VANISHING AT INFINITY ON HOMOGENEOUS SPACES OF REDUCTIVE TYPE

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ABSTRACT. Let G be a real reductive group and $Z = G/H$ a unimodular homogeneous G space. The space Z is said to satisfy VAI if all smooth vectors in the Banach representations $L^p(Z)$ vanish at infinity, $1 \leq p < \infty$. For H connected we show that Z satisfies VAI if and only if it is of reductive type.

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1. Introduction

In many applications of harmonic analysis of Lie groups it is important to study the decay of functions on the group. For example for a simple Lie group G, the fundamental discovery of Howe and Moore ([10], Thm. 5.1), that the matrix coefficients of non-trivial irreducible unitary representations vanish at infinity, is often seen to play an important role. In a more general context it is of interest to study matrix coefficients formed by a smooth vector and a distribution vector. If the distribution vector is fixed by some closed subgroup H of G, these generalized matrix coefficients will be smooth functions on the quotient manifold G/H. This leads to the question which is studied in the present paper, the decay of smooth functions on homogeneous spaces. More precisely, we are concerned with the decay of smooth L^p-functions on G/H.

Let G be a real Lie group and $H \subset G$ a closed subgroup. Consider the homogenous space $Z = G/H$ and assume that it is unimodular, that is, it carries a G-invariant measure μ_Z. Note that such a measure is unique up to a scalar multiple.

For a Banach representation (π, E) of G we denote by E^∞ the space of smooth vectors. In the special case of the left regular representation of G on $E = L^p(Z)$ with $1 \leq p < \infty$, it follows from the local Sobolev lemma that E^∞ is the space of smooth functions on Z, all of whose derivatives belong to $L^p(Z)$ (see [15], Thm. 5.1). Let $C_0^\infty(Z)$ be the space of smooth functions on Z that vanish at infinity. Motivated by the decay of eigenfunctions on symmetric spaces ([17]), the following definition was taken in [12]:

Definition 1.1. *We say Z has the property* VAI *(vanishing at infinity) if for all $1 \leq p < \infty$ we have*

$$L^p(Z)^\infty \subset C_0^\infty(Z).$$

By [15] Lemma 5.1, $Z = G$ has the VAI property for G unimodular and $H = \{\mathbf{1}\}$. The main result of [12] establishes that all reductive symmetric spaces admit VAI. On the other hand, it is easy to find examples of homogeneous spaces without this property. For example, it is clear that a non-compact homogeneous space with finite volume cannot have VAI.

The main result of this article is as follows.

Theorem 1.2. *Let G be a connected real reductive group and $H \subset G$ a closed connected subgroup such that $Z = G/H$ is unimodular. Then* VAI *holds for Z if and only if it is of reductive type.*

Here we recall the following definitions.

Definition 1.3. *Let G be a real reductive group (see* [19]*). We say that H is a* reductive subgroup *and that Z is of* reductive type*, if H is real reductive and the adjoint representation of H in the Lie algebra $\mathfrak{g}$ of G is completely reducible.*

Note that Z is unimodular in that case. If Z is of reductive type and $B \subset G$ is a compact ball, then we show in Section 4 (see also [13]) that

$$\inf_{z \in Z} \operatorname{vol}_Z(Bz) > 0 \,.$$

In view of the invariant Sobolev lemma of Bernstein (see Lemma 3.1) this readily implies that Z has VAI.

The converse implication is established in Proposition 5.1. The main lemma shows that in the non-reductive case the volume of the above mentioned sets Bz can be made exponentially small.

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2. Notation

Throughout G is a connected real reductive group and $H \subset G$ is a closed connected subgroup such that $Z := G/H$ is unimodular. We write μ_Z for a fixed G-invariant measure and vol_Z for the corresponding volume function.

Let $\mathfrak{g}$ be the Lie algebra of G. We fix a Cartan involution θ of G. The derived involution $\mathfrak{g} \to \mathfrak{g}$ will also be called θ. The fixed point set of θ is a maximal compact subgroup K of G whose Lie algebra will be denoted $\mathfrak{k}$. Let $\mathfrak{p}$ denote the -1-eigenspace of θ on $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let κ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ such that

$$\kappa|_{\mathfrak{p}} > 0, \quad \kappa|_{\mathfrak{k}} < 0, \quad \mathfrak{k} \perp \mathfrak{p}.$$

Having chosen κ we define an inner product on $\mathfrak{g}$ by

$$\langle X, Y \rangle = -\kappa(\theta(X), Y).$$

We denote by $\mathfrak{h}$ the Lie algebra of H and by $\mathfrak{q}$ be its orthogonal complement in $\mathfrak{g}$.

Lemma 2.1. *The space Z is of reductive type if and only if there exists a Cartan involution θ of G which preserves H. With such a choice we have $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$.*

Proof. See [9] Exercise VI A8 or [20] Thm. 12.1.4. The last statement follows easily. □

Remark 2.2. Let Z be of reductive type and choose θ and κ as above. Then $[\mathfrak{q},\mathfrak{q}]\subset\mathfrak{h}$ if and only if the pair $(\mathfrak{g},\mathfrak{h})$ is symmetric, that is, if and only if

$$\mathfrak{h}=\{X\in\mathfrak{g}\mid \sigma(X)=X\}$$

for an involution σ of $\mathfrak{g}$. When $\mathfrak{g}$ is semisimple it then follows that

$$\mathfrak{q}=\{X\in\mathfrak{g}\mid \sigma(X)=-X\}.$$

3. VAI versus volume growth

By a ball we will understand a compact symmetric neighborhood B of $\mathbf{1}$ in G. Every ball B determines a weight function $\mathbf{v}_B$ on Z given by

$$\mathbf{v}_B(z):=\operatorname{vol}_Z(Bz)\qquad (z\in Z)\,.$$

The precise shape of the ball B does not matter, as for any two balls B_1, B_2 the quotient $\frac{\mathbf{v}_{B_1}}{\mathbf{v}_{B_2}}$ is bounded from above and below by positive constants (see [1], p. 683).

