THE $SL(2)$-TYPE AND BASE CHANGE

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ABSTRACT. The $SL(2)$-type of any smooth, irreducible and unitarizable representation of GL_n over a p-adic field was defined by Venkatesh. We provide a natural way to extend the definition to all smooth and irreducible representations. For unitarizable representations we show that the $SL(2)$-type of a representation is preserved under base change with respect to any finite extension. The Klyachko model of a smooth, irreducible and unitarizable representation π of GL_n depends only on the $SL(2)$-type of π. As a consequence we observe that the Klyachko model of π and of its base-change are of the same type.

1. INTRODUCTION

Let F be a finite extension of $\mathbb{Q}_p$. In [Ven05], Venkatesh assigned a partition of n, the $SL(2)$-*type* of π, to any smooth, irreducible and unitarizable representation π of $GL_n(F)$. For a representation of Arthur type the $SL(2)$-type encodes the combinatorial data in the Arthur parameter. In general, the $SL(2)$-type is defined in terms of Tadić's classification of the unitary dual.

The reciprocity map for $GL_n(F)$ is a bijection from the set of isomorphism classes of smooth irreducible representations of $GL_n(F)$ to the set of isomorphism classes of n-dimensional Weil-Deligne representations (cf. [HT01] and [Hen00]). Applying the reciprocity map we observe that there is a natural way to extend the definition of the $SL(2)$-type to all smooth and irreducible representations of $GL_n(F)$ (see Theorem 4.1 and Remark 2). The reciprocity map also allows the definition of base change with respect to any finite extension E of F. It is a map $\mathrm{bc}_{E/F}$ from isomorphism classes of smooth irreducible representation of $GL_n(F)$ to isomorphism classes of smooth irreducible representation of $GL_n(E)$ that is the 'mirror image' of restriction with respect to E/F of Weil-Deligne representations. The content of Theorem 6.1, our main result, is that for any smooth, irreducible and unitarizable representation π of $GL_n(F)$ the representations π and $\mathrm{bc}_{E/F}(\pi)$ have the same $SL(2)$-type.

In [OS07], [OS08a], [OS08b] we studied the Klyachko models of smooth irreducible representations of $GL_n(F)$, that is, distinction of a representation with respect to certain subgroups that are a semi direct product of a unipotent and a symplectic group. For unitarizable representations, our results are also described in terms of Tadić's classification and depend, in fact, only on the $SL(2)$-type of a representation. For example, a smooth, irreducible and unitarizable representation π of $GL_{2n}(F)$ is $Sp_{2n}(F)$-distinguished, i.e. it

Date: April 12, 2009.
In this research the first named author is supported by THE ISRAEL SCIENCE FOUNDATION (grant No. 88/08).

satisfies $\mathrm{Hom}_{Sp_{2n}(F)}(\pi,\mathbb{C}) \neq 0$, if and only if the $SL(2)$-type of π consists entirely of even parts (and in this case $\mathrm{Hom}_{Sp_{2n}(F)}(\pi,\mathbb{C})$ is one dimensional [HR90, Theorem 2.4.2]). For unitarizable representations, our results on Klyachko models are reinterpreted here in terms of the $SL(2)$-type. As a consequence we show that Klyachko types (see Definition 1 below) are preserved under base-change with respect to any finite extension. In particular, we have

Theorem 1.1. *Let E/F be a finite extension of p-adic fields. A smooth, irreducible and unitarizable representation π of $GL_{2n}(F)$ is $Sp_{2n}(F)$-distinguished if and only if $\mathrm{bc}_{E/F}(\pi)$ is $Sp_{2n}(E)$-distinguished.*

The rest of this note is organized as follows. After setting some general notation in Section 2, in Section 3 we recall the definition of the reciprocity map. In Section 4 we recall the definition of Venkatesh for the $SL(2)$-type of a unitarizable representation and extend it to all smooth irreducible representations. We recall (and reformulate in terms of the $SL(2)$-type) our results on symplectic (and more generally on Klyachko) models in Section 5. Our main observation Theorem 6.1 and its application to Klyachko models Corollary 6.1 are stated in Section 6 and proved in Section 7. The main theorem says that base change respects $SL(2)$-types and its corollary says that base change respects Klyachko types. Theorem 1.1 is a special case where the Klyachko type is purely symplectic.

