DESCENT CONSTRUCTION FOR GSPIN GROUPS: MAIN RESULTS AND APPLICATIONS

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The purpose of this note is to announce an extension of the descent method of Ginzburg, Rallis, and Soudry to the setting of *essentially* self dual representations. This extension of the descent construction provides a complement to recent work of Asgari and Shahidi [AS06] on the generic transfer for general Spin groups as well as to the work of Asgari and Raghuram [A-R] on cuspidality of the exterior square lift for representations of GL_4. Complete proofs of the results announced in the present note will appear in our forthcoming article(s).

1. Preliminaries

1.1. **GSpin groups and their quasisplit forms.** Let F be a number field. By the classification results in Chapter 16 of [Spr98](see especially 16.3.2, 16.3.3, 16.4.2), and the definition of the L group, there is a unique quasisplit F group G such that the connected component of the identity in ${}^LG$ is $GSp_{2n}(\mathbb{C})$. This is $GSpin_{2n+1}$.

Similarly, there is a 1-1 correspondence between quasisplit F groups G such that ${}^LG^0 = GSO_{2n}(\mathbb{C})$ and homomorphisms from $\mathrm{Gal}(\bar{F}/F)$ to the group with two elements, and hence, by class field theory, with quadratic characters of $\chi : \mathbb{A}_F^\times/F^\times \to \{\pm 1\}$ (the case $n = 4$ is no different, see Section 5). The unique split group G such that ${}^LG^0 = GSO_{2n}(\mathbb{C})$ corresponds to the trivial character. We denote this group $GSpin_{2n}$. The finite Galois form of its L group is $GSO_{2n}(\mathbb{C})$. The form corresponding to the nontrivial character χ we denote by $GSpin_{2n}^\chi$. The finite Galois form of its L group is $GSO_{2n}(\mathbb{C}) \rtimes Gal(E/F)$ where E is the quadratic extension of F corresponding to χ.

1.2. **Liftings.** According to the Langlands functoriality conjecture, one expects a lifting of automorphic representations of $GSpin_{2n+1}(\mathbb{A})$ to automorphic representations of $GL_{2n}(\mathbb{A})$ corresponding to the inclusion

$$GSp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}).$$

Similarly, one expects a lifting of automorphic representations of $GSpin_{2n}(\mathbb{A})$ to automorphic representations of $GL_{2n}(\mathbb{A})$ corresponding to the inclusion

$$GSO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}).$$

For (globally) generic representations, the existence of these liftings is proved in [AS06].

Now consider $GSpin_{2n}^\chi$ for $\chi \neq 1$. Regardless of χ, the L group $GSO_{2n}(\mathbb{C}) \rtimes Gal(E/F)$, with E as above, is isomorphic to $GO_{2n}(\mathbb{C})$, and a specific isomorphism can be fixed by mapping the nontrivial element of $Gal(E/F)$ to

$$\begin{pmatrix} I_{n-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{n-1} \end{pmatrix}.$$

One then expects a lifting of automorphic representations of $GSpin_{2n}^\chi(\mathbb{A})$ to automorphic representations of $GL_{2n}(\mathbb{A})$ corresponding to the inclusion

$$GO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}).$$

We understand that this is the subject of [AS08]. We denote all of these liftings by AS.

2. Main results

2.1. The odd case.

Theorem A. *Let ω be a Hecke character. Suppose $n_1, \dots, n_m \in \mathbb{N}$, and that, for each $1 \le i \le m$, τ_i is an irreducible cuspidal automorphic representation of $GL_{2n_i}(\mathbb{A})$ such that $L^S(s, \tau_i, \wedge^2 \times \omega^{-1})$ has a pole at $s = 1$. Suppose furthermore that the representations τ_i are all distinct. Let $n = n_1 + \dots + n_m$. Then there exists a globally generic irreducible cuspidal automorphic representation σ of $GSpin_{2n+1}(\mathbb{A})$ such that $AS(\sigma) = Ind_{P_{\underline{n}}(\mathbb{A})}^{GL_{2n}(\mathbb{A})}(\tau_1 \otimes \dots \otimes \tau_m)$ (normalized induction), where $P_{\underline{n}}$ is the standard parabolic of GL_{2n} corresponding to the ordered partition $2n = 2n_1 + \dots + 2n_m$ of $2n$. Furthermore, the central character of σ is ω.*

