \mathcal{F} is then precisely $\text{Hom}_{\tilde{\mathcal{H}}}(C_c^\infty(G), \tilde{\theta})$. The action of $\tilde{\mathcal{H}}$ on G is constructible by [BZ76, §6.15, Theorem A]. The second assumption of [BZ76, Theorem 6.9] is precisely (19). It follows that there are no $\tilde{\mathcal{H}}$ -invariant distributions on the sheaf \mathcal{F} , i.e. that (9) holds.

5. THE PROPERTY OF \mathcal{H} -ORBITS MADE EXPLICIT

In order to prove Proposition 2 it will be convenient to reformulate it, by describing more explicitly the property of the \mathcal{H} -orbits that we wish to prove. We begin with this reformulation.

5.1. The property $\mathcal{P}(g, r, r')$. For $g \in G$ let $\mathcal{P}(g, r, r') = \mathcal{P}_n(g, r, r')$ be the following property: either

$$(20) \quad \text{there exists } y \in H_{r,2k} \text{ such that } g^{-1}yg \in \overline{H}_{r',2k'} \text{ and } \theta^{r,r'}(y, g^{-1}yg) \neq 1$$

or $r = r'$ and

$$(21) \quad \text{there exists } y \in H_{r,2k} \text{ such that } g^{-1}y^t g \in \overline{H}_{r,2k} \text{ and } \theta^{r,r}(y, g^{-1}y^t g) = 1.$$

Lemma 1. *For every $g \in G$, $\tilde{\theta}^{r,r'}$ is non-trivial on $\tilde{\mathcal{H}}_g^{r,r'}$ if and only if the property $\mathcal{P}(g, r, r')$ holds.*

Proof. Note that

$$\mathcal{H}_g^{r,r'} = \{(y, g^{-1}yg) : y \in H_{r,2k} \cap g\overline{H}_{r',2k'}g^{-1}\}$$

and therefore (20) holds if and only if $\theta^{r,r'}$ is not trivial on $\mathcal{H}_g^{r,r'}$. If $r \neq r'$ this proves the lemma. If $r = r'$ it remains to show that when $\theta^{r,r}$ is trivial on $\mathcal{H}_g^{r,r}$ then $\tilde{\theta}^{r,r}$ is not trivial on $\tilde{\mathcal{H}}_g^{r,r}$ if and only if we have (21). Note that

$$\{h \in \mathcal{H}^{r,r} : (h, -1) \in \tilde{\mathcal{H}}_g^{r,r}\} = \{(y, g^{-1}y^t g) : y \in H_{r,2k} \cap g\overline{H}_{r,2k}g^t\}.$$

If $y \in H_{r,2k} \cap g\overline{H}_{r,2k}g^t$ then for $h = (y, g^{-1}y^t g) \in \mathcal{H}^{r,r}$ we have $h \cdot^t g = g$, i.e. $(h, -1) \in \tilde{\mathcal{H}}_g^{r,r}$ and therefore by (8) we get that $h\xi(h) \in \mathcal{H}_g^{r,r}$ so that $\theta^{r,r}(h\xi(h)) = 1$. Since $\theta^{r,r} = \theta^{r,r} \circ \xi$ we have $\theta^{r,r}(h) \in \{\pm 1\}$. With this notation (21) is satisfied by y if and only if $\theta^{r,r}(h) = 1$ and $\tilde{\theta}^{r,r}(h, -1) = -1$. The remaining of the lemma follows. \square

We make here another simple observation that will help to shorten some of the arguments in the proof of Proposition 2.

Lemma 2. *If the property $\mathcal{P}(g, r, r')$ holds then $\mathcal{P}(h \cdot g, r, r')$ holds for all $h \in \tilde{\mathcal{H}}$ and $\mathcal{P}(^t g, r', r)$ holds.*

Proof. Note that $\tilde{\mathcal{H}}_{h \cdot g} = h\tilde{\mathcal{H}}_g h^{-1}$ and that $\tilde{\theta}$ is a character. Thus, the first statement is immediate from Lemma 1. If $r = r'$ this argument with $h = (e_{\mathcal{H}}, -1)$ also contains the second statement. If $r \neq r'$ the second statement follows from the fact that $\mathcal{H}_g^{r',r} = \xi(\mathcal{H}_g^{r,r'})$ (that follows from (8)) and the fact that $\theta \circ \xi = \theta$. \square

In light of Lemma 1 in order to show Proposition 2 we need to show that for every $r, r' \leq n$ such that $n - r \equiv n - r' \equiv 0 \pmod{2}$ and for every $g \in G$ we have $\mathcal{P}(g, r, r')$. This will occupy the rest of this paper.

5.2. Two cases where $\mathcal{P}(g, r, r')$ is already known. There are two extremes that are already known. The first is a well known fact concerned with the double coset space $U_n \backslash G / \overline{U}_n$. It can be found in the proof of [GK75, Lemma 4.3.8] (it is essentially the steps (a)-(d) verifying condition 4 of [GK75, Theorem 4.2.10]) and it is applied in order to prove the uniqueness of Whittaker models. We provide a proof here for the sake of completeness.

