DECAY OF MATRIX COEFFICIENTS ON REDUCTIVE HOMOGENEOUS SPACES OF SPHERICAL TYPE

BERNHARD KRÖTZ, EITAN SAYAG AND HENRIK SCHLICHTKRULL

Abstract. Let Z be a homogeneous space $Z = G/H$ of a real reductive Lie group G with a reductive subgroup H. The investigation concerns the quantitative decay of matrix coefficients on Z under the assumption that Z is of spherical type, that is, minimal parabolic subgroups have open orbits on Z.

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1. Introduction

Representation theory provides a concrete way to construct functions on a topological group G via matrix coefficients. For a continuous linear representation (π, E), say on a Banach space E, and for a vector $v \in E$ and a continuous linear functional $\eta \in E^*$ one defines the matrix coefficient

$$m_{v,\eta}(g) := \eta(\pi(g^{-1})v) \qquad (g \in G)\,.$$

One of the first results on matrix coefficients is the Gel'fand-Raikov theorem which asserts that the matrix coefficients of irreducible unitary representations separate points on G.

If H is a closed subgroup of G and η is fixed by H, then the matrix coefficient $m_{v,\eta}$ descends to a function on the homogeneous space $Z := G/H$. Our interest is to obtain sharp upper bounds for such matrix coefficients on Z, under some natural assumptions on G, H and the representation π.

On the geometric side we assume in addition that Z is of *spherical type*, that is, the set PH is open in G for some minimal parabolic subgroup P of G. Standard examples of spaces of spherical type are symmetric spaces, but there are others, for instance triple spaces $Z = G/H$ with $H = \mathrm{SO}(1, n)$ diagonally embedded into $G = H \times H \times H$.

On the analytic side we assume that $E = V^\infty$ is the smooth globalization of a Harish-Chandra module V (according to Casselman-Wallach). Then E is a smooth representation whose dual $V^{-\infty} := E^*$ is rather large and can accommodate non-trivial H-fixed vectors η.

The first main result, Theorem 3.2, concerns a bound for $m_{v,\eta}$ on the subset $PH \subset G$. To be more explicit, let $K \subset G$ be a maximal compact subgroup, $P = MAN$ the Langlands decomposition of the minimal parabolic group P, and $A_P^+ := A^+ \subset A$ the Weyl chamber associated to P. Then for every $v \in V$ there is a constant $C > 0$ such that

$$|m_{v,\eta}(a)| \leq Ca^{\Lambda_V}(1 + \|\log a\|)^{d_V} \qquad (a \in \overline{A^+})\,. \tag{1.1}$$

Here $\Lambda_V \in \mathfrak{a}^*$ (with $\mathfrak{a} = \mathrm{Lie}(A)$) and $d_V \in \mathbb{N}$ are determined by V.

To obtain a bound on the whole of Z a further geometric assumption on Z is needed. Specifically, let $P_1, \ldots, P_l \supset A$ be the finitely many minimal parabolic subgroups containing A so that P_jH is open, then clearly the geometric condition

$$G = \bigcup_{j=1}^{l} K\overline{A_{P_j}^+}H \tag{1.2}$$

allows us to deduce from (1.1) a global bound on $Z = G/H$ (see Corollary 5.3). Homogeneous spaces for which there is a choice of A such that (1.2) holds true we call *strongly spherical*. Symmetric spaces are strongly spherical and likewise the aforementioned triple spaces. It is an open problem whether all spherical spaces are strongly spherical.

Let us mention that strongly spherical spaces satisfy the so-called wave-front lemma (see Lemma 5.4), and are thus well-suited for lattice counting problems along the lines of [12].

Strong sphericity combined with a sufficient supply of finite dimensional H-spherical representations (see condition 5.12) now allows us to obtain a stronger bound in (1.1), in which the constant C is uniform with respect to v. Our second main result, Theorem 5.11, is thus that under these additional assumptions on Z, there exists for each $\eta \in (V^{-\infty})^H$ a continuous norm q on V^∞ such that

$$|m_{v,\eta}(aH)| \leq q(v) a^{\Lambda_V} (1 + \|\log a\|)^{d_V} \qquad (a \in \overline{A_j^+}) \tag{1.3}$$

for all $v \in V^\infty$ and $j = 1, \ldots, l$. In particular, the assumptions on Z are satisfied in the case of symmetric spaces (see Theorem 5.8). It is interesting to observe that even for the 'group case', where $Z = G \times G/G \simeq G$ is a reductive group, the bound improves on known bounds by allowing $v \in V^\infty$ rather than $v \in V$, see Remark 5.10.

Let us comment about some historical developments and the nature of proof of our main results. In the group case the bound (1.1) goes back to Harish-Chandra. There are two different proofs to obtain (1.1) for $Z = G$: one by Casselman-Milicic in [8], which uses that the matrix coefficients satisfy a regular singular system of differential equations on $A_\mathbb{C} \simeq (\mathbb{C}^*)^n$, and an approach by Wallach in [20], just using ODE-techniques which is of stunning simplicity and brevity. Using the first method van den Ban obtained the bound (1.1) for symmetric spaces. Our proof of (1.1) rests on the observation that the condition that PH is open allows an adaptation of the proof of Wallach.

Combining the assumption of strong sphericity and the assumption (5.12) about finite dimensional H-spherical representations, it is possible to construct K-invariant weight functions w on Z with $w(aH) \asymp a^{\Lambda_V}(1 + \|\log a\|)^{d_V}$ on $\overline{A^+}$. Then one can deduce (1.3) from (1.1) using the globalization theorem of Casselman-Wallach. In particular, for symmetric spaces we thus obtain with (1.3) an improvement of van den Ban's bounds on generalized matrix coefficients.

Finally, we mention that in [16] we have studied a more qualitative property of decay of smooth L^p-functions which are not necessarily matrix coefficients of a Harish-Chandra module. More precisely, we showed that on a reductive homogeneous space the smooth vectors in the Banach representation $L^p(Z)$ all belong to the space of continuous functions vanishing at infinity. The results of the present paper, when restricted to unitary representations, provide explicit decay results for generalized matrix coefficients. Therefore, we expect these results to be useful in extending the results of [11] beyond symmetric spaces to the realm of strongly spherical spaces.

2. Homogeneous spaces of spherical type

We will denote Lie groups by upper case $LATIN$ letters, e.g. A, B etc., and their Lie algebras by lower case $\mathfrak{German}$ letters, e.g. $\mathfrak{a}$, $\mathfrak{b}$ etc.

Let G be a real reductive group in the sense of [20], Sect. 2.1. Further let $H < G$ be a subgroup which is reductive in G (as in [16]). With these data we form the homogeneous space of *reductive type* $Z := G/H$. We denote by $z_0 = H$ the standard base point of Z.

We fix a maximal compact subgroup $K < G$ such that $H \cap K$ is a maximal compact subgroup of H and such that the associated Cartan involution θ of G preserves H. We will frequently use the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ of the Lie algebra $\mathfrak{g} = Lie(G)$. Here $\mathfrak{s}$ is the complement of $\mathfrak{k} = Lie(K)$ with respect to a non-degenerate invariant bilinear form on $\mathfrak{g}$, say $\kappa(\cdot, \cdot)$.

The form κ induces an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and the space Z is topologically a vector bundle over $K/H \cap K$. Indeed, by the Mostow decomposition

$$K \times_{K\cap H} (\mathfrak{s} \cap \mathfrak{q}) \to Z, \quad [k, X] \mapsto k \exp(X) \cdot z_0 \tag{2.1}$$

is a diffeomorphism. This decomposition is valid in the generality of reductive homogeneous spaces. A smaller class of homogeneous spaces with a richer geometry is introduced below.

2.1. Spaces of spherical type.

Recall (see [7]) that a complex homogeneous space $G_{\mathbb{C}}/H_{\mathbb{C}}$ is said to be spherical if there exists a Borel subgroup $B_{\mathbb{C}}$ such that $B_{\mathbb{C}}H_{\mathbb{C}}$ is open in $G_{\mathbb{C}}$. The following definition is analogous. Let $Z = G/H$ be a reductive homogeneous space.

Definition 2.1. *The space Z is of* spherical type *if there exists a minimal parabolic subgroup P such that PH is open in G. If in addition* $\dim(P \cap H) = 0$ *then we say that Z is of* pure *spherical type.*

Remark 2.2. For two closed subgroups A, B of a Lie group G, the set AB is open in G if and only if $\mathfrak{a} + \mathfrak{b} = \mathfrak{g}$. Hence the condition that PH is open allows the obvious infinitesimal characterization $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$.

Note that the main intention behind the concept in [7] is the classification of Gel'fand pairs. With that intention one should add to

Definition 2.1 the condition that $(M, M \cap H)$ is a Gel'fand pair. Here $M = Z_K(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in K. However this is not our purpose. The non-symmetric space $\mathrm{Sp}(n,1)/\mathrm{Sp}(n)$, for example, is of spherical type but fails the Gel'fand pair condition.