Remarks 2.1.1.
1. *The notation "$AS(\sigma) = \dots$" requires some justification: Theorem 1.1 of* [AS06] *assures the existence of a weak lift Π of σ, but not its uniqueness. However, for $\tau_1, \dots, \tau_m$ as in our theorem, the induced representation $Ind_{P_{\underline{k}}(\mathbb{A})}^{GL_{2n}(\mathbb{A})}(\tau_1 \otimes \dots \otimes \tau_m)$, is irreducible. In conjunction with Proposition 7.4, of* [AS06] *this implies that for the representation σ which we construct, the transfer is uniquely determined.*
2. *Note that if τ is a representation of $GL_\ell(\mathbb{A})$ and $L^S(s, \tau, \wedge^2 \times \omega^{-1})$ has a pole at $s = 1$ then it follows from* [JS90]*, Theorem 2, p. 224, that ℓ is even, and from* [JS90]*, Theorem 1, p. 213, that $\omega_\tau = \omega^{\frac{\ell}{2}}$.*
3. *If τ is a representation of $GL_\ell(\mathbb{A})$ such that $L^S(s, \tau, sym^2 \times \omega^{-1})$ has a pole at $s = 1$, one may not deduce that ℓ is even. However, one may deduce that $\tau \cong \tilde{\tau} \otimes \omega$, whence $\omega^\ell = \omega_\tau^2$ (where ω_τ is the central character of τ). If ℓ is odd, it then follows that ω is the square of another global character η, and that $\tau' = \tau \otimes \eta^{-1}$ is self dual, with $L^S(s, \tau', sym^2)$ having a pole at $s = 1$. Thus, the case when ω is a square reduces to the self-dual case, and in the case when ω is not a square we can deduce that ℓ is even and that $\omega_\tau / \omega^{\frac{\ell}{2}}$ is quadratic.*

2.2. The even case.

For the statement of the next main result, it will be convenient to define $GSpin_{2n}^\chi := GSpin_{2n}$ when χ is the trivial character.

Theorem B. *Let ω be a Hecke character which is not the square of another Hecke character. Suppose $n_1, \dots, n_m \in \mathbb{N}$, and that, for each i, τ_i is an irreducible cuspidal automorphic representation of $GL_{2n_i}(\mathbb{A})$ such that $L^S(s, \tau_i, sym^2 \times \omega^{-1})$ has a pole at $s = 1$. Suppose furthermore that the representations τ_i are all distinct. Let $n = n_1 + \dots + n_m$, and, for each i, let $\chi_i = \omega_{\tau_i} / \omega^{n_i}$, which is quadratic. Let $\chi = \prod_{i=1}^m \chi_i$. Then there exists a globally generic irreducible cuspidal automorphic representation σ of $GSpin_{2n}^\chi(\mathbb{A})$ such that $AS(\sigma) = Ind_{P_{\underline{n}}(\mathbb{A})}^{GL_{2n}(\mathbb{A})}(\tau_1 \otimes \dots \otimes \tau_m)$ (normalized induction), where $P_{\underline{n}}$ is the standard parabolic of GL_{2n} corresponding to the ordered partition $2n = 2n_1 + \dots + 2n_m$ of $2n$. Furthermore, the central character of σ is ω.*

3. Applications

3.1. The image of the weak lift AS.

We now concentrate on the case of split general Spin groups. In [AS06] , the authors show the existence of functorial lifts from automorphic representations of $GSpin_{2n}(\mathbb{A})$ or $GSpin_{2n+1}(\mathbb{A})$ to $GL_{2n}(\mathbb{A})$. They show that the images consist of automorphic representations which satisfy the essential self-duality condition at almost all places.

Based on the self-dual case, (cf. Theorem A of [GRS01]) one expects that the image of each Asgari-Shahidi lifting consists of isobaric sums of distinct essentially self dual cuspidal representations satisfying the appropriate L-function condition. For example, any representation in the image of the lift from $GSpin_{2n+1}$ should be an isobaric sum of distinct ω-symplectic cuspidals, for some Hecke character ω.

Our results support this expectation. We provide a "lower bound" for the image of the Asgari-Shahidi lifting, by showing that any isobaric sum of distinct essentially self dual cuspidal representations satisfying the appropriate L-function condition is in the image of the appropriate lift.

3.2. **The image of the exterior square lift: GL_4 to GL_6.** The existence of an exterior square lift of a cuspidal automorphic representation of $GL_4(\mathbb{A})$ as an automorphic representation of $GL_6(\mathbb{A})$ was established by Kim in [Kim03]. Recently, Asgari-Raghuram provided an explicit description of those cuspidal automorphic representations of $GL_4(\mathbb{A})$ whose exterior square lift to $GL_6(\mathbb{A})$ is not cuspidal. Among other things their argument requires the following special case of Theorem B.