Lemma 3. *For every $g \in G$ the property $\mathcal{P}_n(g, n, n)$ holds.*

Proof. By the Bruhat decomposition every $\mathcal{H}^{n,n}$ -orbit in G contains an element of the form aw where $w \in W$ and $a = \text{diag}(a_1, \dots, a_n)$. We show that if $\theta^{n,n}$ is trivial on the stabilizer $\mathcal{H}_{aw}^{n,n}$, i.e. if (20) is not satisfied then

$$(22) \quad \text{for all } i=1, \dots, n-1 \text{ either } w^{-1}(i) < w^{-1}(i+1) \\ \text{or } w^{-1}(i) = w^{-1}(i+1) + 1 \text{ and } a_i = a_{i+1}.$$

When (22) is satisfied then there is a partition (n_1, \dots, n_t) of n such that $w = \text{diag}(w_{n_1}, \dots, w_{n_t})$ is the longest element of W_M where M is the standard Levi subgroup of G of type (n_1, \dots, n_t) and a lies in the center of M . In particular, we then have $aw = {}^t(aw)$ and therefore aw satisfies (21) with $y = I_n$. Assume now that $\theta^{n,n}$ is trivial on $\mathcal{H}_{aw}^{n,n}$. Let $u, v \in U_n$ be such that $(u, v^\tau) \in \mathcal{H}_{aw}^{n,n}$. Thus, $w^{-1}a^{-1}uaw = v^\tau \in \overline{U}_n$ and therefore for any $i < j$ if $w^{-1}(i) < w^{-1}(j)$ then $u_{i,j} = 0$ and if $w^{-1}(i) > w^{-1}(j)$ then $(v^\tau)_{w^{-1}(i), w^{-1}(j)} = a_i^{-1}a_j u_{i,j}$. Let $E_{i,j} \in M_{n \times n}(F)$ be the matrix with $(b, c)^{\text{th}}$ entry equal to $\delta_{(i,j), (b,c)}$ and let $u_{i,j}(s) = I_n + s E_{i,j}$, $s \in F$. If $w^{-1}(i) > w^{-1}(i+1)$ then for any $s \in F$ if $u = u_{i,i+1}(s)$ and $v = (w^{-1}a^{-1}uaw)^\tau$ we have $(u, v^\tau) \in \mathcal{H}_{aw}^{n,n}$. If $w^{-1}(i) > w^{-1}(i+1) + 1$ then $\theta(u, v^\tau) = \psi(s)$ and since s may be chosen arbitrarily this leads to a contradiction. Thus $w^{-1}(i) = w^{-1}(i+1) + 1$ and $1 = \theta(u, v^\tau) = \psi(s(1 - a_i^{-1}a_{i+1}))$. It follows that $a_i = a_{i+1}$. The property (22) is therefore satisfied. \square

The second extreme is with respect to the symplectic group. It was proved by Heumus and Rallis [HR90, Proposition 2.3.1] based on results of Klyachko [Kly84, Corollary 5.6]. Recently, Goldstein and Guralnick essentially provided an independent proof over any field [GG07, Proposition 3.1].

Lemma 4. *When n is even for every $g \in G$ the property $\mathcal{P}_n(g, 0, 0)$ holds.*

Proof. We show that when $r = r' = 0$ (21) holds for every $g \in G$. That is, we show that for every $g \in G$ we have ${}^t g \in Sp(J_n)gSp(J_n)$. As observed in the proof of Lemma 2, it is enough to prove that there exists $y \in Sp(J_n)gSp(J_n)$ such that ${}^t y \in Sp(J_n)gSp(J_n)$. Let $n = 2k$ and let

$$J'_n = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} = {}^t \sigma J_n \sigma \text{ where } \sigma = \begin{pmatrix} w_k & 0 \\ 0 & I_k \end{pmatrix}.$$

Thus,

$$Sp(J'_n) = \sigma^{-1} Sp(J_n) \sigma.$$

It follows from [GG07, Proposition 3.1] that there exists $g' \in G_k$ such that $\text{diag}(I_k, g') \in Sp(J'_n)\sigma^{-1}g\sigma Sp(J'_n)$, i.e. that $y = \sigma \text{diag}(I_k, g')\sigma^{-1} \in Sp(J_n)gSp(J_n)$. Since every matrix in G_k is conjugate to its transpose and since $\text{diag}(x, {}^t x) \in Sp(J'_n)$ for every $x \in G_k$ we see that $\text{diag}(I_k, {}^t g') \in Sp(J'_n)\sigma g\sigma Sp(J'_n)$, i.e. that ${}^t y = \sigma \text{diag}(I_k, {}^t g')\sigma^{-1} \in Sp(J_n)gSp(J_n)$. \square

