INVARIANT FUNCTIONALS ON THE SPEH REPRESENTATION

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Abstract. We study $\mathrm{Sp}_{2n}(\mathbb{R})$-invariant functionals on the spaces of smooth vectors in Speh representations of $\mathrm{GL}_{2n}(\mathbb{R})$.

For even n we give an expression for such a functional using an explicit realization of the space of smooth vectors in the Speh representation. Our construction, combined with the argument in [GOSS12], gives a purely local and explicit construction of Klyachko models for all unitary representations of $\mathrm{GL}_{2n}(\mathbb{R})$. Furthermore, we show that this functional is, up to a constant, the unique functional on the Speh representation which is invariant under the Siegel parabolic subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$.

For odd n we show that the Speh representation does not admit an invariant functional with respect to the subgroup $\mathrm{U}(n)$ of $\mathrm{Sp}_{2n}(\mathbb{R})$ consisting of unitary matrices.

1. Introduction

In recent years, there has been considerable interest in periods of automorphic forms in relation to the Langlands program and equidistribution problems ([SV, V10]). The study of periods admits a local counterpart, invariant linear functionals and with it the notion of distinction of a representation π of a reductive group G with respect to a subgroup $H \subset G$. We recall that the representation π is called **distinguished** with respect to a subgroup $H \subset G$ if $\mathrm{Hom}_H(\pi^\infty, \mathbb{C}) \neq 0$. In many interesting cases the pair (G, H) is a Gelfand pair and this allows one to connect the global period integral to local linear functionals. Motivated by the work of Jacquet-Rallis [JR92] and Heumos-Rallis [HR90], the third author together with O. Offen classified in [OS07, OS08a, OS08b, OS09] those unitary representations of $\mathrm{GL}_{2n}(F)$ that are distinguished with respect to the subgroup $\mathrm{Sp}_{2n}(F)$, in the case that F is a non-archimedean local field. The case of Archimedean fields was treated subsequently in [GOSS12, AOS12]. We remark that the pair $\mathrm{Sp}_{2n}(F) \subset \mathrm{GL}_{2n}(F)$ is a Gelfand pair (see [OS08b, AS12, Say]).

The classification of distinguished unitary representations involves the family of unitary representations of $\mathrm{GL}_n(\mathbb{R})$ discovered by B. Speh. We remind that these unitary representations, and their generalizations to $\mathrm{GL}_n(F)$, where F is a local field, play a central role in the classification scheme of the unitary dual of the general linear group over the local field F. Indeed any irreducible representation of $\mathrm{GL}_n(F)$ is a Bernstein-Zelevinski product, in a unique way, of generalized Speh representations and their complementary series counterparts (see [Tad86, Vog86]).

For a discrete series representation σ of $\mathrm{GL}_r(F)$ we denote by $U(\sigma, n)$ the corresponding generalized Speh representation of $\mathrm{GL}_{nr}(F)$. For $|\alpha| < \frac{1}{2}$ we denote by $\pi(\sigma, n, \alpha) = U(\sigma, n)|\cdot|^\alpha \times U(\sigma, n)|\cdot|^{-\alpha}$ the complementary series, which is a unitary representation of $\mathrm{GL}_{2nr}(F)$. Recall that for archimedean F we have $r \leq 2$, and if $F = \mathbb{C}$ then $r = 1$. If $r = 1$ then $U(\sigma, n)$ is a character of $\mathrm{GL}_n(F)$, and $\pi(\sigma, n, \alpha)$ is a Stein complementary series representation of $\mathrm{GL}_{2n}(F)$. We denote by D_m the discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ and by δ_m the corresponding Speh representations of $\mathrm{GL}_{2n}(\mathbb{R})$.

The answer to the distinction is summarized in the next theorem, which in the archimedean case is a combination of [GOSS12, Theorem A] and [AOS12, Theorem 1.1].