Definition 2.3. *Let $P \subset G$ be a parabolic subgroup. The pair (P, H) is called spherical if PH is open in G and for some Langlands decomposition $P = M_P A_P N_P$ we have $\mathfrak{m}_P \cap \mathfrak{s} \subset \mathfrak{h}$. For future reference we write these conditions:*

(1) $\mathfrak{m}_P \cap \mathfrak{s} \subset \mathfrak{h}$,
(2) *PH is open in G.*

Equivalently, for some Levi decomposition $P = LN_P$, the largest non-compact semisimple ideal of $L = Lie(L)$ belongs to $\mathfrak{h}$.

Lemma 2.4. *Let (P, H) be a spherical pair, and let $P_0 \subset P$ be minimal parabolic. Then P_0H is open. In particular, Z is spherical.*

Proof. Write $\mathfrak{m}_P$ as a direct sum of a compact ideal and non-compact ideal. It follows from condition (1) that all non-compact ideals of $\mathfrak{m}_P$ belong to $\mathfrak{h}$. Now if $P_0 = M_0A_0N_0 \subset P$ is a minimal parabolic, then the compact ideals of $\mathfrak{m}_P$ centralize $\mathfrak{a}_0$, hence lie in $\mathfrak{m}_0$. It follows that $M_P \subset M_0H$ and hence $P_0H = PH$. The result follows from condition (2). □

2.2. **Examples.**

2.2.1. *Symmetric spaces.* In a symmetric space Z, all minimal $\sigma\theta$-stable parabolic subgroups P satisfy (1) and (2), see [3]. Hence by Lemma 2.4, Z is of spherical type and P_0H is open for every minimal parabolic $P_0 \subset P$. For more details see remark 3.6.

2.2.2. *Gross-Prasad spaces.* We let G_0 be a reductive group and $H_0 < G_0$ a reductive subgroup. Set $G = G_0 \times H_0$ and $H = \mathrm{diag}(H_0)$. Note that $Z = G/H \simeq G_0$, viewed as a left×right homogeneous space for $G_0 \times H_0$.

We consider the following choices for G_0 and H_0, with which we refer to Z as a Gross-Prasad space (cf. [13]):

- $G_0 = \mathrm{GL}(n+1, \mathbb{F})$ and $H_0 = \mathrm{GL}(n, \mathbb{F})$ for $n \geq 0$.
- $G_0 = \mathrm{U}(p, q+1, \mathbb{F})$ and $H_0 = \mathrm{U}(p, q, \mathbb{F})$ for $p + q \geq 2$.

Here $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

For a parabolic subgroup $P = P_1 \times P_2$ of G the condition that PH is open is equivalent to P_1P_2 is open in G_0, or $\mathfrak{p}_1 + \mathfrak{p}_2 = \mathfrak{g}_0$. The simple verification that this is possible for the above spaces is omitted. The spaces in the first item are pure, but in the second item not in general.

2.2.3. *Triple spaces.* Let G_0 be a reductive group and let $G = G_0^3 := G_0 \times G_0 \times G_0$ and $H = \mathrm{diag}(G_0)$. The corresponding reductive homogeneous space $Z = G/H$ will be referred to as a triple space. In general this space is not spherical as an easy dimension count will show. For G_0 locally $\mathrm{SO}(1,n)$ however, one obtains a spherical space. Dimension count shows that it is pure if and only if $n = 2$ or 3.

2.2.4. *Complex spherical spaces.* Let $G_\mathbb{C}/H_\mathbb{C}$ be a complex spherical space with open Borel orbit $B_\mathbb{C}H_\mathbb{C}$. When we regard the complex groups as real Lie groups, $G_\mathbb{C}/H_\mathbb{C}$ is of spherical type and $(B_\mathbb{C}, H_\mathbb{C})$ is a spherical pair. The complex spherical spaces have been classified (see the lists in [15] and [7]). For example, the triple space of $G_0 = \mathrm{SL}(2,\mathbb{C})$ is a complex spherical space ([15] p. 152).

2.2.5. *Real forms of spherical spaces.* Let $G_\mathbb{C}$, $H_\mathbb{C}$, $B_\mathbb{C}$ be as above, and assume that G is a quasisplit real form of $G_\mathbb{C}$. Then $B_\mathbb{C}$ is the complexification of a minimal parabolic P in G, for which PH is open. Hence G/H is of spherical type. The triple space with $G_0 = \mathrm{SL}(2,\mathbb{R})$ is obtained in this fashion.

2.2.6. Let $G_\mathbb{C}/H_\mathbb{C} = \mathrm{SL}(2n{+}1,\mathbb{C})/\mathrm{Sp}(n,\mathbb{C})$ or $\mathrm{SO}(2n{+}1,\mathbb{C})/\mathrm{GL}(n,\mathbb{C})$. According to [15] p. 143, these are complex spherical spaces as in 2.2.4. Dimension count shows they are pure. The corresponding split or quasisplit real forms in 2.2.5 are

$$\begin{aligned}
&\mathrm{SL}(2n+1,\mathbb{R})/\mathrm{Sp}(n,\mathbb{R})\\
&\mathrm{SU}(n,n+1)/\mathrm{Sp}(k,k), \quad n=2k\\
&\mathrm{SO}(n,n+1)/\mathrm{U}(k,k), \quad n=2k\\
&\mathrm{SU}(n,n+1)/\mathrm{Sp}(k,k+1), \quad n=2k+1\\
&\mathrm{SO}(n,n+1)/\mathrm{U}(k,k+1), \quad n=2k+1.
\end{aligned}$$

Notice that for $n > 3$ the triple spaces with $G_0 = \mathrm{SO}_e(n,1)$ do not correspond to any spaces in 2.2.4 or 2.2.5.

3. Bounds for generalized matrix coefficients

In this section we prove a fundamental bound for generalized matrix coefficients for spaces of spherical type.

To begin with we need to recall a few notions from basic representation theory. Let (π, E) be a Banach representation of G, and let E^∞ denote its space of smooth vectors. As usual we topologize E^∞ by the family of Sobolev norms $\|.\|_k$ for $k = 0, 1, 2, \ldots$, given by

$$\|v\|_k = \sum_{m_1+\cdots+m_n \leq k} \|\pi(X_1^{m_1} \cdots X_n^{m_k})v\|$$

with respect to a fixed basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. Then

$$E^\infty = \cap_{k\in\mathbb{N}} E_k$$

is an intersection Banach spaces, where E_k is the completion of E^∞ with respect to $\|.\|_k$. Likewise the space $E^{-\infty}$ of distribution vectors (i.e. continuous linear forms on E^∞) is the union

$$E^{-\infty} = \cup_{k\in\mathbb{N}} E_{-k},$$

where $(E_{-k}, \|.\|_{-k})$ is the Banach space dual to $(E_k, \|.\|_k)$. For each $k \in \mathbb{N}$ we thus have

$$|\eta(v)| \leq \|\eta\|_{-k}\|v\|_k, \quad (\eta \in E^{-\infty}, v \in E^\infty), \tag{3.1}$$

with $\|\eta\|_{-k} < \infty$ if and only if $\eta \in E_{-k}$.

By continuity of the G-action, we conclude that for each $k \in \mathbb{N}$ there exists $C_k > 0$ and $r > 0$ such that

$$|\eta(\pi(g)v)| \leq C_k\|\eta\|_{-k}\|v\|_k\|g\|^r, \quad (g \in G), \tag{3.2}$$

for all $v \in E^\infty$, $\eta \in E^{-\infty}$. Here $\|\cdot\|$ is a norm on G in the sense of [20] Section 2.A.2, from which Lemma 2.A.2.2 is used. We recall also that if $\mathfrak{a} \subset \mathfrak{s}$ is a maximal abelian subspace and $\mathfrak{a}^+ \subset \mathfrak{a}$ a Weyl chamber, then by [20], Lemma 2.A.2.3, there exist $\delta \in \mathfrak{a}^*$ and $C > 0$ such that

$$\|a\| \leq Ca^\delta \tag{3.3}$$

for all $a \in \overline{A^+}$, the closure of $A^+ = \exp(\mathfrak{a}^+)$.

If V is a Harish-Chandra module for $(\mathfrak{g}, K)$, then we call a Banach representation (π, E) of G a *Banach globalization* of V provided that the K-finite vectors of E are isomorphic to V as $(\mathfrak{g}, K)$-modules. The Casselman-Wallach theorem asserts that E^∞ does not depend on the particular globalization (π, E) of V and thus we may define

$$V^\infty := E^\infty \text{ and } V^{-\infty} := E^{-\infty}.$$

Let $Z = G/H$ be a reductive homogeneous space and V a Harish-Chandra module. For $v \in V^\infty$ and $\eta \in (V^{-\infty})^H$ an H-fixed distribution vector we denote by

$$m_{v,\eta}(gH) := \eta(\pi(g)^{-1}v), \quad (g \in G), \tag{3.4}$$

the corresponding generalized matrix coefficient. It is a smooth function on Z.

Let $P \subset G$ be a minimal parabolic subgroup of G. In this situation, there exists a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$ which is contained in $\mathfrak{p} = \mathrm{Lie}(P)$.