6. PROOF BY INDUCTION OF $\mathcal{P}_n(g, r, r')$

Fix 2 decompositions $n = r + 2k = r' + 2k'$. We prove by induction on n that for every $g \in G$ we have $\mathcal{P}_n(g, r, r')$. If $r = r' = 0$ then this is Lemma 4. We assume from now on that $r + r' > 0$. The induction hypothesis is that for all $n_1 < n$, all $r_1, r'_1 \leq n_1$ such that $n_1 - r_1 \equiv n_1 - r'_1 \equiv 0 \pmod{2}$ and all $g' \in G_{n_1}$ we have $\mathcal{P}_{n_1}(g', r_1, r'_1)$. Set $H = H_{r, 2k}$, $H' = H_{r', 2k'}$, $\mathcal{H} = H \times \overline{H'}$ and $\theta = \theta^{r, r'}$. Let $P = P_{(1^{(r)}, 2k)}$ and $P' = P_{(1^{(r')}, 2k')}$. For $w \in W$ viewed as a permutation in $S[1, n]$ let

$$I_w = \{i \in [1, r] : w^{-1}(i) \in [1, r']\}.$$

6.1. A simple proof for most Bruhat cells.

Lemma 5. *Let $w \in {}_M W_{M'}$ be such that I_w is not empty then the property $\mathcal{P}(g, r, r')$ holds for every $g \in Pw\overline{P'}$.*

Proof. Note that $U \times \overline{U'} \subseteq \mathcal{H}$ and therefore that every \mathcal{H} -orbit in $Pw\overline{P'}$ contains an element of MwM' . In light of Lemma 2 we may assume without loss of generality that $g \in MwM'$.

Assume first that there exists an integer i such that $1 \leq i \leq \min\{r, r'\}$ and $I_w = w^{-1}(I_w) = [1, i]$. We can then write $w = \text{diag}(w_1, w_2)$ for some $w_1 \in S[1, i]$ and $w_2 \in S[i+1, n]$. Thus for $g \in MwM'$ there exists $g_1, g_2 \in G_{n-i}$, and $a = \text{diag}(a_1, \dots, a_i) \in G_i$ such that $g = \text{diag}(I_i, g_1)w \text{diag}(a, g_2) = \text{diag}(w_1 a, g')$ for $g' = g_1 w_2 g_2 \in G_{n-i}$. Let $(u_1, u_2^\tau, \epsilon) \in (\tilde{\mathcal{H}}_i^{i, i})_{w_1 a}$ be such that $\tilde{\theta}^{i, i}(u_1, u_2^\tau, \epsilon) \neq 1$ and let $(h_1, h_2^\tau, \epsilon') \in (\tilde{\mathcal{H}}_{n-i}^{r-i, r'-i})_{g'}$ be such that $\tilde{\theta}^{r-i, r'-i}(h_1, h_2^\tau, \epsilon') \neq 1$. The first exists by Lemma 3. For the second we apply the induction hypothesis to have $\mathcal{P}_{n-i}(g', r-i, r'-i)$. If $\epsilon = 1$ then

$$h = (\text{diag}(u_1, I_{n-i}), \text{diag}(u_2, I_{n-i})^\tau, 1) \in \tilde{\mathcal{H}}_g \text{ and } \tilde{\theta}(h) = \tilde{\theta}^{i, i}(u_1, u_2^\tau, 1) \neq 1.$$

Similarly, if $\epsilon' = 1$ then

$$h = (\text{diag}(I_i, h_1), \text{diag}(I_i, h_2)^\tau, 1) \in \tilde{\mathcal{H}}_g \text{ and } \tilde{\theta}(h) = \tilde{\theta}^{r-i, r'-i}(h_1, h_2^\tau, 1) \neq 1.$$

If on the other hand $\epsilon = \epsilon' = -1$ then

$$h = (\text{diag}(u_1, h_1), \text{diag}(u_2, h_2)^\tau, -1) \in \tilde{\mathcal{H}}_g \text{ and } \tilde{\theta}(h) = -1.$$

We are now left with the case that either I_w or $w^{-1}(I_w)$ is not of the form $[1, i]$ as above. Note that if $g \in Pw\overline{P'}$ then ${}^t g \in P'w^{-1}\overline{P}$ and that $w^{-1} \in {}_M W_M$. It follows from Lemma 2 that it is enough to prove our lemma either for g or for ${}^t g$. We may therefore assume, without loss of generality, that I_w is not of the form $[1, i]$ for any $1 \leq i \leq \min\{r, r'\}$. Since we assume that $g \in MwM'$ there exist $g_1 \in G_{2k}$, $g_2 \in G_{2k'}$ and $a = \text{diag}(a_1, \dots, a_{r'})$ a diagonal matrix in $G_{r'}$ such that $g = \text{diag}(I_r, g_1)w \text{diag}(a, g_2)$. By our assumption on w we have that $[1, r] \setminus I_w$ is not empty. Let $\ell = \min([1, r] \setminus I_w)$. Since $[1, \ell - 1]$ is contained but does not equal I_w the set $[\ell + 1, r] \cap I_w$ is not empty. Let $q = \min([\ell + 1, r] \cap I_w)$. Then $q - 1 \notin I_w$ and $q \in I_w$. In particular, $w^{-1}(q - 1) > r'$ and $w^{-1}(q) \leq r'$. Let $E_{i, j} \in M_{n \times n}(F)$ be