Let $1\leq p<\infty$. For every $k\in\mathbb{N}$ we let $\|\cdot\|_{p,k}$ be a k-th Sobolev norm of $\|\cdot\|_p$, the L^p-norm on $L^p(Z)$ (see [2], Section 2). Note that the collection $\{\|\cdot\|_{p,k} : k\in\mathbb{N}\}$ determine the Fréchet topology on $L^p(Z)^\infty$.

For a subset $\Omega\subset Z$ we write $\|\cdot\|_{p,k,\Omega}$ the semi-norm on $L^p(Z)^\infty$ which is obtained by integrating the derivatives over Ω.

In this context we recall the invariant Sobolev lemma of Bernstein:

Lemma 3.1 ([1], "Key lemma" on p. 686)**.** *Fix $k>\frac{\dim G}{p}$. Then for every ball B there is a constant $C_B>0$ such that*

$$|f(z)|\leq C_B\mathbf{v}_b(z)^{-\frac{1}{p}}\|f\|_{p,k,Bz}\qquad (z\in Z) \tag{3.1}$$

for all smooth functions f on Z.

For $v\in\mathcal{U}(\mathfrak{g})$ and $f\in L^p(Z)^\infty$, as L_vf belongs to $L^p(Z)$, its norm over Bz will be arbitrarily small for z outside a sufficiently large compact set. Hence, for $f\in L^p(Z)^\infty$ with $1\leq p<\infty$ we obtain that

$$\lim_{z\to\infty}\|f\|_{p,k,Bz}=0\,.$$

Hence we have shown that:

Proposition 3.2. *Suppose that there is constant $c>0$ such that $\mathbf{v}_B(z)>c$ for all $z\in Z$. Then* VAI *holds true.*

4. Reductive Spaces are VAI

For G and H both semisimple it was shown with analytic methods in [13] that the assumption of Proposition 3.2 is valid for G/H. In this section we give a geometric proof, which is valid for all spaces of reductive type. Combined with Proposition 3.2 this completes the proof of the implication 'if' of Theorem 1.2.

Lemma 4.1. *Let $Z = G/H$ be of reductive type. Then there exists a constant $c > 0$ such that*

$$\mathbf{v}_B(z) \geq c \tag{4.1}$$

for all $z \in Z$.

Proof. We begin by recalling the Mostow-decomposition of Z. As Z is of reductive type we can and will identify $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{q}$ in an H-equivariant way (see Lemma 2.1). Note that $\mathfrak{q}$ is θ-stable and in particular $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p}$. We denote by $\mathrm{pr}_\mathfrak{q} : \mathfrak{g} \to \mathfrak{q}$ the orthogonal projection. The polar or Mostow decomposition asserts that the polar map

$$\pi : K \times_{H\cap K} (\mathfrak{q} \cap \mathfrak{p}) \to Z, \quad [k, Y] \mapsto k \exp(Y) z_0 \tag{4.2}$$

is a homeomorphism (see [14] or [4], p. 74).

It is no loss of generality to request the balls B to have two additional properties:

$$\theta(B) = B. \tag{4.3}$$

$$KBK = B. \tag{4.4}$$

Note that θ induces an automorphism on Z which is measure preserving. Hence (4.3) implies

$$\mathbf{v}_B(z) = \mathbf{v}_B(\theta(z)) \qquad (z \in Z). \tag{4.5}$$

Furthermore, it follows from (4.4) and the polar decomposition (4.2) that it is sufficient to establish (4.1) for all $z = \exp(tX) \cdot z_0 \in Z$ where $t \in \mathbb{R}$ and $X \in \mathfrak{q} \cap \mathfrak{p}$ has unit length.

Now for every $X \in \mathfrak{q}\cap\mathfrak{p}$ we choose an $\operatorname{ad} X$-stable vector complement $V_X \subset \mathfrak{g}$ to $\mathfrak{h}$, i.e. $\mathfrak{g} = \mathfrak{h} \oplus V_X$. There is a neighborhood $U_X \subset V_X$ of 0 such that $\exp(U_X) \subset B$ and such that

$$U_X \to Z, \quad Y \mapsto \exp(Y) \cdot z_0$$

is a diffeomorphism onto its image, with a Jacobian which is bounded below by a positive number d_X. Let $t \in \mathbb{R}$ and note that

$$\begin{aligned}\mathbf{v}_B(\exp(tX)\cdot z_0) &\geq \mathrm{vol}_Z(\exp(U_X)\exp(tX)\cdot z_0)\\ &= \mathrm{vol}_Z(\exp(e^{-t\,\mathrm{ad}\,X}U_X)\cdot z_0)\\ &\geq \mathrm{vol}_Z(\exp(U_X\cap e^{-t\,\mathrm{ad}\,X}U_X)\cdot z_0).\end{aligned}$$

We decompose V_X into eigenspaces for $\mathrm{ad}\,X$ and choose U_X to be a box along these coordinates. Then $e^{-t\,\mathrm{ad}\,X}$ is an anisotropic scaling along the edges of the box. This shows that there is a $\lambda_X \in \mathbb{R}$ such that for all $t \in \mathbb{R}$ one has

$$\mathbf{v}_B(\exp(tX)\cdot z_0) \geq c_X e^{\lambda_X t} \tag{4.6}$$

where $c_X = d_X\,\mathrm{vol}(U_X)$. As $\theta(X) = -X$ we finally obtain from (4.5) that

$$\mathbf{v}_B(\exp(tX)\cdot z_0) \geq c_X \cosh(t\lambda_X) \geq c_X$$

for all $t \in \mathbb{R}$.