We also use the following notations:

- $\Pi \subset \mathfrak{a}^*$ the set of simple roots.
- $\mathfrak{p}$ the Lie algebra of P
- $\Sigma^+ \subset \mathfrak{a}^*$ for the positive system attached to P.
- $P = MAN$ is a Langlands decomposition of P.
- $A_P^+ := A^+ \subset A$ the Weyl chamber associated to P.
- $\bar{P} = \theta(P)$, an opposite parabolic subgroup.

We will use Iwasawa decomposition in the form $G = KAN$.

More generally, for a subset $F \subset \Pi$ one defines a standard parabolic subalgebra $\mathfrak{p}_F \supset \mathfrak{p}$. Let $\mathfrak{a}_F := \{X \in \mathfrak{a} \mid (\forall \alpha \in F)\alpha(X) = 0\}$ and $\mathfrak{m}_F := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_F)$ be the centralizer of $\mathfrak{a}_F$ in $\mathfrak{g}$. Further let $\Sigma^+ \backslash \langle F \rangle$ be the set of positive roots which do not vanish on $\mathfrak{a}_F$ and let $\mathfrak{n}_F := \bigoplus_{\alpha \in \Sigma^+ \backslash \langle F \rangle} \mathfrak{g}^\alpha$ be the corresponding subalgebra of $\mathfrak{n}$. Then $\mathfrak{p}_F := \mathfrak{m}_F \ltimes \mathfrak{n}_F$ defines a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{p}$. In particular $\mathfrak{p}_\emptyset = \mathfrak{p}$.

Given a Harish-Chandra module V, the quotient $V/\mathfrak{n}_F V$ is non-zero and a Harish-Chandra module for the pair $(\mathfrak{m}_F, K_F)$ with $K_F = Z_K(\mathfrak{a}_F)$, see [20], Lemma 4.3.1. As Harish-Chandra modules are finite under the center of the enveloping algebra, we obtain for every $F \subset \Pi$ a finite subset $E(F, V) \subset \mathfrak{a}_F^*$ and an integer $d_F \in \mathbb{N}_0$ such that

$$V/\mathfrak{n}_F V = \bigoplus_{\lambda \in E(F,V)} (V/\mathfrak{n}_F)_\lambda$$

with the generalized eigenspaces:

$$(V/\mathfrak{n}_F V)_\lambda = \{v \in V/\mathfrak{n}_F V \mid \forall H \in \mathfrak{a}_F : (H - \lambda(H))^{d_F} v = 0\}.$$

We set $E(V) := E(\emptyset, V)$ and record ([20] 4.3.4(2))

$$E(V)|_{\mathfrak{a}_F} = E(F, V).$$

Now label $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ and define $\{H_1, \ldots, H_r\}$ the corresponding dual basis in $\mathfrak{a}$. We follow [20], 4.3.5, and define $\Lambda_V \in \mathfrak{a}^*$ by

$$\Lambda_V(H_j) := \max\{-\operatorname{Re} \lambda(H_j) \mid \lambda \in E(V)\}. \tag{3.5}$$

Furthermore, we let $F_j := \Pi \backslash \{\alpha_j\}$ for $j = 1, \ldots r$ and put

$$d_V := \sum_{j=1}^r d_{F_j}.$$

Remark 3.1. The definition of Λ_V can be motivated as follows. By [8] Thm. 8.22 the leading (with respect to ordering by positive roots) exponents for V belong to $-E(V)$, and hence by [8] Thm. 8.11 every K-finite matrix coefficient of V is bounded on A^+ by a constant multiple of $a^{\Lambda_V}(1+\|\log a\|)^d$ for some $d \in \mathbb{N}$. Furthermore, [17] Thm. 2.1 ensures that Λ_V is sharp for the K-finite matrix coefficients. However, these results from [8] and [17] are not used in what follows.

Theorem 3.2. *Let G be a real reductive group and let $H \subset G$ be reductive in G. Suppose that PH is open in G for a minimal parabolic subgroup $P \subset G$. Let V be a Harish-Chandra module. Then for each $v \in V$ and each $s \in \mathbb{N}$ there exists a constant $C > 0$ such that*

$$|m_{v,\eta}(ka)| \leq C\|\eta\|_{-s}\, a^{\Lambda_V}(1+\|\log a\|)^{d_V} \tag{3.6}$$

for all $k \in K$, $a \in \overline{A_P^+}$ and $\eta \in (V^{-\infty})^H \cap E_{-s}$.

Remark 3.3. Note that (3.6) is uniform with respect to η but not with respect to v. Under some additional hypotheses we give in Section 5 an improvement providing uniformity also with respect to v.

Remark 3.4. The proof will be an adaptation of the proof of Thm. 4.3.5 in [20] to the present situation of generalized matrix coefficients. Note that we have $m_{v,\eta}(a) = \eta(\pi(a)^{-1}v)$ with a K-finite vector v, whereas [20] considers $\mu(\pi(a)v)$ with μ being K-finite. The main difference is then that [20] has $v \in V^\infty$ arbitrary, whereas we have $\eta \in V^{-\infty}$ but H-fixed.

Proof. Since H is invariant under the Cartan involution, the assumed openness of PH is equivalent with the same property for $\bar{P}H$. We shall use the property in this form. The number $s \in \mathbb{N}$ will be fixed throughout the proof.

We first provide the central step where the proof differs from [20]. Our starting point is the following estimate, which follows from (3.2) and (3.3). Let (π, E) be a Banach globalization of V. Then by (3.2) there exists $\delta \in \mathfrak{a}^*$ and $C > 0$ such that

$$|m_{v,\eta}(a)| \leq C\|\eta\|_{-s}\|v\|_s\, a^\delta, \qquad (a \in \overline{A^+}) \tag{3.7}$$

for $v \in V$ and $\eta \in (V^{-\infty})^H$. If δ happens to be $\leq \Lambda_V$ on $\mathfrak{a}^+$ we are done. Otherwise we need to improve the estimate. The proof will be by downwards induction along $\mathfrak{a}^+$.

The key ingredient is as follows. Suppose that $v \in V$ is of the form

$$v = d\pi(X)u \tag{3.8}$$

for some normalized positive root vector $X \in \mathfrak{g}^\alpha \subset \mathfrak{n}$ and some $u \in V$ (this corresponds to the assumption $\mu \in \mathfrak{n}_F V^\sim$ in [20] p.116, where

$V^\sim$ is the contragredient of V). As $\bar{P}H$ is open in G we can write $X = X_1 + X_2$ with $X_1 \in \mathfrak{h}$ and $X_2 \in \mathfrak{a} + \mathfrak{m} + \bar{\mathfrak{n}}$. Now observe that

$$\begin{aligned} m_{v,\eta}(a) &= \eta(\pi(a)^{-1} d\pi(X)u) = a^{-\alpha}\eta(d\pi(X)\pi(a)^{-1}u) \\ &= a^{-\alpha}\eta(d\pi(X_2)\pi(a)^{-1}u) = a^{-\alpha}\eta(\pi(a)^{-1}d\pi(\mathrm{Ad}(a)X_2)u)\,. \end{aligned}$$

As $\mathrm{Ad}(a)$ is contractive on $\mathfrak{a} + \mathfrak{m} + \bar{\mathfrak{n}}$ we can write $d\pi(\mathrm{Ad}(a)X_2)u$ as a linear combination of elements from V with a-dependent coefficients, which are bounded. For vectors of the form (3.8) we thus obtain with (3.7) an improved bound

$$|m_{v,\eta}(a)| \leq C'\|\eta\|_{-s}\, a^{\delta-\alpha} \qquad (a \in \overline{A^+})$$

with a constant C' depending on u.

Having established the key step, the proof will follow as in [20], p. 117-118. For the sake of completeness we provide the adaptation to the present situation. The argument is based on the following simple lemma.

Lemma 3.5. *Let A be a complex $n \times n$-matrix. There exists $C > 0$ such that the following holds. Let $f : \mathbb{R} \to \mathbb{C}^n$ be differentiable and define $g : \mathbb{R} \to \mathbb{C}^n$ by*

$$\frac{df}{dt} - Af = g \tag{3.9}$$

Assume

$$\|f(0)\| \leq 1, \quad \|g(t)\| \leq e^{\nu t} \quad (t \geq 0),$$

for some $\nu \in \mathbb{R}$. Let

$$\mu = \max\{\mathrm{Re}\,\lambda \mid \lambda \text{ an eigenvalue of } A\}$$

and let $p \in \mathbb{N}$ be the maximal algebraic multiplicity of the eigenvalues with $\mathrm{Re}\,\lambda = \mu$. Then

$$\|f(t)\| \leq C(1+t)^p e^{\max\{\mu,\nu\}t}$$

for all $t \geq 0$.