Corollary 3.2.1. *Let Π be a cuspidal representation of $GL_4(\mathbb{A})$ and let ω be any character of $GL_1(F)\backslash GL_1(\mathbb{A})$. Assume that the partial L-function $L^S(s, \Pi, sym^2 \otimes \omega^{-1})$ has a pole at $s = 1$ for a sufficiently large finite set S of places of F. Let $\chi = \omega_\Pi \omega^{-2}$. Then there exists a globally generic cuspidal representation π of $GSpin_4^\chi(\mathbb{A})$ such that π transfers to Π.*

Roughly speaking, Asgari and Raghuram prove that the exterior square lift of a cuspidal representation Π of GL_4 is cuspidal unless Π is isomorphic to a twist of either itself or its contragredient, and that this occurs only if Π is itself in the image of one of four functorial lifts. For the precise statement, see [A-R], Theorem 1.1, p.2. For the precise relationship with our results, see p. 12.

4. Scheme of Proof

The proofs of Theorem **A** and Theorem **B** are obtained by adapting (the special orthogonal group case of) the descent method of Ginzburg, Rallis, and Soudry [GRS99b, GRS99a, GRS99c, GRS01, GRS02]. Some of our arguments are adaptations of unpublished arguments of Ginzburg, Rallis, and Soudry, kindly explained to us by them. The adaptation is reasonably straightforward owing to two observations:

(1) There is a surjective homomorphism, defined over F, from $GSpin_m$ to SO_m, which restricts to an isomorphism between the unipotent subvarieties.
(2) The kernel of this projection is contained in the center of $GSpin_m$. Thus, the action $GSpin_m$ on itself by conjugation factors through the projection.

In what follows we detail the steps needed to prove Theorem **B**. The proof of Theorem **A** is similar and technically simpler.

The input to the construction is a collection $\underline{\tau} = \{\tau_1, \dots, \tau_m\}$ of cuspidal representations τ_i of $GL_{2n_i}(\mathbb{A})$ for $i = 1, \dots, m$, satisfying the assumptions of Theorem B. Let $\chi_{\underline{\tau}} = \omega^{-n} \cdot \prod_{i=1}^m \omega_{\tau_i}$. Then $\chi_{\underline{\tau}}$ is a quadratic character.

We can conveniently describe the method in the following steps:

(1) Construction of a family of descent representations of $GSpin^\chi_{4n+1-2\ell}(\mathbb{A})$ for $\ell \geq n$.
(2) Vanishing of the descent representations for all $\ell > n$ and all $\chi \neq \chi_{\underline{\tau}}$.
(3) Cuspidality and genericity (hence nonvanishing) of the descent representation of $GSpin_{2n}^{\chi_{\underline{\tau}}}(\mathbb{A})$.
(4) Matching of spectral parameters at unramified places.

The construction of the descent representations relies on the notion of Fourier coefficient, as defined in [GRS03], [G] (cf. also the "Gelfand-Graev" coefficients of [So]). For purposes of presenting certain of the global arguments, it seems convenient to embed these Fourier coefficients into a slightly larger family of functionals, which we shall refer to as "unipotent periods."

Suppose that U is a unipotent subgroup of $GSpin_{4n+1}$ and ψ is a character of $U(F)\backslash U(\mathbb{A})$. We define the corresponding *unipotent period* to be the map from smooth, left $U(F)$-invariant functions on $GSpin_{4n+1}(\mathbb{A})$ to smooth, left $(U(\mathbb{A}),\psi)$-equivariant functions, given by

$$\varphi \mapsto \varphi^{(U,\psi)}$$

where

$$\varphi^{(U,\psi)}(g) := \int_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\psi(u)\,du.$$

Each unipotent period has a local analogue at each finite place, which is a twisted Jacquet functor.

Suppose now that U is the unipotent radical of a standard parabolic subgroup, and let M denote the Levi. The characters of $U(F)\backslash U(\mathbb{A})$ may be identified with the points of an F-vector space, so that the stabilizer $Stab_M(\psi)$ makes sense as an algebraic group defined over F. We assume that ψ corresponds to a point in general position. Then the map

$$FC^{\psi} : \varphi \mapsto \varphi^{(U,\psi)}\Big|_{Stab_M(\psi)(\mathbb{A})}$$

is indeed a "Fourier coefficient," as defined (and associated to a nilpotent orbit) in [GRS03, G].It maps smooth functions of moderate growth on $GSpin_{4n+1}(F)\backslash GSpin_{4n+1}(\mathbb{A})$ to smooth functions of moderate growth on $Stab_M(\psi)(F)\backslash Stab_M(\psi)(\mathbb{A})$.