The constant c_X can be chosen to be locally uniform with respect to X. To see this, let $X \in \mathfrak{a}$ where $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. Then we can assume that the complementary subspace V_X has been chosen such that it is $\mathrm{ad}(\mathfrak{a})$-stable. Thus the same subspace V_X can be used for all $X \in \mathfrak{a}$. The map $K \times \mathfrak{a} \to \mathfrak{p}$ given by $(k, X) \mapsto \mathrm{Ad}(k)X$ is open, and $\mathrm{Ad}(k)V_X$ complements $\mathfrak{h}$ for $k \in K$ small as well. This shows that V_X can be chosen so that it depends locally uniformly on X. Moreover it also follows by conjugation with K that local uniformity can be attained for U_X and d_X. By compactness we obtain (4.1). □

Remark 4.2. If G/H is a semisimple symmetric space (see Remark 2.2) then by the same method one obtains strong volume bounds, both below and above, as follows. In this case the polar decomposition (4.2) can be given the more explicit form

$$G = K\overline{A_q^+}H$$

(see [18] Proposition 7.1.3) where $A_q^+ = \exp \mathfrak{a}_q^+$ for a positive Weyl chamber in a maximal abelian subspace $\mathfrak{a}_q \subset \mathfrak{q} \cap \mathfrak{p}$. In the proof above it then suffices to consider elements $X \in \mathfrak{a}_q$. For such elements the complementary subspace V_X can be chosen independently of X. Indeed, if U is the unipotent radical of a minimal $\sigma\theta$-stable parabolic subgroup containing A_q, then $V_X = \mathfrak{q}^{\mathfrak{a}_q} + \mathfrak{u} \subset \mathfrak{g}$ will be such a subspace. The scalings of the box U_X are then exactly determined by the roots of $\mathfrak{u}$, and it follows as in (4.6) that there exists a constant $C_1 > 0$ such that

$$\mathrm{vol}_Z(Ba\cdot z_0) \geq C_1 a^{2\rho}, \qquad (a \in \overline{A_q^+})$$

where $2\rho = \operatorname{tr} \operatorname{ad}_{\mathfrak{u}}$. For an upper bound we use that $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}^{\mathfrak{a}_q} + \mathfrak{u}$, so that $B \subset K \exp(U_X) H^{\mathfrak{a}_q}$ for some ball B. Then by a similar argument we obtain the existence of a constant $C_2 > 0$ such that

$$\operatorname{vol}_Z(Ba \cdot z_0) \leq C_2 a^{2\rho}, \qquad (a \in \overline{A_q^+}).$$

Similar bounds can be obtained for all real spherical spaces by using the results of [11].

Remark 4.3. For a semisimple symmetric space the wave front lemma, Theorem 3.1 of [8], shows that there exists an open neighberhood V of z_0, such that Bz contains a G-translate of V for all $z \in Z$. This implies (4.1) for this case.

5. Non-reductive spaces are not VAI

In this section we prove that VAI does not hold on any homogeneous space $Z = G/H$ of G, which is not of reductive type. We maintain the assumption that G is a connected real reductive group and establish the following result.

Proposition 5.1. *Assume that $H \subset G$ is a closed connected subgroup such that $Z = G/H$ is unimodular and not of reductive type. Then for all $1 \leq p < \infty$ there exists an unbounded function $f \in L^p(Z)^\infty$. In particular,* VAI *does not hold.*

The idea is to show that there is a compact ball $B \subset G$ and a sequence $(g_n)_{n\in\mathbb{N}}$ such that

$$\operatorname{vol}_Z(Bg_n z_0) \leq e^{-n} \text{ for all } n \in \mathbb{N}.$$

Out of these data it is straightforward to construct an unbounded smooth L^p-function.

Before we give a general proof we first discuss the case of unipotent subgroups. The argument in the general case, although more technical, will be modeled after that.

5.1. **Unipotent subgroups.** Let $H = N$ be a unipotent subgroup, that is, $\mathfrak{n} := \mathfrak{h}$ is an ad-nilpotent subalgebra of $[\mathfrak{g}, \mathfrak{g}]$. Now, the situation where $\mathfrak{n}$ is normalized by a particular semi-simple element is fairly straightforward and we shall begin with a discussion of that case.

If $X \in \mathfrak{g}$ is a real semi-simple element, i.e., $\operatorname{ad} X$ is semi-simple with real spectrum, then we denote by $\mathfrak{g}_X^\lambda \subset \mathfrak{g}$ its eigenspace for the eigenvalue $\lambda \in \mathbb{R}$, and by $\mathfrak{g}_X^\pm$ the sum of these eigenspaces for λ positive/negative. We record the triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_X^+ + \mathfrak{z}_{\mathfrak{g}}(X) + \mathfrak{g}_X^- .$$

Here $\mathfrak{z}_{\mathfrak{g}}(X) =: \mathfrak{g}_X^0$ is the centralizer of X in $\mathfrak{g}$.