the matrix with $(b, c)^{th}$ entry equal to $\delta_{(i,j),(b,c)}$ and let $u_{i,j}(s) = I_n + sE_{i,j}$, $s \in F$. Note that $u_{q-1,q}(s) \in U \subseteq H_{r,2k}$ and that $\psi_{r,2k}(u_{q-1,q}(s)) = \psi(s)$. Thus, there exists $s \in F$ such that $\psi_{r,2k}(u_{q-1,q}(s)) \neq 1$. On the other hand,

$$\begin{aligned} g^{-1}u_{q-1,q}(s)g &= \begin{pmatrix} a^{-1} & 0 \\ 0 & g_2^{-1} \end{pmatrix} u_{w^{-1}(q-1),w^{-1}(q)}(s) \begin{pmatrix} a & 0 \\ 0 & g_2 \end{pmatrix} \\ &= \begin{pmatrix} I_{r'} & 0 \\ * & I_{2k'} \end{pmatrix} \in \overline{H}_{r',2k'} \end{aligned}$$

and $\psi_{r',2k'}(g^{-1}u_{q-1,q}(s)g) = 1$. It follows that $h_s = (u_{q-1,q}(s), g^{-1}u_{q-1,q}(s)g) \in \mathcal{H}_g$ and if s is such that $\psi_{r,2k}(u_{q-1,q}(s)) \neq 1$ then $\theta(h_s) \neq 1$. \square

6.2. The closed Bruhat cell. We are now left with the case that I_w is empty. Since this means that w^{-1} maps $[1, r]$ into $[r' + 1, n]$ we must have, in particular, $n \geq r + r'$. It is not difficult to see that there is then a unique such element in $M\overline{W}_{M'}$, namely,

$$w = w^{r,r'} = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix}.$$

Note then that $Pw\overline{P'}$, is the closed Bruhat cell. We remark further that this contains the case that either r or r' is 0. Let $g \in MwM'$. Note that there exist $g_1 \in G_{2k}$ and $g_2 \in G_{2k'}$ such that

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix}.$$

Indeed, for $t \in G_r$, $t' \in G_{r'}$ (and in particular when t and t' are diagonal) if $g'_1 \in G_{2k}$ and $g'_2 \in G_{2k'}$ we have

$$\begin{aligned} &\begin{pmatrix} t & \\ & g'_1 \end{pmatrix} w \begin{pmatrix} t' & \\ & g'_2 \end{pmatrix} \\ &= \begin{pmatrix} I_r & \\ & g'_1 \end{pmatrix} \begin{pmatrix} 0 & t & 0 \\ t' & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} I_{r'} & \\ & g'_2 \end{pmatrix} = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \end{aligned}$$

where $g_1 = g'_1 \text{diag}(t', I_{2k-r'})$ and $g_2 = \text{diag}(t, I_{2k'-r})g'_2$.

In order to show $\mathcal{P}(g, r, r')$ we distinguish between 2 cases. We denote by $\langle v_1, \dots, v_i \rangle$ the subspace of a vector space V spanned by $v_1, \dots, v_i \in V$. Let V be a subspace of the vector space $M_{\ell \times 1}(F)$ for some positive integer ℓ . We say that a skew symmetric matrix $\mathcal{I} \in M_{\ell \times \ell}(F)$ is totally isotropic on V if ${}^t v \mathcal{I} v' = 0$ for all $v, v' \in V$. Denote by e_i the column vector with 1 in the i^{th} row and 0 in each other row. Thus $e_i \in M_{\ell \times 1}(F)$ for an integer ℓ which is implicit in our notation. Let

$$\mathcal{I}_1 = {}^t g_1 J_{2k} g_1 \text{ and } \mathcal{I}_2 = g_2^T J_{2k'} g_2^{-1}.$$

We say that g belongs to the totally isotropic case if both \mathcal{I}_1^{-1} is totally isotropic on $\langle e_1, \dots, e_{r'} \rangle$ and \mathcal{I}_2 is totally isotropic on $\langle e_1, \dots, e_r \rangle$. Otherwise we say that g does not belong to the totally isotropic case. It is easy to verify that this property indeed depends only on g and not on g_1 and g_2 . Note that if g belongs to the totally isotropic case we must have $r' \leq k$ and $r \leq k'$. These inequalities are crucial for the proof of Lemma 11. They follow from the simple observation that a totally

isotropic subspace for a nondegenerate symplectic form \mathcal{I} is of dimension at most half of the rank of \mathcal{I} . We now prove $\mathcal{P}(g, r, r')$ separately in each of the 2 cases.

6.2.1. *When g does not belong to the totally isotropic case.* In this case we prove that g satisfies (20). It will be convenient to make this property more explicit. We say that the 2 skew-symmetric forms $\mathcal{I}_1 \in G_{2k}$ and $\mathcal{I}_2 \in G_{2k'}$ satisfy the property $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ if there exist $u \in U_r$ and $u' \in U_{r'}$ such that $\psi_r(u) \neq \psi_{r'}(u')$ and for some $X \in M_{r \times 2k-r}(F)$, $Y \in M_{r' \times 2k-r'}(F)$ and $D \in G_{n-(r+r')}$ we have

$$\begin{pmatrix} u & X \\ 0 & D \end{pmatrix} \in Sp(\mathcal{I}_2) \text{ and } \begin{pmatrix} {}^t u' & 0 \\ Y & D \end{pmatrix} \in Sp(\mathcal{I}_1).$$

Lemma 6. *Let*

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \in MwM'$$

and let

$$\mathcal{I}_1 = {}^t g_1 J_{2k} g_1 \text{ and } \mathcal{I}_2 = g_2^r J_{2k'} g_2^{-1}.$$

Then g satisfies (20) if and only if $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$.