2. Notation

Let F be a finite extension of $\mathbb{Q}_p$ for some prime number p and let $|\cdot|_F : F^\times \to \mathbb{C}^\times$ denote the standard absolute value normalized so that the inverses of uniformizers are mapped to the size of the residual field. Denote by W_F the Weil group of F and by I_F the inertia subgroup of W_F. We normalize the reciprocity map $T_F : W_F \to F^\times$, given by local class field theory, so that geometric Frobenius elements are mapped to uniformizers. The map T_F defines an isomorphism from the abelianization W_F^{ab} of W_F to $F^\times$ (this is the inverse of the Artin map). Let $|\cdot|_{W_F} = |\cdot|_F \circ T_F$ denote the associated absolute value on W_F.

Denote by $\mathbf{1}_\Omega$ the characteristic function of a set Ω. Let $\mathrm{MS_{fin}}(\Omega)$ be the set of finite multisets of elements in Ω, that is, the set of functions $f : \Omega \to \mathbb{Z}_{\geq 0}$ of finite support. When convenient we will also denote f by $\{\omega_1,\dots,\omega_1,\omega_2,\dots,\omega_2,\dots\}$ where $\omega \in \Omega$ is repeated $f(\omega)$ times. Let $\mathcal{P} = \mathrm{MS_{fin}}(\mathbb{Z}_{>0})$ be the set of partitions of positive integers and let

$$\mathcal{P}(n) = \{f \in \mathcal{P} : \sum_{k=1}^{\infty} k\, f(k) = n\}$$

denote the subset of partitions of n. For n, $m \in \mathbb{Z}_{>0}$ let $(n)_m = m\,\mathbf{1}_n = \{n,\dots,n\}$ be the partition of nm with 'm parts of size n'. Let $\mathrm{odd} : \mathcal{P} \to \mathbb{Z}_{\geq 0}$ be defined by

$$\mathrm{odd}(f) = \sum_{k=0}^{\infty} f(2k+1),$$

i.e. $\mathrm{odd}(f)$ is the number of odd parts of the partition f.

3. Reciprocity and base-change for $GL_n(F)$

3.1. **Weil-Deligne representations.** An n-dimensional *Weil-Deligne* representation is a pair $((\rho, V), N)$ where (ρ, V) is an n-dimensional representation of W_F that decomposes as a direct sum of irreducible representations and $N : V \to V$ is a linear operator such that

$$|w|_{W_F} \; N \circ \rho(w) = \rho(w) \circ N, \;\; w \in W_F.$$

The map $((\rho, V), N) \mapsto ([\rho], f)$, where $[\rho]$ denotes the isomorphism class of the n-dimensional representation (ρ, V) of W_F and $f \in \mathcal{P}(n)$ is the partition of n associated to the Jordan decomposition of N, defines an injective map on isomorphism classes of Weil-Deligne representations. Denote its image by $\mathcal{G}_F(n)$. In this way we identify the set $\mathcal{G}_F(n)$ with the set of isomorphism classes of n-dimensional Weil-Deligne representations. Let $P_{F,n} : \mathcal{G}_F(n) \to \mathcal{P}(n)$ be the projection to the second coordinate. Let $\mathcal{G}_F = \cup_{n=1}^{\infty} \mathcal{G}_F(n)$ be the set of isomorphism classes of all finite dimensional Weil-Deligne representations and let $P_F : \mathcal{G}_F \to \mathcal{P}$ be the map such that $P_{F|\mathcal{G}_F(n)} = P_{F,n}$.