Lemma 5.2. *Assume that* $\mathfrak{n}$ *is normalized by a non-zero real semisimple element* $X \in \mathfrak{g}$ *such that* $\mathfrak{n} \subset \mathfrak{g}_X^+$*. Set* $a_t := \exp(tX)$ *for all* $t \in \mathbb{R}$*. Let* $B \subset G$ *be a compact ball around* $\mathbf{1}$*. Then there exists* $c > 0$ *and* $\gamma > 0$ *such that*

$$\mathrm{vol}_Z(Ba_t z_0) = c \cdot e^{t\gamma} \qquad (t \in \mathbb{R})$$

Proof. Let $A = \exp \mathbb{R}X$ and note that A normalizes N. Thus for all $a \in A$ the prescription

$$\mu_{Z,a}(Bz_0) := \mu_Z(Baz_0) \qquad (B \subset G \text{ measurable})$$

defines a G-invariant measure on Z. By the uniqueness of the Haar measure we obtain that

$$\mu_{Z,a} = J(a)\mu_Z$$

where $J : A \to \mathbb{R}_0^+$ is the group homomorphism $J(a) = \det \mathrm{Ad}(a)|_{\mathfrak{n}}$. The assertion follows. □

Having obtained this volume bound we can proceed as follows. Let us denote by χ_k the characteristic function of $Ba_{-k}z_0 \subset Z$. We claim that the non-negative function

$$\chi := \sum_{k \in \mathbb{N}} k\chi_k \tag{5.1}$$

lies in $L^p(G/H)$. In fact

$$\|\chi\|_p \leq \sum_{k \in \mathbb{N}} k\|\chi_k\|_p \leq c \sum_{k \in \mathbb{N}} ke^{-\gamma k/p}\,.$$

Finally we have to smoothen χ: For that let $\phi \in C_c(G)^\infty$ with $\phi \geq 0$, $\int_G \phi = 1$ and $\operatorname{supp} \phi \subset B$. Then $\tilde{\chi} := \phi * \chi \in L^p(Z)^\infty$ with $\tilde{\chi}(a_{-k}z_0) \geq k$. Hence $\tilde{\chi}$ is unbounded.

In general, given a unipotent subalgebra $\mathfrak{n}$, there does not necessarily exist a semisimple element which normalizes $\mathfrak{n}$. For example if $U \in \mathfrak{g} = \mathfrak{sl}(5, \mathbb{C})$ is a principal nilpotent element, then $\mathfrak{n} = \mathrm{span}\{U, U^2 + U^3\}$ is a 2-dimensional abelian unipotent subalgebra which is not normalized by any semi-simple element of $\mathfrak{g}$. The next lemma offers a remedy out of this situation by finding an ideal $\mathfrak{n}_1 \lhd \mathfrak{n}$ which is normalized by a real semisimple element X with $\mathfrak{n} \subset \mathfrak{g}_X^+ + \mathfrak{g}_X^0$.

Lemma 5.3. *Let* $\mathfrak{n} \subset [\mathfrak{g}, \mathfrak{g}]$ *be an* ad*-nilpotent subalgebra and let* $0 \neq U \in \mathfrak{z}(\mathfrak{n})$*. Then there exists a real semi-simple element* $X \in \mathfrak{g}$ *such that* $[X, U] = 2U$ *and* $\mathfrak{n} \subset \mathfrak{g}_X^+ + \mathfrak{g}_X^0$*.*

Proof. According to the Jacobson-Morozov theorem one finds elements $X, V \in \mathfrak{g}$ such that $\{X, U, V\}$ form an $\mathfrak{sl}_2$-triple, i.e. satisfy the commutator relations $[X, U] = 2U$, $[X, V] = -2V$, $[U, V] = X$. Note that $\mathfrak{n} \subset \mathfrak{z}_{\mathfrak{g}}(U)$ and that $\mathfrak{z}_{\mathfrak{g}}(U)$ is $\operatorname{ad} X$-stable. It is known and in fact easy to see that $\mathfrak{z}_{\mathfrak{g}}(U) \subset \mathfrak{g}_X^+ + \mathfrak{g}_X^0$. All assertions follow. □

Within the notation of Lemma 5.3 we set $\mathfrak{n}_1 = \mathbb{R}U$ and $N_1 = \exp(\mathfrak{n}_1)$. Furthermore we set $Z_1 = G/N_1$ and consider the contractive averaging map

$$L^1(Z_1) \to L^1(Z), \quad f \mapsto \widehat{f}; \quad \widehat{f}(gN) = \int_{N/N_1} f(gnN_1)\, d(nN_1).$$

Let $B \subset G$ be a compact ball around $\mathbf{1}$, of sufficiently large size to be determined later, and let $B_1 = B \cdot B \subset G$. Let χ be the function on Z_1 constructed as in (5.1), using the element X from Lemma 5.3 and the compact set B_1. Let $\widehat{\chi} \in L^1(Z)$ be the average of χ. We claim that $\widehat{\chi}(Ba_{-k}z_0) \geq k$ for all k. In fact let $Q \subset N/N_1$ be a compact neighborhood of $\mathbf{1}$ in N/N_1 with $\operatorname{vol}_{N/N_1}(Q) = 1$. Then for B large enough we have $a_{-k}Qa_k \subset B$ for all k (Lemma 5.3). Hence for $b \in B$,

$$\widehat{\chi}(ba_{-k}z_0) \geq \int_Q \chi(ba_{-k}nN_1)\, d(nN_1) \geq k,$$

proving our claim.

To continue we conclude that $f_p := (\widehat{\chi})^{\frac{1}{p}} \in L^p(Z)$ is a function with $f_p(Ba_{-k}z_0) \geq k$ for all k. Finally we smoothen f_p as before and conclude that VAI does not hold true.

5.2. **The general case of a non-reductive unimodular space.** Finally we shall prove Proposition 5.1 in the general situation where H is a closed and connected subgroup for which $Z = G/H$ is unimodular and not of reductive type.

We fix a Levi-decomposition $\mathfrak{h} = \mathfrak{r} \rtimes \mathfrak{s}$ of $\mathfrak{h}$. As in Section 2 we fix a Cartan involution θ of $\mathfrak{g}$, and by Lemma 2.1 we may assume that it restricts to a Cartan involution of $\mathfrak{s}$.

Proof of Proposition 5.1.