Proof. Let

$$y = \begin{pmatrix} u & Z \\ & h \end{pmatrix} \in H$$

with $u \in U_r$, $h \in Sp(J_{2k})$ and $Z \in M_{r \times 2k}(F)$. To explicate condition (20) we compute $g^{-1}yg$. First note that we have

$$\begin{pmatrix} I_r & \\ & g_1^{-1} \end{pmatrix} \begin{pmatrix} u & Z \\ & h \end{pmatrix} \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} = \begin{pmatrix} u & Zg_1 \\ & g_1^{-1}hg_1 \end{pmatrix}.$$

We write

$$g_1^{-1}hg_1 = \begin{pmatrix} {}^t u' & B \\ Y & D \end{pmatrix} \text{ and } Zg_1 = (Z_1, Z_2)$$

with $u' \in M_{r' \times r'}(F)$, $D \in M_{2k-r' \times 2k-r'}(F)$, $Z_1 \in M_{r \times r'}(F)$ and $Z_2 \in M_{r \times 2k-r'}(F)$.

We then have

$$\begin{pmatrix} 0 & I_{r'} & 0 \\ I_r & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} u & Z_1 & Z_2 \\ 0 & {}^t u' & B \\ 0 & Y & D \end{pmatrix} \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} = \begin{pmatrix} {}^t u' & 0 & B \\ Z_1 & u & Z_2 \\ Y & 0 & D \end{pmatrix}.$$

Therefore,

$$g^{-1}yg = \begin{pmatrix} {}^t u' & (0, B)g_2 \\ g_2^{-1} \begin{pmatrix} Z_1 \\ Y \end{pmatrix} & g_2^{-1} \begin{pmatrix} u & Z_2 \\ 0 & D \end{pmatrix} g_2 \end{pmatrix}.$$

We see that $g^{-1}yg \in \overline{H'}$ if and only if $u' \in U_{r'}$, $B = 0$ and

$$g_2^{-1} \begin{pmatrix} u & Z_2 \\ 0 & D \end{pmatrix} g_2 \in Sp(J_{2k'}).$$

Recall also that

$$\begin{pmatrix} {}^t u' & 0 \\ Y & D \end{pmatrix} \in g_1^{-1}Sp(J_{2k})g_1.$$

With this notation, when $g^{-1}yg \in \overline{H'}$ we have

$$\theta(y, g^{-1}yg) = \psi_r(u)\psi_{r'}((u')^{-1}).$$

Since

$$g_1^{-1}Sp(J_{2k})g_1 = Sp(\mathcal{I}_1) \text{ and } g_2Sp(J_{2k'})g_2^{-1} = Sp(\mathcal{I}_2),$$

the lemma is now immediate. \square

In order to proceed we need the following Lemma of Klyachko [Kly84, §1.3, p. 368, Step 3].

Lemma 7. *Let $\mathcal{I} = -{}^t\mathcal{I} \in G_{2k}$ and let $r \leq 2k$ be such that \mathcal{I} is not totally isotropic on $\langle e_1, \dots, e_r \rangle$ then there exists $u \in U_r$ with $\psi_r(u) \neq 1$ and $X \in M_{r \times 2k-r}(F)$ such that*

$$(23) \quad \begin{pmatrix} u & X \\ 0 & I_{2k-r} \end{pmatrix} \in Sp(\mathcal{I}).$$

Proof. Let $i \in [1, r-1]$ be maximal so that \mathcal{I} is totally isotropic on $\langle e_1, \dots, e_i \rangle$. There is therefore $v_0 \in \langle e_1, \dots, e_i \rangle$ such that ${}^t v_0 \mathcal{I} e_{i+1} \neq 0$. We may further assume that $v_0 \in e_i + \langle e_1, \dots, e_{i-1} \rangle$ since if ${}^t e_i \mathcal{I} e_{i+1} \neq 0$ then we may take $v_0 = e_i$ and otherwise, we may replace v_0 by its sum with any scalar multiple of e_i . Let $V = M_{2k \times 1}(F)$ and for every $s \in F$ define $\lambda_s \in \text{Hom}_F(V, F)$ by $\lambda_s(v) = s {}^t v_0 \mathcal{I} v$. Note that the map $s \mapsto \lambda_s(e_{i+1})$, $s \in F$ is onto F . Identify $GL(V)$ with G_{2k} via the standard basis $\{e_1, \dots, e_{2k}\}$ and define an element $h_s \in G_{2k}$ by

$$h_s(v) = v + \lambda_s(v) v_0.$$

Thus, $h_s \in Sp(\mathcal{I})$ is of the form (23) with $\psi_r(u) = \psi(\lambda_s(e_{i+1}))$. \square

Lemma 8. *Let*

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \in MwM'$$

not belong to the totally isotropic case and let

$$\mathcal{I}_1 = {}^t g_1 J_{2k} g_1 \text{ and } \mathcal{I}_2 = g_2^T J_{2k'} g_2^{-1}.$$

Then we have $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$.