Proof. This is easily obtained from the solution formula for (3.9) and the Jordan form of A. □

For the proof of Theorem 3.2 let us assume that for some $\delta \in \mathfrak{a}^*$ and $d \in \mathbb{N}$ we have established for all $v \in V$ a bound

$$|m_{v,\eta}(a)| \leq C\|\eta\|_{-s}\, a^{\delta}(1 + \|\log a\|)^d \qquad (a \in \overline{A^+}). \tag{3.10}$$

For elements of the form (3.8) we can then improve to

$$|m_{v,\eta}(a)| \leq C'\|\eta\|_{-s}\, a^{\delta-\alpha}(1 + \|\log a\|)^d \qquad (a \in \overline{A^+}) \tag{3.11}$$

by the key argument above. The constants C and C' are allowed to depend on v.

Let us write $\delta = \sum_{j=1}^r \delta_j \alpha_j$ with $\delta_j = \delta(H_j) \in \mathbb{R}$. We fix an index $1 \leq j \leq r$. Note that

$$\overline{\mathfrak{a}^+} = \bigoplus_{k=1}^r \mathbb{R}_{\geq 0} H_k.$$

We consider $F = F_j = \Pi \backslash \{\alpha_j\}$ and note that $\mathfrak{a}_F = \mathbb{R}H_j$. We decompose elements $a \in \overline{A}^+$ as $a = a' a_t$ with $a'^{\alpha_j} = 1$ and $a_t = \exp(tH_j)$.

Let now $v \in V$ and $\overline{v} \in V/\mathfrak{n}_F V$ its coset. Let $\overline{v}_1, \ldots, \overline{v}_p$ be a basis of the finite dimensional space $\mathcal{U}(\mathfrak{a}_F)\overline{v}$ with $\overline{v} = \overline{v}_1$. Define a $p \times p$-matrix $B = (b_{kl})$ by $H_j \overline{v}_k = \sum b_{kl} \overline{v}_l$. Let $v_k \in V$ be representatives of $\overline{v}_k$ and define $w_k := H_j v_k - \sum b_{kl} v_l \in \mathfrak{n}_F V$.

Fix $a' \in \overline{A^+}$ and consider the $\mathbb{C}^p$-valued functions

$$F(t) := \begin{pmatrix} m_{v_1,\eta}(a'a_t) \\ \vdots \\ m_{v_p,\eta}(a'a_t) \end{pmatrix}, \quad G(t) := \begin{pmatrix} m_{w_1,\eta}(a'a_t) \\ \vdots \\ m_{w_p,\eta}(a'a_t) \end{pmatrix}.$$

Then

$$\frac{d}{dt} F(t) = -BF(t) - G(t)$$

and we can apply Lemma 3.5. By (3.5) the eigenvalues of $-B$ have real part $\leq \Lambda_j$ and multiplicity $\leq d_F$. Furthermore, by our a priori bound (3.10) we have

$$\|F(0)\| \leq C_1 \|\eta\|_{-s} \, (a')^\delta (1 + \|\log a'\|)^d$$

for a constant $C_1 > 0$, independent of a'. To estimate $\|G(t)\|$ we note that w_k is of the form (3.8) so that the improved estimate (3.11) can be applied. Since any root of $\mathfrak{n}_F$ restricted to H_j coincides with a positive integer multiple of α_j this yields that

$$\|G(t)\| \leq C_2 \|\eta\|_{-s} \, (a')^\delta (1 + \|\log a'\|)^d e^{t(\delta_j - 1)}$$

for a constant $C_2 > 0$, also independent of a'. Combining matters we arrive at

$$\|F(t)\| \leq C_3 \|\eta\|_{-s} \, (a')^\delta (1 + \|\log a'\|)^d (1+t)^{d_F} e^{t \max\{\Lambda_j, \delta_j - 1\}}.$$

In particular, we conclude that for every $v \in V$ there exists $C > 0$ such that

$$|m_{v,\eta}(a'a_t)| \leq C \|\eta\|_{-s} \, (a')^\delta (1 + \|\log a'\|)^d (1+t)^{d_F} e^{t \max\{\Lambda_j, \delta_j - 1\}} \tag{3.12}$$

for all $t \geq 0$ and all a'. We consider now two cases:

If $\delta_j - 1 \leq \Lambda_j$ it follows that we can replace δ_j by Λ_j and d by $d + d_F$ in our initial bound (3.10).

If $\delta_j - 1 > \Lambda_j$ we remove the logarithms and find

$$|m_{v,\eta}(a'a_t)| \leq C\|\eta\|_{-s}\,(a')^\delta(1+\|\log a\|)^d e^{t(\delta_j-\frac{1}{2})}. \tag{3.13}$$

Thus we may replace δ by $\delta - \frac{1}{2}\alpha_j$ in our initial estimate. In a finite number of steps we arrive in the first case.

After repeating this process for all j, the theorem is proved. □

Remark 3.6. The theorem applies to symmetric spaces. Suppose that Z is symmetric and let $\mathfrak{a}_q \subset \mathfrak{s}\cap\mathfrak{q}$ be a maximal abelian subspace (it is unique up to conjugation by $K\cap H$), and let $\mathfrak{a}\subset\mathfrak{s}$ be maximal abelian with $\mathfrak{a}_q\subset\mathfrak{a}$. Then

$$\mathfrak{a} = \mathfrak{a}_q \oplus \mathfrak{a}_h := (\mathfrak{a}\cap\mathfrak{q})\oplus(\mathfrak{a}\cap\mathfrak{h}).$$

We choose a positive system for the roots of $\mathfrak{a}_q$ in $\mathfrak{g}$, and a compatible ordering for the roots of $\mathfrak{a}$ so that $\mathfrak{a}_q^+ \subset \overline{\mathfrak{a}^+}$ for the positive chambers. Then PH is open for the corresponding minimal parabolic P. Thus (3.6) applies to all $a \in \overline{A_q^+}$. In this situation bounds as (3.6) have previously been established in [2], [5].

4. Homogeneous spaces of polar type

In order to obtain global bounds for the matrix coefficients we need to assume some further properties of $Z = G/H$. First of all we require that it allows a generalized polar decomposition. We recall that for a Riemannian symmetric space $Z = G/K$ the KAK-decomposition of G implies that $Z = KA.z_0$.

Definition 4.1. *Let $Z = G/H$ be a homogeneous space of reductive type. A* polar decomposition *of Z consists of an abelian subspace $\mathfrak{a}\subset\mathfrak{s}$ and a surjective proper map*

$$K\times A \ni (k,a)\mapsto kaH \in Z, \tag{4.1}$$

where $A = \exp\mathfrak{a}$. We say that Z is of polar type *if such a polar decomposition exists.*

Notice that we do not require $\mathfrak{a}\subset\mathfrak{q}$ in Definition 4.1. In general this is not possible with $\mathfrak{a}$ abelian. According to (2.1) every element $z\in Z$ allows a decomposition $z = k\exp(X)H$ with $k\in K$ and $X\in\mathfrak{q}\cap\mathfrak{s}$, but in general the orbits of $K\cap H$ on $\mathfrak{q}\cap\mathfrak{s}$ do not allow a parametrization by an abelian subspace (for instance in Example 4.2.2 below).

Neither do we insist that $\mathfrak{a}$ is maximal abelian, since in general that would conflict with the properness of (4.1).

4.1. **About properness.** By replacing $\mathfrak{a}$ with a subspace complementary to $\mathfrak{a} \cap \mathfrak{h}$, we can arrange $\mathfrak{a} \cap \mathfrak{h} = \{0\}$ without affecting the surjectivity of (4.1). It follows from the corollary below that then the assumption of properness in Definition 4.1 is superfluous.

Lemma 4.2. *Let G/H be a reductive homogeneous space. There exists a finite dimensional representation (π, V) of G and a vector $v_\mathfrak{h} \in V$ such that $\mathfrak{h} = \{X \in \mathfrak{g} \mid d\pi(X)v_\mathfrak{h} = 0\}$.*

Proof. Follows from Sect. 5.6, Th. 3 in [1]. □

Let $\mathfrak{a} \subset \mathfrak{s}$ be an abelian subspace.

Lemma 4.3. *The set AH is closed in G. Furthermore, if $\mathfrak{a} \cap \mathfrak{h} = \{0\}$ then $(a, h) \mapsto ah$ is proper $A \times H \to AH$.*

Proof. We may assume $\mathfrak{a} \cap \mathfrak{h} = \{0\}$ for the entire lemma. We argue by contradiction. Suppose that AH were not closed in G or that $(a, h) \mapsto ah$ were not proper. Then there would exist sequences $(a_n)_{n\in\mathbb{N}}$ in A and $(h_n)_{n\in N}$ in H, both leaving every compact subset, such that $p = \lim_{n\to\infty} a_n h_n$ exists in G.

Let (π, V) be a finite dimensional representation as in Lemma 4.2. Then the limit $\lim_{n\to\infty} \pi(a_n)v_\mathfrak{h}$ exists. Let $X_n = \log(a_n)$. Passing to a subsequence we may assume that

$$X := \lim_{n\to\infty} \frac{X_n}{\|X_n\|} \in \mathfrak{a} - \{0\}$$

exists and $\lim_{n\to\infty} \|X_n\| = \infty$. We will show that $v_\mathfrak{h}$ is fixed under $d\pi(X)$ contradicting the assumption that $\mathfrak{a} \cap \mathfrak{h} = \{0\}$ and $X \neq 0$.

