

$(O(V \oplus F), O(V))$ IS A GELFAND PAIR
FOR ANY QUADRATIC SPACE V OVER A LOCAL FIELD F

AVRAHAM AIZENBUD, DMITRY GOUREVITCH, AND EITAN SAYAG

ABSTRACT. Let V be a quadratic space with a form q over an arbitrary local field F of characteristic different from 2. Let $W = V \oplus Fe$ with the form Q extending q with $Q(e) = 1$. Consider the standard embedding $O(V) \hookrightarrow O(W)$ and the two-sided action of $O(V) \times O(V)$ on $O(W)$.

In this note we show that any $O(V) \times O(V)$ -invariant distribution on $O(W)$ is invariant with respect to transposition. This result was earlier proven in a bit different form in [vD] for $F = \mathbb{R}$, in [AvD] for $F = \mathbb{C}$ and in [BvD] for p -adic fields. Here we give a different proof.

Using results from [AGS], we show that this result on invariant distributions implies that the pair $(O(V), O(W))$ is a Gelfand pair. In the archimedean setting this means that for any irreducible admissible smooth Fréchet representation (π, E) of $O(W)$ we have $\dim \text{Hom}_{O(V)}(E, \mathbb{C}) \leq 1$.

A stronger result for p -adic fields is obtained in [AGRS].

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1. INTRODUCTION

Let F be a local field of characteristic different from 2.

Let (W, Q) be a quadratic space defined over F and fix $e \in W$ a unit vector. Consider the quadratic space $V = e^\perp$ with $q = Q|_V$. Define the standard imbedding $O(V) \hookrightarrow O(W)$ and consider the two-sided action of $O(V) \times O(V)$ on $O(W)$ defined by $(g_1, g_2)h := g_1 h g_2^{-1}$. We also consider the anti-involution τ of O_Q given by $\tau(g) = g^{-1}$. In this paper we prove the following theorem

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Theorem (A). *Any $O(V) \times O(V)$ invariant distribution on $O(W)$ is invariant under τ .*

This theorem has the following corollary in representation theory.

Theorem (B). *Let (π, E) be an irreducible admissible representation of $O(W)$. Then*

$$\dim \text{Hom}_{O(V)}(E, \mathbb{C}) \leq 1$$

Here admissible representation refers to the usual notion in the non-archimedean case and to the notion of admissible smooth Fréchet representation in the archimedean setting.

Our proof for the archimedean and non-archimedean case is uniform, except at one point where the archimedean case requires an extra analysis of a certain normal bundle (see lemma 4.2).

Remark 1.1. *We note that a related result for unitary representations of $SO(V, Q)$ is proved in [BvD] (for p -adic fields) and in [vD] (for the real numbers). In fact, the proof given in those papers implies also theorem **A**. Also, an analogous theorem for unitary groups is proven in [vD2].*

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2. FROM INVARIANT DISTRIBUTIONS TO REPRESENTATION THEORY

In this section we recall a technique due to Gelfand and Kazhdan which allows to deduce theorem **B** from theorem **A**.

Recall the following theorem ([AGS])

Theorem 2.1. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(T) = T$ for all bi H -invariant distributions ¹ on G . Then for any irreducible admissible representation (π, E) of G we have*

$$\dim \text{Hom}_H(E, \mathbb{C}) \cdot \dim \text{Hom}_H(\tilde{E}, \mathbb{C}) \leq 1,$$

where \tilde{E} denotes the smooth contragredient representation.

Note that in the non-archimedean case the same result is proven in [Pra].

To finish the deduction of theorem **B** from theorem **A** we will show that

Theorem 2.2. *Let (π, E) be an irreducible admissible representation of $G = O(V)$. Then $\tilde{E} \cong E$ and in particular*

$$\dim \text{Hom}_H(E, \mathbb{C}) = \dim \text{Hom}_H(\tilde{E}, \mathbb{C})$$

For the proof we recall proposition I.2 (chapter 4) from [MVW]:

Proposition 2.3. *Let V be a quadratic space and let $g \in O(V)$. Then g is conjugate to g^{-1} .*

¹In fact it is enough to check this only for Schwartz distributions.