3.2. **The local Langlands correspondence.** Let $\mathcal{A}_F(n)$ be the set of isomorphism classes of smooth and irreducible representations of $GL_n(F)$ and set $\mathcal{A}_F = \cup_{n=1}^{\infty} \mathcal{A}_F(n)$. For every $\pi \in \mathcal{A}_F$ we denote by ω_π the central character of (any representation in the isomorphism class of) π. Fix a non trivial additive character ψ of F. Due to Harris-Taylor [HT01] and independently to Henniart [Hen00] there exists a unique sequence of bijections

$$\mathrm{rec}_{F,n} : \mathcal{A}_F(n) \to \mathcal{G}_F(n)$$

for all $n \geq 1$ satisfying the following properties:

$$\mathrm{rec}_F(\chi) = \chi \circ T_F; \tag{3.1}$$

$$L(\pi_1 \times \pi_2, s) = L(\mathrm{rec}_F(\pi_1) \otimes \mathrm{rec}_F(\pi_2), s); \tag{3.2}$$

$$\epsilon(\pi_1 \times \pi_2, s, \psi) = \epsilon(\mathrm{rec}_F(\pi_1) \otimes \mathrm{rec}_F(\pi_2), s, \psi); \tag{3.3}$$

$$\det \circ \mathrm{rec}_F(\pi) = rec_F(\omega_\pi); \tag{3.4}$$

$$\mathrm{rec}_F(\pi^\vee) = \mathrm{rec}_F(\pi)^\vee. \tag{3.5}$$

Here $\chi \in \mathcal{A}_F(1)$, π, π_1, $\pi_2 \in \mathcal{A}_F$, $\pi^\vee$ is the contragredient of π, $\mathrm{rec}_F(\pi)^\vee$ is the dual of $\mathrm{rec}_F(\pi)$ and $\mathrm{rec}_F : \mathcal{A}_F \to \mathcal{G}_F$ is such that $\mathrm{rec}_{F|\mathcal{A}_F(n)} = \mathrm{rec}_{F,n}$.

3.3. **Expressing rec_F in terms of rec_F°.** Let $\mathcal{A}_F^\circ(n) \subseteq \mathcal{A}_F(n)$ be the subset of isomorphism classes of supercuspidal representations and let $\mathcal{G}_F^\circ(n) \subseteq \mathcal{G}_F(n)$ be the subset of isomorphism classes $([\rho], f)$ such that ρ is irreducible and $f = \mathbf{1}_n = \{n\}$. The set $\mathcal{G}_F^\circ(n)$ is identified with the set of isomorphism classes of irreducible and n-dimensional representations of W_F. The work of Harris-Taylor, Henniart shows that there exists a unique sequence of bijections

$$\mathrm{rec}_{F,n|\mathcal{A}_F^\circ(n)} = \mathrm{rec}_{F,n}^\circ : \mathcal{A}_F(n) \to \mathcal{G}_F^\circ(n)$$

satisfying (3.1), (3.2), (3.3), (3.4) and (3.5). The work of Zelevinsky [Zel80] allows the extention of rec_F° to the map rec_F on $\mathcal{A}_F$. This is also explained in [Hen85] and we now recall the construction of rec_F in terms of rec_F° .

For $s \in \mathbb{C}$ and every isomorphism class $\varpi = [\pi] \in \mathcal{A}_F$ (resp. $\varrho = ([\rho], f) \in \mathcal{G}_F$) let $\varpi[s] = [\pi \otimes |\det|_F^s]$ (resp. $\varrho[s] = ([\rho \otimes |\cdot|_{W_F}^s], f)$). A *segment* in $\mathcal{A}_F^\circ$ (resp. $\mathcal{G}_F^\circ$) is a set of the form

$$\Delta[\sigma, r] = \{\sigma[\frac{1-r}{2}], \sigma[\frac{3-r}{2}], \dots, \sigma[\frac{r-1}{2}]\}$$

(resp.

$$\Delta[\rho, r] = \{\rho[\frac{1-r}{2}], \rho[\frac{3-r}{2}], \dots, \rho[\frac{r-1}{2}]\})$$