Theorem. *Let π be an irreducible unitary representation of $\mathrm{GL}_{2n}(F)$. Write $\pi = \times_{i=1}^k U(\sigma_i, n_i) \times \times_{j=1}^l \pi(\sigma'_j, m_j, \alpha_j)$ with*

- *$\sigma_1, ..., \sigma_k$ discrete series representations of $\mathrm{GL}_{p_1}(F), ..., \mathrm{GL}_{p_k}(F)$ respectively*
- *$\sigma'_1, ..., \sigma'_l$ discrete series representations of $\mathrm{GL}_{q_1}(F), ..., \mathrm{GL}_{q_l}(F)$ respectively*
- *$\alpha_1, ..., \alpha_l$ real numbers in $(-\frac{1}{2}, \frac{1}{2})$.*

Date: May 29, 2014.
2010 *Mathematics Subject Classification.* 20G05,20G20,22E45,46T30.

and $2n = \sum_{i=1}^{k} n_i p_i + \sum_{j=1}^{l} 2m_j q_j$.

Then π *is* $Sp_{2n}(F)$*-distinguished if and only if all* n_i *and all* m_j *are even.*

One of the key steps in the proof that the generalized Speh representations $U(\sigma, n)$ with even n are distinguished by the symplectic group. The proof of this result in [OS07] and [GOSS12] is based on a global argument involving periods of residues of automorphic Eisenstein series.

In [SaSt90] Speh representations δ_m of $\mathrm{GL}_{2n}(\mathbb{R})$ have been constructed explicitly as a natural Hilbert space of distributions on matrix space. The paper [SaSt90] also describes and uses a construction of the Speh representation as a quotient of a degenerate principal series representation induced from a character of the (n, n) standard parabolic subgroup (see §2.2 below).

In the present paper we use the explicit constructions of [SaSt90] and give a direct proof that the spaces of $\mathrm{Sp}_{2n}(\mathbb{R})$-invariant functionals on the Speh representations of $\mathrm{GL}_{2n}(\mathbb{R})$ are zero if n is odd and one-dimensional if n is even. We also analyze functionals invariant with respect to subgroups of $\mathrm{Sp}_{2n}(\mathbb{R})$.

To describe our result we need some further notation. Let $G := G_{2n}$ denote the group $\mathrm{GL}_{2n}(\mathbb{R})$. Let ω_{2n} be the standard symplectic form on $\mathbb{R}^{2n}$. More explicitly ω_{2n} is given by $\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$ and let $H := H_{2n} = Sp_{2n}(\mathbb{R}) < G_{2n}$ denote the stabilizer of this form. Let

$$P := \left\{ \begin{pmatrix} g & X \\ 0 & g^{-t} \end{pmatrix} \mid g \in \mathrm{GL}_n(\mathbb{R}), X \in \mathrm{Mat}_{n\times n}(\mathbb{R}), X = X^t \right\} < H$$

denote the Siegel parabolic subgroup. Let $\mathrm{U}(n) < H_{2n} < G_{2n}$ be the unitary group.

In this paper we prove the following result.

Theorem A. *(i) If* n *is even then*

$$\mathrm{Hom}_H(\delta_m^\infty, \mathbb{C}) = \mathrm{Hom}_P(\delta_m^\infty, \mathbb{C}) \simeq \mathbb{C}$$

(ii) If n *is odd then*

$$\mathrm{Hom}_H(\delta_m^\infty, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(n)}(\delta_m^\infty, \mathbb{C}) = 0.$$

It is known that the restriction of δ_m to $\mathrm{SL}_{2n}(\mathbb{R})$ decomposes as a direct sum of two irreducible components. It follows from Theorem A that exactly one of them admits an H-invariant functional. In Lemma 4.2 we determine which one does.

It is easy to see that if n is odd and m is even then there are no functionals on δ_m^∞ invariant with respect to $-\mathrm{Id} \in H$, and thus neither P-invariant nor U_n -invariant functionals exist (see Remark 6.1).