Proof. If \mathcal{I}_2 is not totally isotropic on $\langle e_1, \dots, e_{r'} \rangle$ then by Lemma 7 there exist $u \in U_{r'}$ and $X \in M_{r' \times 2k'-r'}$ such that $\psi_{r'}(u) \neq 1$ and

$$\begin{pmatrix} u & X \\ 0 & I_{2k'-r'} \end{pmatrix} \in Sp(\mathcal{I}_2).$$

Then $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ is satisfied with $Y = 0$, $u' = I_{r'}$ and $D = I_{n-(r+r')}$. Note further that $Sp(\mathcal{I}_1^{-1}) = \{{}^t g : g \in Sp(\mathcal{I}_1)\}$. Thus, if \mathcal{I}_1^{-1} is not totally isotropic on $\langle e_1, \dots, e_r \rangle$ then by Lemma 7 applied to \mathcal{I}_1^{-1} there exist $u' \in U_r$ and $Y \in M_{2k-r' \times r'}$ such that $\psi_r(u') \neq 1$ and

$$\begin{pmatrix} {}^t u' & 0 \\ Y & I_{2k-r'} \end{pmatrix} \in Sp(\mathcal{I}_1).$$

Thus, $\mathcal{Q}(\mathcal{I}_1, \mathcal{I}_2, r, r')$ is satisfied with $X = 0$, $u = I_r$ and $D = I_{n-(r+r')}$. \square

6.2.2. When g belongs to the totally isotropic case. Assume from now on that both \mathcal{I}_2 is totally isotropic on $\langle e_1, \dots, e_{r'} \rangle$ and \mathcal{I}_1^{-1} is totally isotropic on $\langle e_1, \dots, e_r \rangle$. Recall that, in particular, we then have $r \leq k'$ and $r' \leq k$. In the case at hand $\mathcal{H} \cdot g$ contains an element of a rather simple form that will allow us the inductive argument. In order to bring g to this simpler form we need the following lemma.

Lemma 9. *Let $\ell \leq m$ and $Q = P_{(\ell, 2m-\ell)}$. Then*

$$Sp(J_{2m})Q = \{g \in G_{2m} : {}^t g J_{2m} g \text{ is totally isotropic on } \langle e_1, \dots, e_\ell \rangle\}.$$

Proof. If $h \in Sp(J_{2m})$ and $q \in Q$ then ${}^t(hq)J_{2m}hq = {}^tqJ_{2m}q$. Since q preserves the space $\langle e_1, \dots, e_\ell \rangle$ and since J_{2m} is totally isotropic on $\langle e_1, \dots, e_\ell \rangle$ we get that ${}^tqJ_{2m}q$ is also totally isotropic on $\langle e_1, \dots, e_\ell \rangle$. To prove the other direction let $g \in G_{2m}$ be such that ${}^tgJ_{2m}g$ is totally isotropic on $\langle e_1, \dots, e_\ell \rangle$. Then

$$x = {}^tgJ_{2m}g = \begin{pmatrix} 0_\ell & A \\ -{}^tA & D \end{pmatrix} \in G_{2m}$$

for some $D = -{}^tD \in M_{2m-\ell \times 2m-\ell}(F)$. We must show that there exists $q \in Q$ such that ${}^tqxq = J_{2m}$. Since x is invertible and $\ell \leq 2m - \ell$ the matrix A is of rank ℓ . Performing elementary operations, there exists $\alpha \in G_\ell$ and $\gamma \in G_{2m-\ell}$ such that ${}^t\alpha A \gamma = (0_{\ell \times 2(m-\ell)}, w_\ell)$. It follows that for $q = \text{diag}(\alpha, \gamma) \in Q$, tqxq has the form

$$\begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & b \\ -w_\ell & -{}^tb & d \end{pmatrix}$$

where $a = -{}^ta \in G_{2(m-\ell)}$ and $d = -{}^td \in M_{\ell \times \ell}(F)$. Write $\beta = (\beta_1, \beta_2)$ with $\beta_1 \in M_{\ell \times 2(m-\ell)}(F)$ and $\beta_2 \in M_{\ell \times \ell}(F)$. Note that

$$\begin{aligned} & \begin{pmatrix} I_\ell & 0 & 0 \\ {}^t\beta_1 & I_{2(m-\ell)} & 0 \\ {}^t\beta_2 & 0 & I_\ell \end{pmatrix} \begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & b \\ -w_\ell & -{}^tb & d \end{pmatrix} \begin{pmatrix} I_\ell & \beta_1 & \beta_2 \\ 0 & I_{2(m-\ell)} & 0 \\ 0 & 0 & I_\ell \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & b + {}^t\beta_1 w_\ell \\ -w_\ell & -{}^tb - w_\ell \beta_1 & d + {}^t\beta_2 w_\ell - w_\ell \beta_2 \end{pmatrix}. \end{aligned}$$