for some $\sigma \in \mathcal{A}_F^\circ$ (resp. $\rho \in \mathcal{G}_F^\circ$) and $r \in \mathbb{Z}_{>0}$. Let $\mathcal{S}$ (resp. $\mathcal{S}'$) denote the set of all segments in $\mathcal{A}_F^\circ$ (resp. $\mathcal{G}_F^\circ$) and let $\mathcal{O} = \mathrm{MS}_{\mathrm{fin}}(\mathcal{S})$ (resp. $\mathcal{O}' = \mathrm{MS}_{\mathrm{fin}}(\mathcal{S}')$). The bijection $\mathrm{rec}_F^\circ : \mathcal{A}_F^\circ \to \mathcal{G}_F^\circ$ defines a bijection $\mathrm{rec}_F^\circ : \mathcal{S} \to \mathcal{S}'$ given by $\mathrm{rec}_F^\circ(\Delta[\sigma, r]) = \Delta[\mathrm{rec}_F^\circ(\sigma), r]$ and a bijection $\mathrm{rec}_F^\circ : \mathcal{O} \to \mathcal{O}'$ given by $\mathrm{rec}_F^\circ(a)(\mathrm{rec}_F^\circ(\Delta)) = a(\Delta)$, $\Delta \in \mathcal{S}$.

In [Zel80, Section 6.5] Zelevinsky defines a bijection $a \mapsto \langle a \rangle$ from $\mathcal{O}$ to $\mathcal{A}_F$. The Zelevinsky involution is defined in [Zel80, Section 9.12] as an involution on the Grothendieck group associated with $\mathcal{A}_F$. It is proved by Aubert [Aub95], [Aub96] and independently by Procter [Pro98] that the Zelevinsky involution restricts to a bijection from $\mathcal{A}_F$ to itself that we denote by $\pi \mapsto \pi^t$. In [Zel80, Section 10.2] Zelevinsky defines a bijection $\tau : \mathcal{O}' \to \mathcal{G}_F$ as follows. For a segment $\Delta[\rho, r] \in \mathcal{S}'$ where $\rho \in \mathcal{G}_F^\circ(k)$ let

$$\tau(\Delta[\rho, r]) = (\oplus_{i=1}^r \rho, (r)_k)$$

and for $a' \in \mathcal{O}'$ set

$$\tau(a') = \oplus_{\Delta' \in \mathcal{O}'} \tau(\Delta')$$

where for $([\rho_1], f_1), \dots, ([\rho_m], f_m) \in \mathcal{G}_F$ the direct sum is given by

$$([\rho_1], f_1) \oplus \cdots \oplus ([\rho_m], f_m) = ([\rho_1 \oplus \cdots \oplus \rho_m], f_1 + \cdots + f_m).$$

The reciprocity map rec_F is given by

$$\mathrm{rec}_F(\langle a \rangle^t) = \tau(\mathrm{rec}_F^\circ(a)),\ a \in \mathcal{O}.$$

4. The $SL(2)$-type of a representation

Denote by $\mathcal{A}_F^u(n)$ the subset of $\mathcal{A}_F(n)$ consisting of all isomorphism classes of unitarizable representations and let $\mathcal{A}_F^u = \cup_{n=1}^\infty \mathcal{A}_F(n)$. For $[\pi_1], \dots, [\pi_m] \in \mathcal{A}_F$ we denote by $\pi_1 \times \cdots \times \pi_m$ the representation parabolically induced from $\pi_1 \otimes \cdots \otimes \pi_m$ and by $[\pi_1] \times \cdots \times [\pi_m]$ its isomorphism class.

For $\sigma \in \mathcal{A}_F^\circ$ and integers n, $r > 0$ let

$$\delta[\sigma, n] = \langle \Delta[\sigma, n] \rangle^t,$$

$$a(\sigma, n, r) = \{\Delta[\sigma[\frac{1-r}{2}], n], \Delta[\sigma[\frac{3-r}{2}], n], \cdots, \Delta[\sigma, n](\frac{r-1}{2})\} \in \mathcal{O}$$

and

$$U(\delta[\sigma, n], r) = \langle a(\sigma, n, r) \rangle.$$

Tadić's classification of the unitary dual of $GL_n(F)$ [Tad86] implies that if $\sigma \in \mathcal{A}_F^\circ \cap \mathcal{A}_F^u$ then $U(\delta[\sigma, n], r) \in \mathcal{A}_F^u$ and that for any $\pi \in \mathcal{A}_F^u$ there exist $\sigma_1, \dots, \sigma_m \in \mathcal{A}_F^\circ$ and integers $n_1, \dots, n_m, r_1, \dots, r_m > 0$ such that

$$\pi = U(\delta[\sigma_1, n_1], r_1) \times \cdots \times U(\delta[\sigma_m, n_m], r_m). \tag{4.1}$$

It further follows from [Tad95, Lemma 3.3] that

$$U(\delta[\sigma, n], r)^t = U(\delta[\sigma, r], n). \tag{4.2}$$

Remark 1. A representation π as in (4.1) is unitarizble if and only if the representations σ_i with a non-unitary central character come in pairs $\sigma[\alpha]$ and $\sigma[-\alpha]$ where $0 < \alpha < \frac{1}{2}$. In particular, (4.1) takes the complimentary series into account.