Let S be a set of unipotent periods. We will say that another unipotent period (U,ψ) *is spanned by S* if

$$\left(\varphi^{(N,\vartheta)} \equiv 0\ \forall (N,\vartheta)\in S\right) \quad \Longrightarrow \quad \varphi^{(U,\psi)} \equiv 0.$$

We are now ready to describe each of the four steps listed above in more detail.

Step one: Construction of the descent representations

Using $\tau_1,\dots,\tau_m$, a space of Eisenstein series $E_{\underline{\tau},\omega}(g,\underline{s})$ on $GSpin_{4n+1}(\mathbb{A})$ is constructed– corresponding to a representation induced from the standard parabolic subgroup $P = MU$ of $GSpin_{4n+1}$ for which $M \cong GL_{2n_1}\times\cdots\times GL_{2n_m}\times GL_1$. The partial L functions

$$L^S(s,\tau_i, sym^2\times\omega^{-1}) \qquad i = 1 \text{ to } m$$

appear in the constant terms of elements of this space. As a consequence, some of them have non-vanishing multi-residues at a certain point $\underline{s}_0$, *precisely because* of the L-function hypothesis on $\underline{\tau}$. In this fashion we obtain a residual representation– which lies in the discrete spectrum of $L^2(GSpin_{4n+1}(F)\backslash GSpin_{4n+1}(\mathbb{A}))$– the nontriviality of which depends intrinsically on this L-function condition. We denote this representation by $\mathcal{E}_{-1}(\underline{\tau},\omega)$.

Now, $GSpin_{4n+1}$ contains a family of parabolic subgroups $Q_\ell = L_\ell N_\ell,\ \ell = 1$ to $2n$, with L_ℓ isomorphic to $GL_1^\ell \times G_{4n-2\ell+1}$, having the crucial property that for each character ψ of N_ℓ in general position, the identity component of the group $Stab_{L_\ell}(\psi)$ is isomorphic to one of the groups $GSpin^{\chi}_{4n-2\ell}$. Fixing specific isomorphisms, we may pull back each Fourier coefficient

$$FC^{\psi}(\mathcal{E}_{-1}(\underline{\tau},\omega))$$

as described above, to a space of functions defined on $GSpin^{\chi}_{4n-2\ell}(\mathbb{A})$. There are many characters ψ for a given value of ℓ and χ, but they comprise a single orbit for the action of $L_\ell(F)$ by conjugation, and the various spaces $FC^{\psi}(\mathcal{E}_{-1}(\underline{\tau},\omega))$ all pull back to the same space of functions on $GSpin^{\chi}_{4n-2\ell}(\mathbb{A})$, regardless of the choice of ψ in this orbit and regardless of the choice of isomorphism $GSpin^{\chi}_{4n-2\ell} \to Stab_{L_\ell}(\psi)^0$. (For this, we require the extension of meromorphic continuation of Eisenstein series to non-K-finite sections, provided in [La08].)

In this manner we obtain a space of functions on $GSpin^{\chi}_{4n-2\ell}(\mathbb{A})$ for each value of χ. The family of representations thus obtained comprises the descent representations.

Step two: Vanishing of other descents

For $\ell > n$, one shows that the above Fourier coefficients vanish identically on our residue representation $\mathcal{E}_{-1}(\underline{\tau}, \omega)$. The reason is local: the corresponding twisted Jacquet module of the unramified constituent of the corresponding local induced representation vanishes. The same is true if $\ell = n$, at any unramified place v such that the identity component of $Stab_{L_n}(\psi)$ is not isomorphic to $GSpin_{2n}^{\chi}$ over F_v.

The remaining descent representation, corresponding to $\ell = n$ and $\chi = \chi_{\underline{\tau}}$, may now be referred to as "the" descent without ambiguity.

Step Three: Cuspidality and genericity of the descent

Next we appeal to global arguments which may be presented in terms of "identities of unipotent periods."