We will argue by induction on $\dim \mathfrak{g}$, the base of the induction being clear. We will perform a number of reduction steps (which may involve the induction hypothesis) that will lead us to a simplified situation which is described in Step 9 of the proof.

Step 1: $\mathfrak{h}$ is not contained in any reductive proper subalgebra of $\mathfrak{g}$.

Indeed, otherwise $\mathfrak{h}$ is contained in a proper subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$, which is reductive in $\mathfrak{g}$. Then $\mathfrak{h}$ is not reductive in $\tilde{\mathfrak{h}}$ ([6], §6.6 Cor. 2). By induction $\tilde{H}/H$ is not VAI, in the strong sense that for every $1 \leq p < \infty$ there exists an unbounded function $f \in L^p(\tilde{H}/H)^\infty$. We claim that G/H is not VAI in the same strong sense. Let $\tilde{\mathfrak{q}} \subset \mathfrak{g}$ be the orthogonal complement to $\tilde{\mathfrak{h}}$ in $\mathfrak{g}$. Then for a small neighborhood $V \subset \tilde{\mathfrak{q}}$ of 0 the tubular map

$$V \times \tilde{H} \to G, \quad (X, h) \mapsto \exp(X)h$$

is diffeomorphic. The Haar measure on G is expressed by $J(X)dXdh$ with $J > 0$ a bounded positive function. Since $\tilde{H}$ normalizes $\tilde{\mathfrak{q}}$, this allows us to extend smooth L^p-functions from $\tilde{H}/H$ to G/H and we see that G/H is not VAI in the strict sense. Hence we may assume as stated in Step 1.

Step 2: $\mathfrak{h}$ is contained in a maximal parabolic subalgebra $\mathfrak{p}_0$.

Indeed, by the characterization of maximal subalgebras of $\mathfrak{g}$ (see [7], Ch. 8, §10, Cor. 1), a maximal subalgebra is either a maximal parabolic subalgebra or it is a maximal reductive subalgebra. Hence it follows from Step 1 that $\mathfrak{h}$ is contained in a maximal parabolic subalgebra $\mathfrak{p}_0$.

Step 3: $\mathfrak{s} \subset \mathfrak{l}_0$, the Levi part of $\mathfrak{p}_0$

We write $\mathfrak{p}_0 = \mathfrak{n}_0 \rtimes \mathfrak{l}_0$ where $\mathfrak{n}_0$ is the unipotent radical of $\mathfrak{p}_0$. Note that $\mathfrak{l}_0$ is reductive in $\mathfrak{g}$ and hence $\mathfrak{h}$ is not contained in $\mathfrak{l}_0$. In addition we may assume that $\mathfrak{s} \subset \mathfrak{l}_0$ ([6] §6.8 Cor. 1).

Step 4: $\mathfrak{z}(\mathfrak{l}_0)$ is not contained in $\mathfrak{h}$. We may assume that $\mathfrak{g}$ is semisimple. Then, as $\mathfrak{p}_0$ is maximal, we have $\mathfrak{z}(\mathfrak{l}_0) = \mathbb{R}X_0$ and the spectrum of $\operatorname{ad} X_0$ on $\mathfrak{n}_0$ is either entirely negative or positive. Suppose that $X_0 \in \mathfrak{h}$. Since G/H is unimodular, $|\det \operatorname{Ad}(h)|_{\mathfrak{h}}| = 1$ for $h \in H$ and in particular $|\det e^{\operatorname{ad} X_0}|_{\mathfrak{r}}| = 1$. This would imply that $\mathfrak{h} \subset \mathfrak{l}_0$, contradicting Step 1.

Step 5: Decomposition $\mathfrak{p}_0 = \mathfrak{l}_1 \oplus (\mathfrak{h} + \mathfrak{n}_0)$.

Indeed, define the subspace $\mathfrak{l}_1 \subset \mathfrak{l}_0$ by

$$\mathfrak{l}_1 = \operatorname{pr}_{\mathfrak{l}_0}(\mathfrak{h})^\perp$$

where $\operatorname{pr}_{\mathfrak{l}_0} : \mathfrak{p}_0 \to \mathfrak{l}_0$ is the projection along $\mathfrak{n}_0$. Then $\mathfrak{l}_1 \oplus (\mathfrak{h} + \mathfrak{n}_0) = \mathfrak{p}_0$.

Step 6: The case $\mathfrak{p}_0 = \mathfrak{l}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_0$.

In this case $\mathfrak{h} \cap \mathfrak{n}_0 = \{0\}$ and thus the projection $\operatorname{pr}_{\mathfrak{l}_0}|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{l}_0$, which is a Lie-algebra homomorphism, is injective. Write $\mathfrak{h}_0$ for the homomorphic image of $\mathfrak{h}$ in $\mathfrak{l}_0$. The analysis will be separated in two subcases.

Case 6a: $\mathfrak{h}_0$ is not reductive in $\mathfrak{l}_0$. Let H_0 and L_0 be the connected subgroups of G corresponding to $\mathfrak{h}_0$ and $\mathfrak{l}_0$. As G/H is unimodular and H is homomorphic to H_0, it follows that G/H_0 and thus L_0/H_0 is unimodular. By induction we find for every $1 \leq p < \infty$ an unbounded function $f \in L^p(L_0/H_0)^\infty$. As before in the case of $\tilde{H}/H$ we extend f to a smooth vector in $L^p(G/H)$ (note that $P_0/H \to L_0/H_0$ is a fibre bundle, and we first extend f to a function on P_0/H and then to a function on G/H).