Indeed, write v_h as a sum of joint eigenvectors for $\mathfrak{a}$, say

$$v_h = \sum_{\mu\in\mathfrak{a}^*} v_\mu$$

Applying $\pi(a_n)$ yields

$$\pi(a_n)v_\mathfrak{h} = \sum_{\mu} e^{\mu(X_n)} v_\mu$$

Since $\pi(a_n)v_\mathfrak{h}$ is convergent, it follows that $\sup_n \mu(X_n) < \infty$ for every μ for which $v_\mu \neq 0$. With $\lim_{n\to\infty} ||X_n|| = \infty$ we get that

$$\mu(X) = \lim_{n\to\infty} \frac{\mu(X_n)}{||X_n||} \leq 0$$

for all such μ.

Applying the same argument to the convergent sequence $\theta(a_n h_n)$ we find likewise that $\mu(X) \geq 0$ for all μ with $v_\mu \neq 0$. Thus we obtain that $v_\mathfrak{h}$ is fixed under $d\pi(X)$ which finishes the proof.

□

Corollary 4.4. *The set KAH is closed in G, and if $\mathfrak{a} \cap \mathfrak{h} = \{0\}$ then $(k, a, h) \mapsto kah$ is proper $K \times A \times H \to KAH$.*

4.2. **Examples.** We provide some examples of homogeneous spaces of polar type.

4.2.1. *Symmetric spaces.* Symmetric spaces are of polar type. In fact let $\mathfrak{a}_q \subset \mathfrak{s} \cap \mathfrak{q}$ be maximal abelian, as in Remark 3.6. Then $G = KA_qH$ (see [19] p. 117).

4.2.2. *Triple spaces.* Let $G/H = G_0^3/\operatorname{diag}(G_0)$ with G_0 reductive, as in Example 2.2.3. Let $K_0 < G_0$ be a maximal compact subgroup. We fix an Iwasawa decomposition $G_0 = K_0A_0N_0$ and let $P_0 = M_0A_0N_0$ be the associated minimal parabolic subgroup. Set $K = K_0 \times K_0 \times K_0$.

Proposition 4.5. *Suppose that $B_0 \subset G_0$ is a subset such that*

$$G_0 = A_0M_0B_0K_0.$$

Then, for $A = A_0 \times A_0 \times B_0$, one has $G = KAH$.

Proof. Let $(g_1, g_2, g_3) \in G$. From the KAH-decomposition of the symmetric space $G_0 \times G_0/\operatorname{diag}(G_0)$ we obtain

$$(g_1, g_2) = (g, g)(a_1, a_2)(k_1, k_2)$$

for some $g \in G_0$, $a_1, a_2 \in A_0$ and $k_1, k_2 \in K_0$. Now choose $m \in M_0$, $a_0 \in A_0$, $b_0 \in B_0$ and $k_0 \in K_0$ such that $g^{-1}g_3 = a_0m_0b_0k_0$. Then

$$(g_1, g_2, g_3) = (ga_0m_0, ga_0m_0, ga_0m_0)(a_0^{-1}a_1, a_0^{-1}a_2, b_0)(m_0^{-1}k_1, m_0^{-1}k_2, k_0)$$

as asserted. □

The proposition applies in the following cases:

- $G_0 = \mathrm{SL}(2, \mathbb{R})$ and

$$A_0 = \Big\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \Big\}, \quad B_0 = \Big\{ \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \mid s \in \mathbb{R} \Big\}$$

- $G_0 = \mathrm{SO}_e(1, n)$ and $B_0 = \exp(\mathbb{R}X)$ for some $0 \neq X \in \mathfrak{s}_0 \cap \mathfrak{a}_0^{\perp}$.

Note that it also applies to $B_0 = N_0$ for general G_0,

Corollary 4.6. *Let $G_0 = \mathrm{SL}(2, \mathbb{R})$ or $G_0 = \mathrm{SO}_e(1, n)$ for $n \geq 2$ and $Z = G_0^3/\operatorname{diag}(G_0)$. Then Z is of polar type.*

4.2.3. *Gross-Prasad spaces.* Let $G/H = G_0 \times H_0/\operatorname{diag}(H_0)$ be one of the Gross-Prasad spaces considered in Example 2.2.2.

Lemma 4.7. *G/H is of polar type.*

Proof. We first treat the case $(G_0, H_0) = (\mathrm{GL}(n+1,\mathbb{F}), \mathrm{GL}(n,\mathbb{F}))$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let us embed H_0 in G_0 as the lower right corner.

We define a two-dimensional non-compact torus of $\mathrm{GL}(2,\mathbb{F})$ by

$$B = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R}, a > |b| \right\}$$

In $\mathrm{GL}(2k,\mathbb{F})$ we define a $2k$-dimensional non-compact torus A_{2k} by k block matrices of form B along the diagonal. Explicitly,

$$A_{2k} = \{diag(b_1, ..., b_k) : b_j \in B\}$$

In $\mathrm{GL}(2k+1,\mathbb{F})$ we define A_{2k+1} to consist of similar blocks together with a positive number in the last diagonal entry. Explicitly,

$$A_{2k+1} = \{diag(b_1, ..., b_k, b) : b_j \in B, b \in \mathbb{R}^*\}$$

With these choices, when we consider $A_{2k} \subset H_0$ as a subgroup of G_0 using the lower right corner embedding, we get $A_{2k} \cap A_{2k+1} = \{1\}$.

Finally we let

$$A = A_{n+1} \times A_n \subset G = \mathrm{GL}(n+1,\mathbb{F}) \times \mathrm{GL}(n,\mathbb{F}).$$

With $K = \mathrm{U}(n+1,\mathbb{F}) \times \mathrm{U}(n,\mathbb{F})$ we claim that

$$G = KAH\,,$$

or, equivalently,

$$\mathrm{GL}(n+1,\mathbb{F}) = \mathrm{U}(n+1,\mathbb{F}) A_{n+1} A_n\, \mathrm{U}(n,\mathbb{F})\,.$$

We proceed by induction on n. The case $n = 0$ is clear. We shall use the known polar decomposition for the almost symmetric pair $(\mathrm{GL}(n+1,\mathbb{F}), \mathrm{GL}(n,\mathbb{F}))$:

$$\mathrm{GL}(n+1,\mathbb{F}) = \mathrm{U}(n+1,\mathbb{F}) B_1\, \mathrm{GL}(n,\mathbb{F})$$

where B_1 is the two-dimensional torus of form B located in in the upper left corner. Now insert for $\mathrm{GL}(n,\mathbb{F})$ by induction, but in opposite order:

$$\mathrm{GL}(n,\mathbb{F}) = \mathrm{U}(n-1,\mathbb{F}) A_{n-1} A_n\, \mathrm{U}(n,\mathbb{F})$$

and observe that $\mathrm{U}(n-1,\mathbb{F})$ commutes with B_1.

The case with $G_0 = \mathrm{U}(p, q+1,\mathbb{F})$ where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ is similar. Choose non-compact Cartan subspaces for $\mathfrak{g}_0$ and $\mathfrak{h}_0$ along antidiagonals, and note that the overlap between these, as subspaces of $\mathfrak{g}_0$, is trivial. Now proceed by induction as before. □

4.2.4. *The spaces* $G/H = \mathrm{Sp}(n,\mathbb{R})/(\mathrm{Sp}(n-1,\mathbb{R})\times \mathrm{U}(1))$.

Consider $G = \mathrm{Sp}(n,\mathbb{R})$ with maximal compact subgroup $K = \mathrm{U}(n)$. Let $H \subset L \subset G$, where

$$L = L_1 \times L_2 := \mathrm{Sp}(n-1,\mathbb{R}) \times \mathrm{Sp}(1,\mathbb{R}),$$

and $H_1 = L_1$, $H_2 = \mathrm{U}(1) \subset L_2$. The intermediate quotients G/L and $L/H = L_2/H_2$ are both symmetric.

We use the standard model

$$\mathfrak{sp}(n,\mathbb{R}) = \left\{ [X_1,X_2,X_3] := \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^t \end{pmatrix} \,\middle|\, \begin{matrix} X_1,X_2,X_3 \in M(n,\mathbb{R}) \\ X_2,X_3 \text{ symmetric} \end{matrix} \right\}$$

for $\mathfrak{g}$. Then $\mathfrak{l}_1$ consists of similar blocks of size one less, embedded as the upper left corners of X_1, X_2 and X_3 . Likewise, $\mathfrak{l}_2 = \mathfrak{sp}(1,\mathbb{R})$ consists of the matrices from $\mathfrak{sl}(2,\mathbb{R})$ of which the entries embed in the lower right corners of X_1,X_2 and X_3.