Let $\pi \in \mathcal{A}_F^u$ be of the form (4.1) where $\sigma_i \in \mathcal{A}_F^\circ(k_i)$, $i = 1, \dots, m$. The $SL(2)$-type of π is defined in [Ven05, Definition 1] to be the partition

$$\{(r_1)_{k_1 n_1}, \dots, (r_m)_{k_m n_m}\}. \tag{4.3}$$

Theorem 4.1. *The $SL(2)$-type of a representation $\pi \in \mathcal{A}_F^u$ equals $P_F(\mathrm{rec}_F(\pi^t))$.*

Remark 2. Theorem 4.1 allows us to define the $SL(2)$-type of any $\pi \in \mathcal{A}_F$ by the formula $P_F(\mathrm{rec}_F(\pi^t))$. Since for representations of Arthur type the $SL(2)$-type is determined by the partition associated with the second $SL_2(\mathbb{C})$ component of the Arthur parameter and since the Zelevinsky involution interchanges between the two $SL_2(\mathbb{C})$ components (cf. [Ban06, (1)] and the references there) this is a natural extension for the definition of the $SL(2)$-type. Note further that given a reciprocity map (local Langlands conjecture), this provides a recipe to define the $SL(2)$-type of an irreducible representation for any reductive group!

Proof. Based on Tadić's classification of the unitary dual of $GL_n(F)$, the proof of Theorem 4.1 is merely a matter of following the definitions. For convenience, we provide the proof. The key is in the following simple observations.

Lemma 4.1. *Let $\pi \in \mathcal{A}_F^u$ be of the form* (4.1). *Then*

$$\mathrm{rec}_F(\pi) = \oplus_{i=1}^m \oplus_{j=1}^{r_i} \tau(\Delta[\sigma_i[\frac{r_i+1}{2} - j], n_i]) \tag{4.4}$$

and

$$\pi^t = U(\delta[\sigma_1, r_1], n_1) \times \cdots \times U(\delta[\sigma_m, r_m], n_m) \in \mathcal{A}_F^u. \tag{4.5}$$

Proof. Let $a_i = a(\sigma_i, r_i, n_i)$. It follows from (4.2) that

$$\pi = \langle a_1 \rangle^t \times \cdots \times \langle a_m \rangle^t = (\langle a_1 \rangle \times \cdots \times \langle a_m \rangle)^t \tag{4.6}$$

and since t is an involution on $\mathcal{A}_F$ that $\langle a_1 \rangle \times \cdots \times \langle a_m \rangle \in \mathcal{A}_F$. Thus, it follows from [Zel80, Proposition 8.4] that $\langle a_1 \rangle \times \cdots \times \langle a_m \rangle = \langle a_1 + \cdots + a_m \rangle$. In other words $\pi = \langle a_1 + \cdots + a_m \rangle^t$ and therefore by definition

$$\mathrm{rec}_F(\pi) = \tau(\mathrm{rec}_F^\circ(a_1 + \cdots + a_m)) = \oplus_{i=1}^m \tau(\mathrm{rec}_F^\circ(a_i)).$$

The identity (4.4) now follows from the definition of $\tau(\mathrm{rec}_F^\circ(a_i))$. Equation (4.6) implies the identity in (4.5) and the classification of Tadić therefore implies that π^t is indeed unitarizable. ∎

Applying (4.4) to π^t and comparing with (4.3) Theorem 4.1 follows from the definitions. $\square$

From now on for every $\pi \in \mathcal{A}_F$ we denote by

$$\mathcal{V}(\pi) = P_F(\mathrm{rec}_F(\pi^t)) \tag{4.7}$$

the $SL(2)$-type of π.