Proof of Theorem 2.2. For non-archimedean fields this is a theorem from [MVW] page 91. For archimedean fields we use the Harish-Chandra regularity theorem and the proposition that any element in $g \in O(V)$ is conjugate in $O(V)$ to g^{-1} . Thus, the characters of E and \tilde{E} are the same and hence $\tilde{E} \cong E$. \square

Remark 2.4. *A related result for the groups $SO(V)$ can be found in [GP], proposition 5.3.*

3. BASIC RESULTS ON INVARIANT DISTRIBUTIONS

In this paper we consider distributions over l -spaces and over smooth manifolds. l -spaces are locally compact totally disconnected topological spaces (see [BZ], section 1).

For X a smooth manifold or an l -space we denote by $\mathcal{D}(X)$ the space of distributions on X . When X is an l -space this means that $\mathcal{D}(X) = S(X)^*$ where $S(X)$ is the space of locally constant functions with compact support on X . For smooth X , we let $\mathcal{D}(X) = C_c^\infty(X)^*$.

The basic tools to study invariant distributions on a G -space X are Bruhat filtration, Frobenius reciprocity ([BZ], [Bar] and [AGS]) and the Bernstein's localization principle ([Ber] and [AG]). Let us remind the statements.

For the simplicity of formulation we provide, for each principle, two versions: for l -spaces and for smooth manifolds.

3.1. Bruhat Filtration. Although we will not need the non-archimedean version of this principle, we formulate it for completeness. It is a simple consequence of proposition 1.8 in [BZ].

Theorem 3.1. *Let an l -group G act on an l -space X . Let $X = \bigcup_{i=0}^l X_i$ be a G -invariant stratification of X . Let χ be a character of G . Suppose that $\mathcal{D}(X_i)^{G \cdot \chi} = 0$. Then $\mathcal{D}(X)^{G \cdot \chi} = 0$.*

To formulate the archimedean version we let X be a smooth manifold and $Y \subset X$ a smooth submanifold. We remind the definition of the conormal bundle CN_Y^X . For this denote by T_X the tangent bundle of X and by $N_Y^X := (T_X|_Y)/T_Y$ the normal bundle to Y in X . The conormal bundle is defined by $CN_Y^X := (N_Y^X)^*$. Denote by $Sym^k(CN_Y^X)$ the k -th symmetric power of the conormal bundle.

Theorem 3.2 (Bruhat). *Let a Lie group G act on a smooth manifold X . Let $X = \bigcup_{i=0}^l X_i$ be a smooth G -invariant stratification of X . Let χ be a character of G . Suppose that for any $k \in \mathbb{Z}_{\geq 0}$, $\mathcal{D}(X_i, Sym^k(CN_{X_i}^X))^{G \cdot \chi} = 0$. Then $\mathcal{D}(X)^{G \cdot \chi} = 0$.*

For proof see [AGS].

3.2. Frobenius reciprocity. For l -space, the following version of Frobenius reciprocity is proven in [Ber]:

Theorem 3.3 (Frobenius reciprocity). *Let a unimodular l -group G act transitively on an l -space Z . Let $\varphi : X \rightarrow Z$ be a G -equivariant continuous map. Let $z \in Z$. Suppose that its stabilizer $\text{Stab}_G(z)$ is unimodular. Let X_z be the fiber of z . Let χ be a character of G . Then $\mathcal{D}(X)^{G \cdot \chi}$ is canonically isomorphic to $\mathcal{D}(X_z)^{\text{Stab}_G(z) \cdot \chi}$.*

An archimedean version is considered in [Bar]. Here is a slight generalization (see [AGS]):