Case 6b: $\mathfrak{h}_0$ *is reductive in* $\mathfrak{l}_0$. In particular it is a reductive Lie algebra, hence so is $\mathfrak{h}$. In the Levi decomposition $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{r}$ we now know that $\mathfrak{r}$ is the center of $\mathfrak{h}$. Let $\mathfrak{u}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{r}$ and $\theta(\mathfrak{r})$, then $\mathfrak{s} + \mathfrak{u}$ is a direct Lie algebra sum. Moreover, $\mathfrak{s} + \mathfrak{u}$ is θ-invariant, hence reductive in $\mathfrak{g}$, and hence in fact $= \mathfrak{g}$ by our previous assumption on $\mathfrak{h}$. Thus $\mathfrak{s}$ is an ideal in $\mathfrak{g}$ which we may as well assume is 0. Now $\mathfrak{h} = \mathfrak{r}$ is an abelian subalgebra which together with $\theta(\mathfrak{r})$ generates $\mathfrak{g}$. We shall reduce to the case where $\mathfrak{r}$ is nilpotent in $\mathfrak{g}$, which we already treated in Section 5.1.

Every element $X \in \mathfrak{r}$ has a Jordan decomposition $X_n + X_s$ (in $\mathfrak{g}$), and we let $\mathfrak{o}_1, \mathfrak{o}_2$ be the subalgebras generated by the X_n's and X_s's, respectively. Then $\mathfrak{o} = \mathfrak{o}_1 \oplus \mathfrak{o}_2$ is abelian and $\mathfrak{o}_2$ consists of semisimple elements. The centralizer of $\mathfrak{o}_2$ is reductive in $\mathfrak{g}$ and contains $\mathfrak{r}$, hence equal to $\mathfrak{g}$. Hence $\mathfrak{o}_2$ is central in $\mathfrak{g}$, and we may assume that it is θ-stable. Let $\mathfrak{g}_1$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{o}_1$ and $\theta(\mathfrak{o}_1)$. It is reductive in $\mathfrak{g}$, and $(\mathfrak{g}_1, \mathfrak{o}_1)$ is of the type already treated, hence not VAI. Since $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{o}_2$ we can now conclude that $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{g}, \mathfrak{h})$ is not VAI either.

Step 7: An element $X \in \mathfrak{z}(\mathfrak{l}_0)$.

Using the result of Step 4, there exists $X \in \mathfrak{z}(\mathfrak{l}_0) \backslash \mathfrak{h}$ such that $\mathfrak{n}_0 \subset \mathfrak{g}_X^+$. As before we set $a_t := \exp(tX)$ and observe that $a_t z_0 \to \infty$ in Z for $|t| \to \infty$ (this is because $a_t[L_0, L_0]N_0$ tends to infinity in $G/[L_0, L_0]N_0$.)

Step 8: A decomposition $\mathfrak{p}_0 = \mathfrak{l}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_1$

We construct an $\operatorname{ad} X$-invariant subspace $\mathfrak{n}_1 \subset \mathfrak{n}_0$ such that $\mathfrak{h} + \mathfrak{n}_0 = \mathfrak{h} \oplus \mathfrak{n}_1$, as follows. If $\mathfrak{n}_0 \subset \mathfrak{h}$, then $\mathfrak{n}_1 = \{0\}$. Otherwise we choose an $\operatorname{ad} X$-eigenvector, say Y_1, in $\mathfrak{n}_0$ with largest possible eigenvalue, such that $\mathfrak{h} + \mathbb{R}Y_1$ is a direct sum. If this sum contains $\mathfrak{n}_0$, we set $\mathfrak{n}_1 = \mathbb{R}Y_1$. Otherwise we continue that procedure until a complementary subspace is reached. Now $\mathfrak{l}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_1 = \mathfrak{p}_0$ and by Step 6 we can assume $\mathfrak{n}_1 \subsetneq \mathfrak{n}_0$.

Step 9: We summarize the situation we have reduced to:

- $\mathfrak{h} = \mathfrak{r} \rtimes \mathfrak{s}$ is a Levi decomposition of $\mathfrak{h}$.
- $\mathfrak{h} \subset \mathfrak{p}_0 = \mathfrak{n}_0 \rtimes \mathfrak{l}_0$, a maximal parabolic subalgebra of $\mathfrak{g}$.
- $\mathfrak{s} \subset \mathfrak{l}_0$, the Levi part of $\mathfrak{p}_0$.
- $\mathfrak{l}_1 := \operatorname{pr}_{\mathfrak{l}_0}(\mathfrak{h})^\perp \subset \mathfrak{l}_0$ with $\operatorname{pr}_{\mathfrak{l}_0} : \mathfrak{p}_0 \to \mathfrak{l}_0$ the projection along $\mathfrak{n}_0$.
- $\mathfrak{p}_0 = \mathfrak{l}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_1$ with $\mathfrak{l}_1 \subset \mathfrak{l}_0$ and $\mathfrak{n}_1 \subsetneq \mathfrak{n}_0$.
- $X \in \mathfrak{z}(\mathfrak{l}_0)$ and
  (1) $\mathfrak{n}_0 \subset \mathfrak{g}_X^+$.

(2) $\mathfrak{n}_1$ is invariant with respect to $\mathrm{ad}(X)$.
(3) With $a_t = \exp(tX)$ we have $a_t z_0 \to \infty$ in Z for $|t| \to \infty$.

We will construct (for any $1 \leq p < \infty$) a smooth function χ in $L^p(Z)$ which does not decay. For this we need some auxilary functions Φ_t which we now construct.