5. Klyachko models

For positive integers r and k denote by U_r the subgroup of upper triangular unipotent matrices in $GL_r(F)$ and by $Sp_{2k}(F)$ the symplectic group in $GL_{2k}(F)$. Fix a decomposition $n = r + 2k$. Let

$$H_{r,2k} = \{\begin{pmatrix} u & X \\ 0 & h \end{pmatrix} : u \in U_r,\ X \in M_{r\times 2k}(F),\ h \in Sp_{2k}(F)\}.$$

Let ψ be a non trivial character of F. For $u = (u_{i,j}) \in U_r$ let

$$\psi_r(u) = \psi(u_{1,2} + \cdots + u_{r-1,r})$$

and let $\psi_{r,2k}$ be the character of $H_{r,2k}$ defined by

$$\psi_{r,2k}\begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_r(u).$$

We refer to the space

$$\mathcal{M}_{r,2k} = \mathrm{Ind}_{H_{r,2k}}^{GL_n(F)}(\psi_{r,2k})$$

as a *Klyachko model* for $GL_n(F)$. Here Ind denotes the functor of non-compact smooth induction.

In [OS08b, Corollary 1] we showed that for any $\pi \in \mathcal{A}_F^u(n)$ there exists a unique decomposition

$$n = r(\pi) + 2k(\pi)$$

such that

$$\mathrm{Hom}_{GL_n(F)}(\pi, \mathcal{M}_{r(\pi),2k(\pi)}) \neq 0$$

and that in fact $\dim_{\mathbb{C}}(\mathrm{Hom}_{GL_n(F)}(\pi, \mathcal{M}_{r(\pi),2k(\pi)})) = 1$.

Definition 1. For $\pi \in \mathcal{A}_F^u$, the *Klyachko type* of π is the ordered pair $(r(\pi), 2k(\pi))$.

In fact, for $\mathcal{A}_F^u$ [OS08a, Theorem 8] provides a recipe for reading the Klyachko type off from Tadić's classification. Based on (4.3), our results can be reinterpreted by the formula

$$r(\pi) = \mathrm{odd}(\mathcal{V}(\pi)),\ \pi \in \mathcal{A}_F^u. \tag{5.1}$$

6. Base change-The main results

Let E be a finite extension of F. Denote by $\mathrm{res}_{E/F,n} : \mathcal{G}_F(n) \to \mathcal{G}_E(n)$ the map defined by $\mathrm{res}_{E/F,n}(([\rho], f)) = ([\rho_{|W_E}], f)$. For $n \geq 1$ the *base change* $\mathrm{bc}_{E/F}(\pi) \in \mathcal{A}_E(n)$ of $\pi \in \mathcal{A}_F(n)$ is defined by

$$\mathrm{rec}_E(\mathrm{bc}_{E/F}(\pi)) = \mathrm{res}_{E/F}(\mathrm{rec}_F(\pi)).$$

Theorem 6.1. *Let E/F be a finite extension of p-adic fields and let π be a smooth, irreducible and unitarizable representation of $GL_n(F)$. Then $\mathrm{bc}_{E/F}(\pi)$ is a smooth, irreducible and unitarizable representation of $GL_n(E)$ and*

$$\mathcal{V}(\pi) = \mathcal{V}(\mathrm{bc}_{E/F}(\pi)),$$

i.e. π and $\mathrm{bc}_{E/F}(\pi)$ have the same $SL(2)$-type.

As a consequence we have the following.

Corollary 6.1. *Under the assumptions of Theorem 6.1 we have*

$$r(\pi) = r(\mathrm{bc}_{E/F}(\pi)),$$

i.e. π and $\mathrm{bc}_{E/F}(\pi)$ have the same Klyachko type.

Corollary 6.1 is straightforward from Theorem 6.1 and (5.1).