Theorem 3.4 (Frobenius reciprocity). *Let a unimodular Lie group G act transitively on a smooth manifold Z . Let $\varphi : X \rightarrow Z$ be a G -equivariant smooth map. Let $z \in Z$. Suppose that its stabilizer $\text{Stab}_G(z)$ is unimodular. Let X_z be the fiber of z . Let χ be a character of G . Then $\mathcal{D}(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}(X_z)^{\text{Stab}_G(z),\chi}$. Moreover, for any G -equivariant bundle E on X , $\mathcal{D}(X, E)^{G,\chi}$ is canonically isomorphic to $\mathcal{D}(X_z, E|_{X_z})^{\text{Stab}_G(z),\chi}$.*

3.3. Bernstein's Localization principle. For l -spaces it is taken from [Ber]:

Theorem 3.5 (Localization principle). *Let X and T be l -spaces and $\phi : X \rightarrow T$ be a continuous map. Let an l -group G act on X preserving the fibers of ϕ . Let χ be a character of G . Suppose that for any $t \in T$, $\mathcal{D}(\phi^{-1}(t))^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.*

For real smooth algebraic varieties, the following theorem is proven in [AG], Corollary 4.0.8:

Theorem 3.6 (Localization principle). *Let a real reductive group G act on a smooth affine real algebraic variety X . Let Y be a smooth real algebraic variety and $\phi : X \rightarrow Y$ be an algebraic G -invariant submersion. Suppose that for any $y \in Y$ we have $\mathcal{D}(\phi^{-1}(y))^{G,\chi} = 0$. Then $\mathcal{D}(X)^{G,\chi} = 0$.*

4. PROOF OF THEOREM A

Recall the setting. (W, Q) is a quadratic space over F , $e \in W$ with $Q(e) = 1$. Also (V, q) is defined by $V = e^\perp$ and $q = Q|_V$.

We need some further notations.

- $O_q = O(V, q)$ is the group of isometries of the quadratic space (V, q) .
- $G_q = O(V, q) \times O(V, q)$.
- $\Delta : O_q \rightarrow G_q$ the diagonal. $H_q = \Delta(O_q) \subset G_q$.
- $\tau(g_1, g_2) = (g_2, g_1)$.
- $\widetilde{G}_q = G_q \rtimes \{1, \tau\}$, same for \widetilde{H}_q
- $\chi : \widetilde{G}_q \rightarrow \{+1, -1\}$ the non trivial character with $\chi(G_q) = 1$.
- \widetilde{G}_Q acts on O_Q by $(g_1, g_2)x = g_1xg_2^{-1}$ and $\tau(x) = x^{-1}$.

Clearly Theorem A follows from the following theorem:

Theorem 4.1. $\mathcal{D}(O_Q)^{\widetilde{G}_q,\chi} = 0$

4.1. Proof of theorem 4.1. We denote by $\Gamma = \{w \in W : Q(w) = 1\}$. Note that by Witt's theorem Γ is an O_Q transitive set and therefore $\Gamma \times \Gamma$ is a transitive \widetilde{G}_Q set where the action of G_Q is the standard action on $W \oplus W$ and τ acts by flip.

Applying Frobenius reciprocity (3.3, 3.4) to projections of $O_Q \times \Gamma \times \Gamma$ first on $\Gamma \times \Gamma$ and then on O_Q we have

$$\mathcal{D}(O_Q)^{\widetilde{G}_q,\chi} = \mathcal{D}(O_Q \times \Gamma \times \Gamma)^{\widetilde{G}_Q,\chi}$$

and also that

$$\mathcal{D}(O_Q \times \Gamma \times \Gamma)^{\widetilde{G}_Q,\chi} = \mathcal{D}(\Gamma \times \Gamma)^{\widetilde{H}_Q,\chi}$$

In what follows we will abuse notation and write $Q(u, v)$ for the bilinear form defined by Q . Define a map $D : \Gamma \times \Gamma \rightarrow Z$ where $Z = \{(v, u) \in W \oplus W : Q(v, u) = 0, Q(v + u) = 4\}$ by

$$D(x, y) = (x + y, x - y).$$

D defines an \tilde{G}_Q -equivariant homeomorphism and thus we need to show that

$$\mathcal{D}(Z)^{\tilde{H}_Q, \chi} = 0$$

Here, the action of G_Q on $Z \subset W \oplus W$ is the restriction of its action on $W \oplus W$ while the action of τ is given by $\tau(v, u) = (v, -u)$.