Let $\mathfrak{a} \subset \mathfrak{s}$ be the two-dimensional abelian space of matrices $[X_1,0,0]$ with $X_1 \in \langle E_{11}+E_{nn}, E_{n1}+E_{1n}\rangle$. We claim that

$$G = KAH \tag{4.2}$$

holds for $A = \exp\mathfrak{a}$. Since $[E_{11},0,0] \in \mathfrak{h}$ it is equivalent that (4.2) holds for the non-abelian space of matrices $[X_1,0,0]$, with $X_1 \in \langle E_{nn}, E_{n1} + E_{1n}\rangle$. Let

$$Y_1 = [E_{n1}+E_{n1},0,0], \quad Y_2 = [E_{nn},0,0]$$

and $A_i = \exp\mathbb{R}Y_i$, then our claim amounts to $G = KA_1A_2H$.

The intermediate symmetric spaces both allow a polar decomposition. Specifically, $G = KA_1L$ and $L = (L\cap K)A_2H$. Hence

$$G = KA_1(L\cap K)A_2H,$$

and it remains to remove the middle factor. This is accomplished by showing that it allows the product decomposition

$$L\cap K = (L\cap K)^{\mathfrak{a}_1}(L\cap K\cap H)^{\mathfrak{a}_2}.$$

As $L_1 \subset H^{\mathfrak{a}_2}$, it suffices to decompose elements from $L_2 \cap K$. This is done on the level of Lie algebras as follows

$$[0,E_{nn},-E_{nn}] = [0,-E_{11}+E_{nn},E_{11}-E_{nn}] + [0,E_{11},-E_{11}].$$

An elementary computation shows that the two terms commute with Y_1 and Y_2, respectively. Hence (4.2) follows and thus G/H is of polar type with a 2-dimensional A.

Remark 4.8. Note that in 4.2.2 and 4.2.3 all polar decompositions $G = KAH$ are with $\mathfrak{a}$ a full Cartan subspace in $\mathfrak{s}$. In 4.2.4 this is only the case when $n = 2$.

4.3. **Relation to spherical spaces.** It seems that there is a close connection between spherical spaces and polar spaces. Here we provide an indicator why spherical might imply polar.

Lemma 4.9. *Let (P, H) be a spherical pair with Langlands decomposition $P = MAN$ of P. Then there exists $a \in A$ such that*

$$\mathfrak{k}^a + \mathfrak{a} + \mathfrak{h} = \mathfrak{g} \tag{4.3}$$

where $\mathfrak{k}^a := \mathrm{Ad}(a^{-1})(\mathfrak{k})$.

Note that in view of Sard's theorem an equivalent formulation of the conclusion is that KAH has an interior point.

Proof. Otherwise $L(a) := \mathrm{Ad}(a)\mathfrak{k} + \mathfrak{a} + \mathfrak{h}$ is a proper subspace of $\mathfrak{g}$ for all $a \in A$. For $t \mapsto a_t$ a ray tending to infinity in A^+ we note that $\lim_{t\to\infty} L(a_t) = \mathfrak{m}\cap\mathfrak{k}+\mathfrak{a}+\mathfrak{n}+\mathfrak{h}$ in the Grassmann variety of all subspaces of $\mathfrak{g}$. By (1) and (2) in Definition 2.3 we obtain $\lim_{t\to\infty} L(a_t) = \mathfrak{g}$. In particular, for large enough t we have $L(a_t) = \mathfrak{g}$, a contradiction. □

5. Strongly spherical spaces

The estimate in Theorem 3.2 for matrix coefficients on $Z = G/H$ yields bounds for the $m_{v,\eta}$ on subsets of the form $K\overline{A_P^+}.z_0 \subset Z$ where the associated minimal parabolic $P \supset A$ satisfies PH is open. In order to derive a global estimate on Z we therefore need that Z admits not only the polar decomposition $Z = KA.z_0$ but also the stronger

$$Z = \bigcup_{P\supset A,\, PH \text{ open}} K\overline{A_P^+}.z_0\,. \tag{5.1}$$

Unfortunately (5.1) is not true for every A which admits $G = KAH$.

Example 5.1. Let Z be the triple space of $G_0 = \mathrm{SL}(2,\mathbb{R})$ and $A = A_0 \times A_0 \times B_0$, with A_0 the diagonal matrices with positive diagonal entries, and with $B_0 = \mathrm{SO}_e(1,1)$. We have already seen in Proposition 4.5 that $G = KAH$. It is not hard to see that (5.1) fails for this A. Let P_0 and Q_0 be parabolics containing A_0 and B_0, respectively, and let $\overline{P}_0$ be opposite to P_0. Then $P' = P_0 \times \overline{P}_0 \times Q_0$ is a parabolic subgroup which intersects H trivially, hence $P'H$ is open, whereas $P'' := P_0 \times P_0 \times Q_0$ intersects H in positive dimension and $P''H$ is not open. However, to attain that $Z = \cup K\overline{A_P^+}.z_0$ we need the union to encompass parabolics of both types P' and P''.

To remedy the situation we propose a new notion which combines polar and spherical types compatibly.

Definition 5.2. *A reductive homogeneous space $Z = G/H$ is of* strong spherical type *if the following holds. There exists a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and minimal parabolic subgroups $P_1, \ldots, P_l \supset A = \exp \mathfrak{a}$ such that $P_j H$ is open for all $1 \leq j \leq l$ and such that*

$$Z = \bigcup_{j=1}^{l} K\overline{A_{P_j}^{+}}.z_0 \,. \tag{5.2}$$

In particular, Z is then both spherical and polar. The following is now an immediate consequence of Theorem 3.2.

Let V be a Harish-Chandra module and let let $\Lambda_{j,V} \in \mathfrak{a}^*$ be defined by (3.5) with respect to P_j, that is, if $P_j = s_j P s_j^{-1}$ for s_j in the normalizer of $\mathfrak{a}$ in K, then $\Lambda_{j,V} = \Lambda_V \circ \mathrm{Ad}(s_j^{-1})$.

Corollary 5.3. *Assume that $Z = G/H$ is of strong spherical type. Then for each $v \in V$ and each $s \in \mathbb{N}$ there exists a constant $C > 0$ such that*

$$|m_{v,\eta}(ka.z_0)| \leq C\|\eta\|_{-s}\, a^{\Lambda_{j,V}}(1 + \|\log a\|)^{d_V}, \tag{5.3}$$

for all $k \in K$, $a \in \overline{A_{P_j}^{+}}$ and $\eta \in (V^{-\infty})^H \cap E_{-s}$.

5.1. **Examples.**

5.1.1. *Symmetric spaces are strongly spherical.* As in Remark 3.6 let $\mathfrak{a}_q \subset \mathfrak{s} \cap \mathfrak{q}$ and $\mathfrak{a} \subset \mathfrak{s}$ be maximal abelian subspaces with $\mathfrak{a}_q \subset \mathfrak{a}$, then we have seen in Example 4.2.1 that

$$Z = KA_q.z_0.$$

Furthermore, let $\mathfrak{a}_{qj}^{+}$ for $j = 1, \ldots, l$ be the Weyl chambers of $\mathfrak{a}_q$ corresponding to all (up to $K \cap H$-conjugacy) the positive systems Σ_{qj}^{+} for the roots of $\mathfrak{a}_q$ in $\mathfrak{g}$. For each j we choose a compatible positive system Σ_j^{+} for the roots of $\mathfrak{a}$ in $\mathfrak{g}$, and denote by P_j the corresponding minimal parabolic subgroup of G. Then P_j is contained in the minimal $\sigma\theta$-stable parabolic subgroup corresponding to Σ_{qj}^{+}, and it follows from Example 2.2.1 that $P_j H$ is open. Finally

$$A_q = \bigcup_{j=1}^{l} \overline{A_{qj}^{+}}, \tag{5.4}$$

and since $\overline{A_{qj}^{+}} \subset \overline{A_{P_j}^{+}}$, we obtain (5.2).

5.1.2. *Gross-Prasad spaces.* The simplest of these spaces is

$$G/H = G_0 \times H_0 / \operatorname{diag}(H_0)$$

where $G_0 = \mathrm{GL}(2,\mathbb{R})$ and $H_0 = \mathrm{GL}(1,\mathbb{R})$ (see Sections 2.2.2, 4.2.3). This space is strongly spherical with A chosen as in the proof of Lemma 4.7. We expect other Gross-Prasad spaces are strongly spherical.

5.1.3. *Triple space.* The triple space attached to $G_0 = \mathrm{SL}(2,\mathbb{R})$ is strongly spherical. In [10] we show that for any choice of $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$ with $\mathfrak{a}_i \subset \mathfrak{s}_0$ one-dimensional subspaces and not all equal, one has $G = KAH$. Furthermore, if all the one-dimensional subspaces are different from each other, then PH is open for all parabolics P containing A. Thus Z is strongly spherical.

5.2. **The wave front lemma.** The following result was proved for symmetric spaces in [12], Thm. 3.1, under the name of "wavefront lemma". It plays a crucial role in that paper.

Let $Z = G/H$ be of strong spherical type, and let $P_1, \ldots, P_l \supset A$ be as in Definition 5.2 so that $G = \cup_{j=1}^{l} K\overline{A_{P_j}^{+}}H$.