Let $\overline{\mathfrak{n}}_0$ be the nilradical of the parabolic opposite to $\mathfrak{p}_0$ and consider the $\mathrm{ad}\, X$-invariant vector space

$$\mathfrak{v} := \overline{\mathfrak{n}}_0 \times \mathfrak{l}_1 \times \mathfrak{n}_1 \subset \mathfrak{g}$$

which is complementary to $\mathfrak{h}$. For fixed $t \in \mathbb{R}$ we define the differentiable map

$$\Phi = \Phi_t : \mathfrak{v} \to Z,$$

by the formula

$$\Phi(Y^-, Y^0, Y^+) = \exp(Y^-) \exp(Y^0) \exp(Y^+) a_t z_0 \,.$$

The main property which we need of these functions Φ_t is expressed in the following lemma. For $Y = (Y^-, Y^0, Y^+) \in \mathfrak{v}$ we put

$$y^{\pm,0} = \exp(Y^{\pm,0}) \in G$$

and $y = y^- y^0 y^+$, and we identify the tangent space $T_{\Phi_t(Y)} Z$ with $\mathfrak{v}$ via the map

$$T_{\Phi_t(Y)} Z \to \mathfrak{v}, \quad d\tau_{y a_t}(z_0)(X + \mathfrak{h}) \mapsto \pi_{\mathfrak{v}}(X + \mathfrak{h}), \quad (X \in \mathfrak{g})$$

where $\pi_{\mathfrak{v}} : \mathfrak{g} \to \mathfrak{v}$ is the projection along $\mathfrak{h}$.

Lemma 5.4. *Let the data summarized under Step 9 above be given. Then there exists a constant $\gamma > 0$ with the following property. For every sufficiently small compact neighborhood Q of 0 in $\mathfrak{v}$, there exist constants $c_Q, C_Q > 0$ such that*

$$c_Q e^{t\gamma} \leq \sup_{Y \in Q} |\det d\Phi_t(Y)| \leq C_Q e^{t\gamma} \qquad (t \leq 0).$$

In particular $\Phi_t|_Q$ is a chart for all $t \leq 0$.

The proof, which is computational, is postponed to the end of this section. The construction of the function χ is now easy to describe. Let $Q \subset \mathfrak{v}$ be as above. We fix a function $\psi \in C_c^\infty(Q)$ with $0 \leq \psi \leq 1$ and $\psi(0) = 1$. For all $t < 0$ define $\chi_t \in C_c^\infty(Z)$ by $\chi_t(z) = \psi(\Phi_t^{-1}(z))$ and set

$$\chi := \sum_{n \in \mathbb{N}} n \chi_{-n} \,.$$

It is clear that $\chi \in C^\infty(Z)$ and that χ is unbounded. We claim that $\chi \in L^p(Z)^\infty$.

It follows immediately from the definition that $\chi_t \in L^p(Z)$ for all $1 \leq p < \infty$ and $t \leq 0$, and it follows from the estimate of the differential of Φ in Lemma 5.4 that $\|\chi_t\|_p \leq Ce^{t\gamma/p}$ for some $C > 0$ not depending on t (but possibly on p). Hence

$$\chi = \sum_{n \in \mathbb{N}} n\chi_{-n} \in L^p(Z)$$

for all $1 \leq p < \infty$, and it only remains to be seen that also the derivatives of χ belong to $L^p(Z)$. The proof of this fact depends in addition on the following estimate, which will be proved together with Lemma 5.4.

Lemma 5.5. *Define*

$$M_t := \sup_{U \in \mathfrak{g}, \|U\|=1} \| \operatorname{Ad}(a_t)\pi_{\mathfrak{v}} \operatorname{Ad}(a_t)^{-1}U\|$$

Then $\sup_{t<0}(M_t) < \infty$.

We now complete the proof of the Proposition 5.1 by proving that the left derivatives of χ by elements $U \in \mathfrak{g}$, up to all orders, belong to $L^p(Z)$.

We first show this for first order derivatives. Let $U \in \mathfrak{g}$ and consider the derivative $L(U)\chi_t$. At $z = \Phi_t(Y)$ this is given by

$$L(U)\chi_t(z) = d/ds|_{s=0}\, \chi_t(\exp(sU)ya_tz_0).$$

For Y in a compact set, we can replace U by its conjugate by y without loss of generality, and thus we may as well consider the derivatives of

$$\chi_t(y \exp(sU)a_tz_0).$$

We rewrite this as

$$\chi_t(ya_t \exp(s \operatorname{Ad}(a_t)^{-1}U)z_0)$$

and apply the projection along $\mathfrak{h}$. It follows that the derivative can be rewritten as

$$d/ds|_{s=0}\, \chi_t(ya_t \exp(s\pi_{\mathfrak{v}} \operatorname{Ad}(a_t)^{-1}U)z_0)$$

and then finally also as

$$d/ds|_{s=0}\, \chi_t(y \exp(s \operatorname{Ad}(a_t)\pi_{\mathfrak{v}} \operatorname{Ad}(a_t)^{-1}U)a_tz_0).$$

Note that $\operatorname{Ad}(a_t)\pi_{\mathfrak{v}} \operatorname{Ad}(a_t)^{-1}U \in \mathfrak{v}$. We conclude that the derivative is a linear combination of derivatives of ψ on Q, with coefficients that depend smoothly on Y. Furthermore, it follows from Lemma 5.5 that the coefficients are bounded for $t < 0$. As before we conclude $L(U)\chi_t \in L^p(Z)$ for all $t \leq 0$, with exponentially decaying p-norms. It follows that $L(U)\chi \in L^p(Z)$.

By repeating the argument for higher derivatives we finally see that $\chi \in L^p(Z)^\infty$. □

It remains to verify Lemmas 5.4 and 5.5. We first prove the latter.