Consider the unipotent period on $C^{\infty}(GSpin_{4n+1}(F)\backslash GSpin_{4n+1}(\mathbb{A}))$ which consists of taking the constant term with respect to the maximal parabolic with Levi isomorphic to $GL_{2n} \times GL_1$, and then taking a Whittaker integral on the GL_{2n} Levi. It can be shown that this unipotent period does not vanish on our residue representation $\mathcal{E}_{-1}(\underline{\tau}, \omega)$. One shows that this unipotent period is, in fact, spanned by the periods corresponding to Whittaker integrals on the descent representations (as $\ell \geq n$ and χ vary).

Having proved by local arguments that these periods vanishes identically on the residue representation $\mathcal{E}_{-1}(\underline{\tau}, \omega)$, whenever $\ell > n$ or $\chi \neq \chi_{\underline{\tau}}$, we deduce that they do not vanish identically when $\ell = n$ and $\chi = \chi_{\underline{\tau}}$. This shows that the space $FC^{\psi}(\mathcal{E}_{-1}(\underline{\tau}, \omega))$ is not only nontrivial, but supports a nontrivial global Whittaker integral.

Next, consider the unipotent periods on $C^{\infty}(GSpin_{4n+1}(F)\backslash GSpin_{4n+1}(\mathbb{A}))$ which consist of taking the constant term with respect to the maximal parabolic with Levi isomorphic to $GL_k \times GSpin_{4n-2k+1}$ for some k, and then, if k is even, performing the integral one would use to define a descent representation of $GSpin_{4n-2k+1}$, with some value of ℓ larger than $n - \frac{k}{2}$. Combining the vanishing results of Step two with well-known facts from the theory of Eisenstein series, we deduce that all of these periods vanish identically on the residue representation $\mathcal{E}_{-1}(\underline{\tau}, \omega)$. We then show that the unipotent period which corresponds to taking the constant term of one of the functions in $FC^{\psi}(\mathcal{E}_{-1}(\underline{\tau}, \omega))$ is in their span.

It follows that all the functions in the descent are cuspidal. At this point, we may deduce that the descent representation decomposes discretely as a direct sum of irreducible cuspidal automorphic representations, at least one of which is generic. We select one such component for the representation σ of Theorem B. What remains is to show that σ lifts weakly to $Ind_{P_{\underline{n}}(\mathbb{A})}^{GL_{2n}(\mathbb{A})}(\tau_1 \otimes \cdots \otimes \tau_m)$.

Step Four: Matching of spectral parameters at unramified places

For $\ell = n$, at an unramified place, where the identity component of $Stab_{L_n}(\psi)$ is isomorphic to $GSpin_{2n}^{\chi_{\underline{\tau}}}$, the twisted Jacquet module of the unramified constituent of the local induced representation is isomorphic, as a $Stab_{L_n}(\psi)(F_v)$-module to a certain induced representation of $Stab_{L_n}(\psi)(F_v)$. When restricted to the identity component, this representation may not be irreducible. Nevertheless, we are able to deduce that any nonzero irreducible component of the Fourier coefficient must lift weakly to $Ind_{P_{\underline{n}}(\mathbb{A})}^{GL_{2n}(\mathbb{A})}(\tau_1 \otimes \cdots \otimes \tau_m)$.

5. Final Remarks

(1) When considering the identification of $GO_{2n}(\mathbb{C})$ with $GSO_{2n}(\mathbb{C}) \rtimes Gal(E/F)$, one could also map the nontrivial element to

$$\begin{pmatrix} -I_{n-1} & & & \\ & & -1 & \\ & -1 & & \\ & & & -I_{n-1} \end{pmatrix}.$$

This produces a slightly different functorial lift corresponding to the twist of the one we have chosen above by the quadratic character χ. Theorem B is, of course, true for this "alternate" lifting, as well, since one may "untwist."

(2) These are essentially the only distinct extensions of the inclusion $GSO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ to $GSO_{2n}(\mathbb{C}) \rtimes Gal(E/F)$ in the following sense. Suppose V_1 and V_2 are two $2n$ dimensional representations of $GSO_{2n}(\mathbb{C}) \rtimes Gal(E/F)$ such that the restriction of either to $GSO_{2n}(\mathbb{C})$ is the standard representation. Then one may show that V_2 is isomorphic to either V_1 or the twist of V_1 by the unique nontrivial character of $Gal(E/F)$.

(3) A natural question arises in the case $n = 4$: **does the 3-fold symmetry of the Dynkin diagram of $GSpin_8$ lead to additional quasi-split forms?** The answer is no, because the 3-fold symmetry of the D_4 root *system* does not extend to a symmetry of the root *data* of GSO_8 and $GSpin_8$.

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