Now we cover $Z = U_1 \cup U_2$ where

$$U_1 = \{(v, u) \in Z : Q(v) \neq 0\}$$

and

$$U_2 = \{(v, u) \in Z : Q(u) \neq 0\}$$

We will show $\mathcal{D}(U_1)^{\tilde{H}_Q, \chi} = 0$, and the proof for U_2 is analogous. This will finish the proof.

Lemma 4.2. $\mathcal{D}(U_1)^{\tilde{H}_Q, \chi} = 0$

Proof for non-archimedean F . Consider $\ell_1 : U_1 \rightarrow F - \{0\}$ defined as $\ell_1(v, u) = Q(v)$. By the localization principle, it is enough to show $\mathcal{D}(U_1^\alpha)^{\tilde{H}_Q, \chi} = 0$ where $U_1^\alpha = \ell_1^{-1}(\alpha)$, for any $\alpha \in F - \{0\}$. But

$$U_1^\alpha = \{(v, u) | Q(v) = \alpha, Q(u) = 4 - \alpha, Q(v, u) = 0\}$$

Let $W^\alpha = \{w \in W | Q(w) = \alpha\}$ and let $p_1 : U_1^\alpha \rightarrow W^\alpha$ be given by $p_1(v, u) = v$.

On W^α our group acts transitively. Fix a vector $v_0 \in W^\alpha$.

Denote $H(v_0) := H_{(Q|_{v_0^\perp})}$ and $\tilde{H}(v_0) := \tilde{H}_{(Q|_{v_0^\perp})}$.

The stabilizer in \tilde{H}_Q of v_0 is $\tilde{H}(v_0)$. The fiber $p_1^{-1}(v_0) = \{a \in v_0^\perp | Q(a) = 4 - \alpha\}$. Frobenius reciprocity implies that

$$\mathcal{D}(U_1^\alpha)^{\tilde{H}_Q, \chi} = \mathcal{D}(p_1^{-1}(v_0))^{\tilde{H}(v_0), \chi}$$

But clearly $\mathcal{D}(p_1^{-1}(v_0))^{\tilde{H}(v_0), \chi} = 0$ as $-Id \in H(v_0)$. □

Proof for archimedean F . Now let us consider the archimedean case. Define $U := \{(v, u) \in U_1 | u \neq 0\}$. Note that the map $\ell_1|_U$ is a submersion, so the same argument as in the non-archimedean case shows that $\mathcal{D}(U)^{\tilde{H}_Q, \chi} = 0$. Let $Y := \{(v \in W | Q(v) = 4\} \times \{0\}$ be the complement to U in U_1 . By theorem 3.2, it is enough to prove $\mathcal{D}(Y, \text{Sym}^k(CN_Y^{U_1}))^{\tilde{H}_Q, \chi} = 0$.

Note that the action of \tilde{H}_Q on Y is transitive, and fix a point $(v, 0) \in Y$. The stabilizer in \tilde{H}_Q of $(v, 0)$ is $\tilde{H}(v)$, and the normal space to Y at $(v, 0)$ is v^\perp . So Frobenius reciprocity (theorem 3.4) implies that

$$\mathcal{D}(Y, \text{Sym}^k(CN_Y^{U_1}))^{\tilde{H}_Q, \chi} = \text{Sym}^k(v^\perp)^{\tilde{H}(v), \chi}$$

But clearly $\text{Sym}^k(v^\perp)^{\tilde{H}(v), \chi} = 0$ as $-Id \in H(v)$. □

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E-mail address: aizenr@yahoo.com

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, THE WEIZMANN INSTITUTE OF SCIENCE
POB 26, REHOVOT 76100, ISRAEL

E-mail address: guredim@yahoo.com

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, THE WEIZMANN INSTITUTE OF SCIENCE
POB 26, REHOVOT 76100, ISRAEL

E-mail address: eitan.sayag@gmail.com

INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904,
ISRAEL