7. Proof of the main result

Lemma 7.1. *Let E/F be a finite extension. For $\sigma \in \mathcal{A}_F^\circ \cap \mathcal{A}_F^u$ there exist $\sigma_1, \dots, \sigma_m \in \mathcal{A}_E^\circ \cap \mathcal{A}_E^u$ such that*

$$\mathrm{bc}_{E/F}(\sigma) = \sigma_1 \times \cdots \times \sigma_m.$$

Proof. Recall that a representation in $\mathcal{A}_F^\circ$ is unitarizable if and only if its central character is unitary. Let ρ be the irreducible representation of W_F such that $\mathrm{rec}_F(\sigma) = ([\rho], \mathbf{1}_n)$. It follows from (3.4) that ρ has a unitary central character and therefore it has a unitary structure. Thus, the restriction $\rho_{|W_E}$ to W_E also has a unitary structure and therefore each of its irreducible componencts has a unitary central character. The lemma follows by applying (4.4) to $\mathrm{res}_{E/F}(\mathrm{rec}_F(\sigma))$. ☐

Proposition 7.1. *Let E/F be a finite extension and let $\pi \in \mathcal{A}_F^u$ then $\mathrm{bc}(\pi) \in \mathcal{A}_E^u$ and*

$$\mathrm{bc}_{E/F}(\pi^t) = \mathrm{bc}_{E/F}(\pi)^t. \tag{7.1}$$

Proof. Let $\pi \in \mathcal{A}_F^u$ be of the form (4.1). It follows from Lemma 7.1 there exist $\sigma_{i,k} \in \mathcal{A}_E^\circ$, $i = 1, \dots, m$, $k = 1, \dots, t_i$ (not necessarily with a unitary central character) such that

$$\mathrm{bc}_{E/F}(\sigma_i) = \sigma_{i,1} \times \cdots \times \sigma_{i,t_i}.$$

Let $\rho_i = \mathrm{rec}_F^\circ(\sigma_i)$ and $\rho_{i,k} = \mathrm{rec}_E^\circ(\sigma_{i,k})$. Thus,

$$\mathrm{res}_{E/F}(\rho_i) = \oplus_{k=1}^{t_i} \rho_{i,k}.$$

It follows from (4.4) that

$$\mathrm{res}_{E/F}(\mathrm{rec}_F(\pi)) = \oplus_{i=1}^m \oplus_{j=1}^{r_i} \oplus_{k=1}^{t_i} \tau(\Delta[\sigma_{i,k}[\frac{r_i+1}{2} - j], n_i]). \tag{7.2}$$

On the other hand, let

$$\Pi = \times_{i=1}^{m} \times_{k=1}^{t_i} U(\delta[\sigma_{i,k}, n_i], r_i).$$

Since $\pi \in \mathcal{A}_F^u$, the classification of Tadić implies that $\Pi \in \mathcal{A}_E^u$ and by (4.4) applied to E instead of F we have

$$\mathrm{rec}_E(\Pi) = \oplus_{i=1}^{m} \oplus_{j=1}^{r_i} \oplus_{k=1}^{t_i} \tau(\Delta[\sigma_{i,k}[\frac{r_i+1}{2} - j], n_i]). \tag{7.3}$$

Comparing (7.2) with (7.3) we obtain that $\Pi = \mathrm{bc}_{E/F}(\pi)$ and in particular that $\mathrm{bc}_{E/F}(\pi) \in \mathcal{A}_E^u$. Applying this to π^t expressed by (4.5) gives

$$\mathrm{bc}_{E/F}(\pi^t) = \times_{i=1}^{m} \times_{k=1}^{t_i} U(\delta[\sigma_{i,k}, r_i], n_i).$$

Applying (4.5) now to $\mathrm{bc}_{E/F}(\pi)^t$ we obtain the identity (7.1). □

It is straightforward from the definitions that

$$P_F(\mathrm{rec}_F(\pi)) = P_E(\mathrm{rec}_E(\mathrm{bc}_{E/F}(\pi))), \ \pi \in \mathcal{A}_F. \tag{7.4}$$

For $\pi \in \mathcal{A}_F^u$, applying (7.4) to π^t and then (7.1) we get that

$$P_F(\mathrm{rec}_F(\pi^t)) = P_E(\mathrm{rec}_E(\mathrm{bc}_{E/F}(\pi)^t)).$$

The identity $\mathcal{V}(\pi) = \mathcal{V}(\mathrm{bc}_{E/F}(\pi))$ is now immediate from the definition (4.7). This completes the proof of Theorem 6.1.

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