Remark. *Although the pair* (G_{2n}, P) *is* **not** *a Gelfand pair for simple geometric reasons, we show that the Speh representation* δ_m *still admits at most one* P*-invariant functional (at least for even* n*). The reason we suspected this result to hold is that, as shown in* [SaSt90]*, Speh representations stay irreducible when restricted to a standard maximal parabolic subgroup* $Q \subset G$ *satisfying* $Q \cap H = P$*. It is possible that* (Q, P) *is a generalized Gelfand pair, i.e. the space of* P*-invariant functionals on the space of smooth vectors of any irreducible unitary representation of* Q *is at most one dimensional. However, this statement does not imply our uniqueness result, since the space of* G*-smooth vectors of* δ_m *could a priori have more functionals.*

1.1. **Klyachko models.** For any n, any even $k \leq n$ and any field F, [Kly84] defines a subgroup Kl_k of $\mathrm{GL}_n(F)$ and a generic character ψ_k of Kl_k. In particular, $Kl_n = \mathrm{Sp}_n(F)$ (if n is even) and Kl_0 is the group of upper unitriangular matrices. For local fields F, it is shown in [HR90, OS07, OS08a, OS08b, OS09, GOSS12, AOS12] that for any irreducible unitary representation π of $GL_n(F)$ there exists a non-zero (Kl_k, ψ_k)-equivariant functional on π^∞ for exactly one k. The uniqueness of such functional is known only over non-archimedean fields (see [OS08b]).

The proof of existence of k for $F = \mathbb{R}$, given in [GOSS12], is done by reduction to the statement that certain representations are H-distinguished. This case is reduced, using the Vogan classification of the unitary dual, to the proof of existence of H-invariant functionals on Speh representations (for even n). This existence is proved using a global (adelic) argument. In this paper we give an explicit local construction of such a functional. Together with [GOSS12] this gives a proof of existence of Klyachko models which uses only the representation theory of $\mathrm{GL}_n(\mathbb{R})$ (and the theory of distributions).

1.2. **Structure of the proof.** We use the realization of δ_m^∞ as the image of a certain intertwining differential operator $\square^m : \pi_{-m} \to \pi_m$, where π_{-m} and π_m are certain degenerate principal series induced from characters of a fixed (n,n)-parabolic subgroup $\overline{Q} \subset G$ (see §2.2).

The study of the even case is divided into two parts. In §3 we first use the realization of δ_m as a quotient of the degenerate principal series π_{-m} to lift a linear P-invariant functional on δ_m to an equivariant distribution on G. More precisely, we study $P \times \overline{Q}$ equivariant distributions on G. The technical heart is Corollary 3.3, which shows that such distributions do not vanish on the open cell $N\overline{Q}$. This is based on the techniques of [AGS08], classical invariant theory and a careful analysis of the double cosets $P \setminus G/\overline{Q}$, which is postponed to §5. Then we analyze the distributions supported on the open cell by identifying them with the space of distributions on N with a certain equivariance property. Identifying N with its Lie algebra and using the Fourier transform we show that this space is at most one dimensional for even n. This finishes the proof of Proposition 3.1 which states that there exists at most one invariant P-invariant functional in the n even case.

In the second part (§4) we construct an H-invariant functional as an $H \times \overline{Q}$-equivariant distribution on G. For that we fix an explicit $H \times \overline{Q}$-equivariant polynomial p, consider the meromorphic family of distributions $|p|^\lambda$ (cf. [Ber72]) and take the principal part of this family at $\lambda = (n-m)/2$. This distribution defines an H-invariant functional on π_m^∞. To show that the restriction of this functional to δ_m^∞ is non-zero (Lemma 4.1) we use Corollary 3.3 along with another lemma from §3 on non-existence of equivariant distributions with certain support. The uniqueness of P-invariant functionals and the existence of H-invariant ones imply that the two spaces are equal. Our proof shows that the spaces of such functionals are equal and one-dimensional also for the (reducible) representations π_m and π_{-m}.