We may now take $\beta_1 = -w_\ell {}^tb$. Any skew symmetric matrix in $M_{\ell \times \ell}(F)$ can be written as a difference $X - {}^tX$ for some $X \in M_{\ell \times \ell}(F)$. Thus, there also exists β_2 such that ${}^t\beta_2 w_\ell - w_\ell \beta_2 = -d$. We get that there exists $q \in Q$ such that

$${}^tqxq = \begin{pmatrix} 0 & 0 & w_\ell \\ 0 & a & 0 \\ -w_\ell & 0 & 0 \end{pmatrix}.$$

Let $y \in G_{2(m-\ell)}$ be such that ${}^tyay = J_{2(m-\ell)}$. Thus $q' = q \text{diag}(I_\ell, y, I_\ell) \in Q$ and ${}^tq'xq' = J_{2m}$. \square

For $x \in G_\ell$ let

$$\tilde{x} = w_\ell x^\tau w_\ell.$$

The following property of the group $Sp(J_{2m})$ will be used several times in the proof of $\mathcal{P}(g, r, r')$. Assume that $\ell \leq m$.

(24) For all $x \in G_\ell$, $s \in Sp(J_{2(m-\ell)})$ and y there exists y^* uniquely determined by x , s and y and dependent linearly on y and there exists z such that

$$\begin{pmatrix} x & y^* & z \\ 0 & s & y \\ 0 & 0 & \tilde{x} \end{pmatrix} \text{ (resp. } \begin{pmatrix} x & 0 & 0 \\ y^* & s & 0 \\ z & y & \tilde{x} \end{pmatrix}) \text{ lies in } Sp(J_{2m}).$$

We now choose a convenient representative for g .

Lemma 10. *Let*

$$g = \begin{pmatrix} I_r & \\ & g_1 \end{pmatrix} w \begin{pmatrix} I_{r'} & \\ & g_2 \end{pmatrix} \in MwM'$$

belong to the totally isotropic case. Then there exists $\gamma \in G_{n-(r+r')}$ such that

$$\begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in \mathcal{H} \cdot g.$$

Proof. Since $-\mathcal{I}_1^{-1} = g_1^{-1} J_{2k} g_1^T$ is totally isotropic on $\langle e_1, \dots, e_{r'} \rangle$ and $\mathcal{I}_2 = g_2^T J_{2k} g_2^{-1}$ is totally isotropic on $\langle e_1, \dots, e_r \rangle$, it follows from Lemma 9 that

$$g_1 \in Sp(J_{2k}) \begin{pmatrix} \alpha_1 & 0 \\ \beta'_1 & \gamma_1 \end{pmatrix} \text{ and } g_2 \in \begin{pmatrix} \alpha_2 & \beta'_2 \\ 0 & \gamma_2 \end{pmatrix} Sp(J_{2k'})$$

for some $\alpha_1 \in G_{r'}$, $\gamma_1 \in G_{2k-r'}$, $\alpha_2 \in G_r$, $\gamma_2 \in G_{2k'-r}$ and β'_1 and β'_2 of the appropriate size. Therefore,

$$\begin{pmatrix} 0 & \alpha_2 & \beta'_2 \\ \alpha_1 & 0 & 0 \\ \beta'_1 & 0 & \gamma_1 \gamma_2 \end{pmatrix} = \begin{pmatrix} I_r & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & \beta'_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} I_{r'} & 0 & 0 \\ 0 & \alpha_2 & \beta'_2 \\ 0 & 0 & \gamma_2 \end{pmatrix} \in \mathcal{H} \cdot g.$$

Note that $\text{diag}(\alpha_1, I_{2(k-r')}, \tilde{\alpha}_1) \in Sp(J_{2k})$ and $\text{diag}(\alpha_2, I_{2(k'-r)}, \tilde{\alpha}_2) \in Sp(J_{2k'})$ and therefore that

$$h = \text{diag}(I_r, \alpha_1^{-1}, I_{2(k-r')}, \tilde{\alpha}_1^{-1}) \in H \text{ and } h' = \text{diag}(I_{r'}, \alpha_2^{-1}, I_{2(k'-r)}, \tilde{\alpha}_2^{-1}) \in \overline{H'}.$$

Thus,

$$h \begin{pmatrix} 0 & \alpha_2 & \beta'_2 \\ \alpha_1 & 0 & 0 \\ \beta'_1 & 0 & \gamma_1 \gamma_2 \end{pmatrix} h' = \begin{pmatrix} 0 & I_r & \beta_2 \\ I_{r'} & 0 & 0 \\ \beta_1 & 0 & \gamma \end{pmatrix} \in \mathcal{H} \cdot g$$

for some $\gamma \in G_{n-(r+r')}$, β_1 and β_2 . Now note that

$$\begin{pmatrix} I_r & \beta_2 \gamma^{-1} \beta_1 & -\beta_2 \gamma^{-1} \\ 0 & I_{r'} & 0 \\ 0 & 0 & I_{n-(r+r')} \end{pmatrix} \begin{pmatrix} 0 & I_r & \beta_2 \\ I_{r'} & 0 & 0 \\ \beta_1 & 0 & \gamma \end{pmatrix} \begin{pmatrix} I_{r'} & 0 & 0 \\ 0 & I_r & 0 \\ -\gamma^{-1} \beta_1 & 0 & I_{n-(r+r')} \end{pmatrix} = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in \mathcal{H} \cdot g.$$

□

Lemma 11. Let $\gamma \in G_{n-(r+r')}$ and let

$$g = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

then $\mathcal{P}(g, r, r')$.