Lemma 5.4. *For every neighborhood V of $\mathbf{1}$ in G, there exists a neighborhood U of $\mathbf{1}$ such that*

$$Vg.z_0 \supset gU.z_0$$

for all $g \in \cup_{j=1}^{l} K\overline{A_{P_j}^{+}}$.

Proof. We may assume that V is $\mathrm{Ad}(K)$-invariant. By (5.2) we reduce to the case $g = a \in \overline{A_{P_j}^{+}}$ for a minimal parabolic $P_j \supset A$ such that P_jH (and hence also $\bar{P}_jH$) is open. Let U_j be a neighborhood of $\mathbf{1}$ in $\bar{P}_j$ which is contained in V and which is stable under conjugation with elements from $\overline{A_{P_j}^{+}}$. As $\bar{P}_jH$ is open, we see that $U_j.z_0$ is a neighborhood of z_0. Then

$$Va.z_0 \supset U_ja.z_0 = aU_j.z_0 \supset aU.z_0$$

where $U = \cap_{j=1}^{l} U_jH$. □

5.3. **Weights on Z.** In this subsection we let $Z = G/H$ be a reductive homogeneous space. In the context of strongly spherical spaces we aim for a more quantitative bound in Theorem 3.2, in which the constant C depends continuously on v in the V^∞-topology. For that the concept of weight will be useful.

We fix a norm $\|\cdot\|$ on G (see [20], Section 2.A.2). By a *weight* on $Z = G/H$ we shall understand a locally bounded function $w : Z \to \mathbb{R}_{>0}$

such that there exists constants $C > 0$, $N \in \mathbb{N}$ with

$$w(gz) \leq C\|g\|^N w(z) \qquad (g \in G, z \in Z)\,.$$

The following is an easy way to construct a weight on Z.

Lemma 5.5. *Let $w(g.z_0) := \inf_{h\in H} \|gh\|$ for $g \in G$. Then w is a weight on Z. Furthermore, there exist constants $c_1, c_2, C_1, C_2 > 0$ such that*

$$C_1 e^{c_1\|X\|} \leq w(k\exp(X).z_0) \leq C_2 e^{c_2\|X\|} \tag{5.5}$$

for all $k \in K$ and $X \in \mathfrak{s} \cap \mathfrak{q}$.

Note that (5.5) applies to every element in Z by (2.1).

Proof. We have $w \geq 1$ since $\|g\| \geq 1$ for all $g \in G$. As $\|xy\| \leq \|x\|\|y\|$ for $x, y \in G$ the first statement follows.

There exist constants $c_1, c_2, C_1, C_2 > 0$ such that

$$C_1 e^{c_1\|Y\|} \leq \|\exp(Y)\| \leq C_2 e^{c_2\|Y\|}$$

for all $Y \in \mathfrak{s}$. Hence the second inequality in (5.5) is clear. For the first inequality we need to show that

$$C_1 e^{c_1\|X\|} \leq \|\exp(X)h\|$$

for all $h \in H$. By Cartan decomposition of H we reduce to $h = \exp(T)$ where $T \in \mathfrak{s} \cap \mathfrak{h}$. Let $Y \in \mathfrak{s}$ be determined by $\exp(X)\exp(T) \in K\exp(Y)$, then $\|X\| \leq \|Y\|$ since the sectional curvatures of $K\backslash G$ are ≤ 0 (see [14], p. 73) and $X \perp Y$. Now

$$C_1 e^{c_1\|X\|} \leq C_1 e^{c_1\|Y\|} \leq \|\exp(Y)\| = \|\exp(X)\exp(T)\|$$

as claimed. □

Let w be a weight on Z. From the definition we readily obtain that w^{-1} is a weight. More generally $w_\alpha(z) := w(z)^\alpha$ defines a weight for all $\alpha \in \mathbb{R}$. Further if w and w' are weights then so is $w \cdot w'$. If $w(z) \geq c$ for some $c > 1$ and all $z \in Z$, then $\log w$ is a weight as well.

A more refined construction of weights than that of Lemma 5.5 goes as follows. Let U be a finite dimensional G-module with a non-zero H-fixed vector $u_H \in U$ (see Lemma 4.2). Such a representation will be referred to as *H-spherical.* Set

$$w_U(g.z_0) := \|g \cdot u_H\| \qquad (g \in G)\,, \tag{5.6}$$

then w_U is a weight.

Lemma 5.6. *Let U be H-spherical and irreducible. Let $P = M_P A_P N_P$ be a parabolic subgroup of G for which PH is open. Let $\lambda \in \mathfrak{a}_P^*$ be the highest $\mathfrak{a}_P$-weight of U and $A_P^+ \subset A_P$ the positive chamber, both with respect to P. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 a^\lambda \leq w_U(ka.z_0) \leq C_2 a^\lambda \quad (a \in \overline{A_P^+}, k \in K)\,. \tag{5.7}$$

Proof. We may choose an inner product on U such that

$$\langle X.u, v\rangle = \langle u, -\theta(X).v\rangle$$

for $X \in \mathfrak{g}$. In particular, the norm is then K-invariant. Let $U_\lambda \subset U$ be the λ-weight space for $\mathfrak{a}_P$, then $U = \mathcal{U}(\bar{n}_P)U_\lambda$ and hence all the $\mathfrak{a}_P$-weights μ in U are obtained from λ by subtracting positive combinations of positive $\mathfrak{a}_P$-roots. It follows that $a^\mu \leq a^\lambda$ for all $a \in \overline{A_P^+}$. By expanding u_H into $\mathfrak{a}_P$-weights we conclude the second inequality of (5.7).

Note that u_H cannot be orthogonal to U_λ. Otherwise, as U_λ is P-invariant, $\pi(g)u_H$ would be orthogonal to U_λ for all g in the open set $\bar{P}H$, contradicting irreducibility. Hence $\langle u_H, u\rangle \neq 0$ for some $u \in U_\lambda$. Now

$$a^\lambda |\langle u_H, u\rangle| = |\langle u_H, a\cdot u\rangle| = |\langle a\cdot u_H, u\rangle| \leq \|u\| w_U(a).$$

for $a \in A_P$, and the first inequality of (5.7) follows. □

5.4. **Symmetric spaces.** In this section we assume that $Z = G/H$ is a symmetric space and use the notation from Example 5.1.1. In particular

$$A_q = \cup_{j=1}^{l} \overline{A_{qj}^+}. \tag{5.8}$$

We fix a chamber A_q^+ (for example A_{q1}^+) for reference, and choose Weyl group elements s_j such that $A_{qj}^+ = \mathrm{Ad}(s_j)A_q^+$ for $j = 1, \dots, l$.

Lemma 5.7. *Assume that G/H is a symmetric space. For each $\Lambda \in \mathfrak{a}_q^*$ and all $d \in \mathbb{Z}$ there exists a weight w and a constant $C > 0$ such that*

$$a^{s_j\Lambda}(1 + \|\log a\|)^d \leq w(ka.z_0) \leq C a^{s_j\Lambda}(1 + \|\log a\|)^d \tag{5.9}$$

for all $k \in K$, $a \in \overline{A_{qj}^+}$ and $j = 1, \dots, l$.

Proof. It follows from the work of Hoogenboom (see [4], Section 5) that $\mathfrak{a}_q^*$ is spanned by the restrictions of the highest weights of H-spherical representations. Hence $\Lambda = c_1\lambda_1 + \cdots + c_k\lambda_k$ for some $c_1, \dots, c_k \in \mathbb{R}$, where $\lambda_1, \dots, \lambda_k \in \mathfrak{a}_q^*$, are highest weights with respect to A_q^+ of irreducible H-spherical representations $U_1, \dots, U_k$. The highest weight of U_i with respect to A_{qj}^+ is then $s_j\lambda_i$. It follows from Lemma 5.6 that

$$C_1 a^{s_j\lambda_i} \le w_{U_i}(ka.z_0) \le C_2 a^{s_j\lambda_i}$$

for $a \in A_{qj}^+$. With w a multiple of $\Pi_i w_{U_i}^{c_i}$ we obtain (5.9) for $d = 0$.

Select $\lambda_0 \in \mathfrak{a}_q^*$ such that $\lambda_0(X) \ge \|X\|$ for all X in the cone $\mathfrak{a}_q^+$. By applying the proved version of (5.9) we see that there exist a weight w_0 and a constant $C_0 > 0$ such that

$$a^{s_j\lambda_0} \le w_0(ka.z_0) \le C_0 a^{s_j\lambda_0}$$

for $a \in A_{qj}^+$ and all j, and hence

$$e^{\|\log a\|} \le w_0(ka.z_0) \le C_0 e^{\|\lambda_0\|\|\log a\|}$$

for all $a \in A$. In particular, $w_0 \ge 1$, hence $\log(cw_0)$ is a weight for every $c > 1$. Taking logarithms we thus find a weight w_1 and a constant $C_0' > 0$ for which

$$1 + \|\log a\| \le w_1(ka.z_0) \le C_0'(1 + \|\log a\|)$$

for all $a \in A$. Now (5.9) follows by multiplication of the previously found weight with w_1^d. □

To any weight w we associate the Banach space

$$E_w := \{f \in C(Z) \mid \|f\|_w := \sup_{z\in Z} w(z)|f(z)| < \infty\}.$$

The group G acts on E_w by left displacements in the arguments, say $\pi(g)f(z) := f(g^{-1}z)$ and we have $\|\pi(g)\| \le C\|g\|^N$. Thus the smooth vectors E_w^∞ form an SF-representation of G in the sense of [6] (that is a smooth Fréchet representation of moderate growth).