Proof of Lemma 5.5. For $U \in \mathfrak{v}$ we have

$$\mathrm{Ad}(a_t)\pi_{\mathfrak{v}}\,\mathrm{Ad}(a_t)^{-1}U = U,$$

hence we may assume $U \in \mathfrak{h}$. Since $\mathfrak{h} \subset \mathfrak{p}_0$ we can write U as a combination of an element $Y_0 \in \mathfrak{l}_0$ and possibly some $\mathrm{ad}\,X$-eigenvectors Y_λ with eigenvalues $\lambda > 0$. Then

$$\mathrm{Ad}(a_t)^{-1}U = Y_0 + \sum e^{-\lambda t}Y_\lambda = U + \sum (e^{-\lambda t} - 1)Y_\lambda$$

(possibly with an empty sum). If $Y_\lambda \in \mathfrak{n}_1$ then

$$\mathrm{Ad}(a_t)\pi_{\mathfrak{v}}(e^{-\lambda t} - 1)Y_\lambda = (1 - e^{\lambda t})Y_\lambda \to Y_\lambda$$

as $t \to -\infty$. On the other hand if Y_λ is not in $\mathfrak{h}$, then it is the sum of an element from $\mathfrak{h}$ and some eigenvectors $V_\mu \in \mathfrak{n}_1$. If one of the eigenvalues μ of these vectors is strictly smaller than λ, then it follows from the definition of $\mathfrak{n}_1$ (see Step 8) that Y_λ must belong to $\mathfrak{n}_1$ (as it will have been preferred before this V_μ). Thus, if Y_λ is not in $\mathfrak{n}_1$, then all the V_μ contributing to Y_λ must have eigenvalues $\mu \geq \lambda$. Then

$$\mathrm{Ad}(a_t)\pi_{\mathfrak{v}}(e^{-\lambda t} - 1)Y_\lambda = \sum e^{\mu t}(e^{-\lambda t} - 1)V_\mu$$

(possibly with an empty sum), which stays bounded for $t \to -\infty$. Our claim is thus established. □

To prepare the proof of Lemma 5.4 we establish the following lemma. To simplify the main formula below we denote

$$\beta(T) = \frac{\mathbf{1} - e^{-\mathrm{ad}\,T}}{\mathrm{ad}\,T} \in \mathrm{End}(\mathfrak{g})$$

for $T \in \mathfrak{g}$. Note that $\beta(0) = \mathbf{1}$.

Lemma 5.6. *Let $Y = (Y^-, Y^0, Y^+) \in \mathfrak{v}$.*

(1) Let $X = (X^-, X^0, X^+) \in \mathfrak{v}$, then $d\Phi_t(Y)(X) \in \mathfrak{v}$ is given by

$$d\Phi_t(Y)(X) = \pi_{\mathfrak{v}} \circ \mathrm{Ad}(a_t)^{-1}(S_{Y,X})$$

where $S_{Y,X} \in \mathfrak{g}$ is the element

$$\mathrm{Ad}(y_0y^+)^{-1}\beta(Y^-)(X^-) + \mathrm{Ad}(y^+)^{-1}\beta(Y^0)(X^0) + \beta(Y^+)(X^+).$$

(2) There exists a linear map $L(Y) : \mathfrak{v} \to \mathfrak{g}$ such that

$$d\Phi_t(Y) = \mathrm{Ad}(a_t)^{-1}(\mathbf{1}_{\mathfrak{v}} + \mathrm{Ad}(a_t)\pi_{\mathfrak{v}}\,\mathrm{Ad}(a_t)^{-1}L(Y))$$

for all $t \leq 0$, and such that $\|L(Y)\| \to 0$ for $Y \to 0$.

Proof. We get for the differential of Φ:

$$d\Phi(Y^-, Y^0, Y^+)(X^-, X^0, X^+) = d\tau_{y^- y^0 y^+ a_t}(z_0) \circ \mathrm{Ad}(a_t)^{-1}(S_{Y,X})$$

with $S_{Y,X}$ as above. Using the identification of the tangent space with $\mathfrak{v}$ this is exactly the statement of item (1).

Defining $L(Y)$ by $L(Y)(X) = S_{Y,X} - X$ for $X \in \mathfrak{v}$, we obtain the expression in item (2). It is easily seen that $\|L(Y)\| \to 0$ for $Y \to 0$. □

Proof of Lemma 5.4. Finally, it follows from Lemma 5.5 that $\mathrm{Ad}(a_t)^{-1}\mathbf{1}_{\mathfrak{v}}$ dominates in the expression in item (2) above, for $Y \in \mathfrak{v}$ sufficiently small. Since $\mathfrak{n}_1$ is proper in $\mathfrak{n}_0$, it follows that

$$\gamma := \mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{n}_0}) - \mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{n}_1}) > 0$$

and with this we obtain Lemma 5.4. □

5.3. **Final remarks.** **1.** We did not address here the case where G is not reductive. One might expect for G and H algebraic and G general, that Z has VAI if and only if the nilradical of H is contained in the nilradical of G.

2. The following may be an alternative approach to Theorem 1.2 for algebraic groups G and H. To be more specific, assume G and $H < G$ to be complex algebraic groups and $Z = G/H$ to be unimodular and quasi-affine. Under these assumptions we expect that there is a rational G-module V, and an embedding $Z \to V$ such that the invariant measure μ_Z, via pull-back, defines a tempered distribution on V. Note that if Z is of reductive type, then there exists a V such that the image of $Z \to V$ is closed, and hence μ_Z defines a tempered distribution on V. If Z is not of reductive type, then by Matsushima's criterion ([5], Thm. 3.5) all images $Z \to V$ are non-closed and the expected embedding would imply that VAI does not hold. This is supported by a result in [16], which asserts that for a reductive group G and $X \in \mathfrak{g} := \mathrm{Lie}(G)$ the invariant measure on the adjoint orbit $Z := \mathrm{Ad}(G)(X) \subset \mathfrak{g}$ defines a tempered distribution on $\mathfrak{g}$. Various particular results in the theory of prehomogeneous vector spaces provide additional support (see [3]).

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