Proof. Recall that $r + r' > 0$. Let

$$\sigma_1 = \begin{pmatrix} I_{2(k-r')} \\ w_{r'} \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} I_{2(k'-r)} \\ w_r \end{pmatrix}.$$

For $x = \sigma_1^{-1}\gamma\sigma_2$ we have by the induction hypothesis $\mathcal{P}_{n-(r+r')}(x, r, r')$. Fix $y \in H_{r,2(k'-r)}$ such that either

$$(25) \quad x^{-1}yx \in \overline{H}_{r',2(k-r')} \text{ and } \theta(y, x^{-1}yx) \neq 1$$

or

$$(26) \quad r = r', x^{-1}y {}^t x \in \overline{H}_{r',2(k-r')} \text{ and } \theta(y, x^{-1}y {}^t x) = 1.$$

For every invertible matrix z denote by z^* the matrix z if y satisfies (25) and the matrix ${}^t z$ otherwise. Note that if (26) holds then $\sigma_1 = \sigma_2$ and therefore in either case we have

$$x^* = \sigma_1^{-1}\gamma^*\sigma_2.$$

There exist $s' \in Sp(J_{2(k-r')})$, $u' \in U_{r'}$ and $\varrho' \in M_{r' \times 2(k-r')}(F)$ such that

$$\sigma_1 y \sigma_1^{-1} = \begin{pmatrix} s' & \\ \varrho' & {}^t(\tilde{u}') \end{pmatrix}$$

and there exist $s \in Sp(J_{2(k'-r)})$, $u \in U_r$ and $\varrho \in M_{2(k'-r) \times r}(F)$ such that

$$\gamma^{-1}\sigma_1 y \sigma_1^{-1} \gamma^* = \sigma_2 x^{-1} y x^* \sigma_2^{-1} = \begin{pmatrix} s & \varrho \\ & \tilde{u} \end{pmatrix}.$$

Note then that

$$(27) \quad \theta(y, x^{-1}yx^*) = \psi_r(u)\psi_{r'}(u')^{-1}.$$

By (24) there exist $(\varrho')^* \in M_{2(k-r') \times r'}(F)$, $\varrho^* \in M_{r \times 2(k'-r)}(F)$, z' and z such that

$$h = \begin{pmatrix} {}^t u' & 0 & 0 \\ (\varrho')^* & s' & 0 \\ z' & \varrho' & {}^t \tilde{u}' \end{pmatrix} \in Sp(J_{2k}) \text{ and } h' = \begin{pmatrix} u & \varrho^* & z \\ 0 & s & \varrho \\ 0 & 0 & \tilde{u} \end{pmatrix} \in Sp(J_{2k'}).$$

Note that

$$g^* = \begin{pmatrix} 0 & I_r & 0 \\ I_{r'} & 0 & 0 \\ 0 & 0 & \gamma^* \end{pmatrix}.$$

Let

$$\zeta_1 = (\varrho^*, z)(\gamma^*)^{-1} \text{ and } \zeta = (0_{r \times r'}, \zeta_1)$$

then

$$Y = \begin{pmatrix} u & \zeta \\ 0 & h \end{pmatrix} \in H, g^{-1}Yg^* = \begin{pmatrix} {}^t u' & 0 \\ \zeta' & h' \end{pmatrix} \in \overline{H'}$$

where

$$\zeta' = \begin{pmatrix} 0_{r \times r'} \\ \zeta_1' \end{pmatrix}, \zeta_1' = \gamma^{-1} \begin{pmatrix} (\varrho')^* \\ z' \end{pmatrix} \text{ and } \theta(Y, g^{-1}Yg^*) = \psi_r(u)\psi_{r'}(u')^{-1}.$$

The property $\mathcal{P}_n(g, r, r')$ therefore follows from (27) and the fact that either (25) holds or (26) holds. \square

6.3. Conclusion. For $g \in G$, by (7) there exists $w \in {}_M W_{M'}$ such that $g \in Pw\overline{P'}$. If I_w is not empty then $\mathcal{P}(g, r, r')$ is proved in Lemma 5. If I_w is empty then we separated in §6.2 the statement $\mathcal{P}(g, r, r')$ into 2 cases. If g belongs to the totally isotropic case then $\mathcal{P}(g, r, r')$ follows from Lemma 2, Lemma 10 and Lemma 11. Otherwise $\mathcal{P}(g, r, r')$ follows from Lemma 6 and Lemma 8. It follows that for every $g \in G$ we have $\mathcal{P}(g, r, r')$. Proposition 2 now follows from Lemma 1. Therefore, Proposition 1 follows from §4.1 and Theorem 1 follows from §3.1.

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