In the following theorem the linear form Λ_V is defined by (3.5) with respect to an open chamber of A, which is compatible with the fixed chamber A_q^+.

Theorem 5.8. *Suppose that $Z = G/H$ is symmetric. Let V be a Harish-Chandra module and fix $\eta \in (V^{-\infty})^H$. Then there exists a continuous norm q on V^∞ such that*

$$|m_{v,\eta}(a)| \le q(v) a^{\Lambda_V} (1 + \|\log a\|)^{d_V} \tag{5.10}$$

for all $a \in \overline{A_q^+}$, and $v \in V^\infty$.

Note that for this case it is known that $\dim(V^{-\infty})^H < \infty$ (see Corollary 2.2 of [2]).

Proof. We use the parametrization of the chambers of A_q from (5.8). According to (5.3) we obtain for all $k \in K$, $a \in \overline{A_{q,j}^+}$ and $j = 1, \ldots, l$ that

$$|m_{v,\eta}(ka.z_0)| \leq C_v a^{s_j^{-1}\Lambda_V}(1 + \|\log a\|)^{d_V}. \tag{5.11}$$

It follows from Lemma 5.7 that there exists a weight w on Z such that

$$w(ka.z_0) \asymp a^{-s_j^{-1}\Lambda_V}(1 + \|\log a\|)^{-d_V} \qquad (k \in K, a \in \overline{A_{q,j}^+}),$$

for all j, and hence the product $wm_{v,\eta}$ is bounded on Z for all $v \in V$. Hence we obtain an embedding

$$V \hookrightarrow E_w^\infty, \quad v \mapsto m_{v,\eta}.$$

By the Casselman-Wallach globalization theorem (see [20], Thm. 11.6.7 or [6]) this embedding extends to a continuous embedding of Fréchet spaces $V^\infty \hookrightarrow E_w^\infty$. In particular, there exists a continuous norm q on V^∞ such that

$$\|m_{v,\eta}\|_w \leq q(v) \qquad (v \in V^\infty).$$

Unwinding the definition of the norm in E_w we retrieve (5.11) with C_v replaced by $q(v)$. Now (5.10) follows. □

5.5. **Group Case.** We explicate Theorem 5.8 for the group case $Z = G \times G/G$. For that let W be a Harish-Chandra module for $(\mathfrak{g}, K)$ and $\tilde{W}$ its contragredient. With that we form the Harish-Chandra module $V := W \otimes \tilde{W}$ for $(\mathfrak{g} \times \mathfrak{g}, K \times K)$. We view V as a submodule of $\mathrm{End}(W)$ and identify V^∞ as a subspace of $\mathrm{End}(W^\infty)$. To be more precise V^∞ identifies with the rapidly decreasing matrices as follows: Choose a Hilbert globalization E of W and with respect to the Hilbert structure an orthonormal basis $v_1, v_2, \ldots$ of E consisting of vectors v_i which belong to K-types $\tau_i \in \widehat{K}$ with $\tau_i \leq \tau_j$ for $i \leq j$. This identifies W^∞ with the standard nuclear Fréchet space

$$s(\mathbb{N}) := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^\mathbb{N} \mid \sup_{n\in\mathbb{N}} n^k |x_n| < \infty, \forall k \in \mathbb{N}\}$$

of rapidly decreasing sequences (see [9], p. 290). A continuous linear map $T : s(\mathbb{N}) \to s(\mathbb{N})$ is thus given by a matrix $T = (t_{n,m})_{n,m}$ and we say that T is rapidly decreasing provided that $\|T\|_k := \sup_{n,m} |t_{n,m}|(n+m)^k < \infty$ for all $k \in \mathbb{N}$. Now V^∞ is the space of such maps, and its topology is defined by the norms $\|\,.\,\|_k$ In particular the trace map

$$\eta : V^\infty \to \mathbb{C}, \quad T \mapsto \mathrm{tr}(T) = \sum_n t_{nn}$$

is a continuous linear functional on V^∞, which is fixed by the diagonal subgroup $H := \mathrm{diag}(G) < G \times G$.

Let $\mathfrak{a} \subset \mathfrak{s}$ be maximal abelian, and put

$$\mathfrak{a}_q = \{(X, -X) \in \mathfrak{g} \times \mathfrak{g} \mid X \in \mathfrak{a}\}$$

then $\mathfrak{a}_q$ is a subspace for $Z = G \times G/G$ as chosen in Remark 3.6. The element $\Lambda_V \in \mathfrak{a}^* \times \mathfrak{a}^*$ is identified as $\Lambda_V = (\Lambda_W, -\Lambda_W)$, and likewise $d_V = 2d_W$. If we write π for the action of G on W^∞, then it follows that the bound in Theorem 5.8 asserts for all $a \in \overline{A^+}$ and $T \in V^\infty$ that

$$|\mathrm{tr}(\pi(a^{-1})T)| \leq a^{\Lambda_W}(1 + \|\log a\|)^{2d_W} q(T)$$

with q a continuous norm on V^∞. Let us specialize to the case where T is a rank one operator $T(w) := \tilde{u}(w)u$ for $u, w \in W^\infty$ and $\tilde{u} \in \tilde{W}^\infty$. We conclude:

Corollary 5.9. *Let W be a Harish-Chandra module for G. Then there exist $d \in \mathbb{N}$ and continuous norms p on W^∞ and $\tilde{p}$ on $\tilde{W}^\infty$ such that*

$$|\tilde{u}(\pi(a^{-1})u)| \leq a^{\Lambda_W}(1 + \|\log a\|)^d p(u)\tilde{p}(\tilde{u})$$

for all $u \in W^\infty$, $\tilde{u} \in \tilde{W}^\infty$ and $a \in \overline{A^+}$.

Remark 5.10. The corollary generalizes the estimate in [20], Thm. 4.3.5, where the matrix coefficient is required to be K-finite on one side. However, it should be emphasized that our proof of Theorem 5.8 invokes the globalization theorem, which is not available at that stage in the exposition of [20].

5.6. **Other strongly spherical spaces.** We now return to the general assumption that $Z = G/H$ is a reductive homogeneous space with H connected, and discuss the generalization of Theorem 5.8. We assume that Z is strongly spherical, so that (5.2) is valid. Recall that $\mathfrak{a} \subset \mathfrak{s}$ is maximal abelian. We use the standard isomorphism between $\mathfrak{a}$ and its dual space $\mathfrak{a}^*$, and let $\mathfrak{a}_{hw} \subset \mathfrak{a}$ be the subspace such that $\mathfrak{a}_{hw}^*$ is the span of all the H-spherical highest weights $\lambda \in \mathfrak{a}^*$.

Theorem 5.11. *Let V be a Harish-Chandra module and let $\eta \in (V^{-\infty})^H$. Let $P_1, ..., P_\ell$ be minimal parabolic subgroups that contains $A = exp(\mathfrak{a})$ with $P_j H$ open for each $1 \leq j \leq \ell$ and such that $Z = \bigcup_{j=1}^{\ell} K\overline{A_{P_j}^+}.z_0$.*

1. *Then for any $v \in V^\infty$ there exists a constant C_v such that*
$$|m_{v,\eta}(a)| \leq C_v a^{\Lambda_{j,V}}(1 + \|\log a\|)^{d_V}$$
for all $a \in \overline{A_{P_j}^+}$.
2. *Suppose that*

$$Z = \cup_{j=1}^{l} K(\overline{A_{P_j}^+} \cap A_{hw}).z_0\,. \tag{5.12}$$

Then there exists a continuous norm q on V^∞ such that

$$|m_{v,\eta}(a)| \leq q(v) a^{\Lambda_{j,V}} (1 + \|\log a\|)^{d_V}$$

for all $a \in \overline{A_{P_j}^+} \cap A_{hw}$, $v \in V^\infty$.

Proof. The first point in the theorem is a direct consequence of Corollary 5.3. The second point follows from the first and the assumption (5.12) using the arguments given in theorem 5.8. □

Remark 5.12. The triple space for $\mathrm{SL}(2,\mathbb{R})$ satisfies the assumptions with $\mathfrak{a}_{hw} = \mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$. If we denote by δ the defining representation of $\mathrm{SL}(2,\mathbb{R})$, then $\delta \times \delta \times \mathbf{1}$, $\delta \times \mathbf{1} \times \delta$ and $\mathbf{1} \times \delta \times \delta$ are H-spherical representations, and their highest weights span $\mathfrak{a}$.

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