For odd n we prove that already a $\mathrm{U}(n)$-invariant functional does not exist (Corollary 6.4). We do that by analyzing the $\mathrm{O}(2n)$-types of δ_m described in [HL99, Sah95] and showing that none of those have a $\mathrm{U}(n)$-invariant vector.

To summarize, Theorem A follows from Proposition 3.1 on uniqueness of P-invariant functionals for even n, Lemma 4.1 on existence of H-invariant functionals for even n and Corollary 6.4 on non-existence of $\mathrm{U}(n)$-invariant functionals for odd n.

1.3. **Acknowledgements.** The authors thank the Hausdorff Institute in Bonn for perfect working conditions during the summer of 2007 where the initial collaboration on this project started. They further thank Avraham Aizenbud, Joseph Bernstein and Omer Offen for fruitful discussions on the subject matter of this paper.

D.G. was partially supported by ISF grant 756/12.

E.S. was partially supported by ISF grant 1138/10.

2. Preliminaries

2.1. **Notation.** Recall the notation $G = G_{2n} = \mathrm{GL}_{2n}(\mathbb{R})$, and $H = H_{2n} = \mathrm{Sp}_{2n}(\mathbb{R}) \subset G$. Let

$$Q := \left\{ \begin{pmatrix} a & c \\ 0 & d \end{pmatrix} \in G \right\} \quad \overline{Q} := \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \in G \right\} \quad N := \left\{ \begin{pmatrix} \mathrm{Id}_n & c \\ 0 & \mathrm{Id}_n \end{pmatrix} \in G \right\}.$$

Recall that P denotes $Q \cap H$ and let

$$M := \left\{ \begin{pmatrix} g & 0 \\ 0 & g^{-t} \end{pmatrix} \right\} \quad \text{and} \quad U := \left\{ \begin{pmatrix} \mathrm{Id}_n & B \\ 0 & \mathrm{Id}_n \end{pmatrix} \mid B = B^t \right\}$$

denote the Levi subgroup and the unipotent radical of P.

For $g \in \mathrm{Mat}_{i\times i}(\mathbb{R})$ we denote $|g| := |\det(g)|$ and $\mathrm{sgn}(g) := \mathrm{sign}(\det(g))$.

For $q = \begin{pmatrix} A & 0 \\ B & D \end{pmatrix} \in \overline{Q}$ we denote $\gamma(q) := |A||D|^{-1}$ and $\varepsilon(q) := \mathrm{sgn}(D)$.

For any integer m let L_m denote the character of $\overline{Q}$ given by $L_m := \varepsilon^{m+1}\gamma^{-(n+m)/2}$. Let π_m denote the (unnormalized) induced representation $\mathrm{Ind}_{\overline{Q}}^{G}(L_m)$. Considering N as an open subset of $G/\overline{Q}$, one can restrict smooth vectors of π_m to N. This restriction is an embedding since N is an open subset of $G/\overline{Q}$. We sometimes identify N and its Lie algebra $\mathfrak{n}$ with $\mathrm{Mat}_{n\times n}(\mathbb{R})$ in the obvious way. This enables us to define the Fourier transform on $\mathfrak{n}$. Denote by M_n^+ (respectively M_n^-) the subset of $Mat_{n\times n}$ consisting of matrices with nonnegative (resp. nonpositive) determinant. For $f \in \pi_m^\infty$ we denote its restriction to $\mathfrak{n}$ by $f|_{\mathfrak{n}}$. We denote the space of all smooth functions obtained in this way by $\pi_m^\infty|_{\mathfrak{n}}$.

2.2. **Sahi-Stein realization of the Speh representations.** For any $m \in \mathbb{Z}_{\geq 0}$ define

$$\widehat{H}_m := \{f \in \mathcal{S}^*(\mathfrak{n}) \,|\, \widehat{f} \in L^2(\mathfrak{n}, |x|^{-m}dx)\} \text{ and } \widehat{H}_m^\pm := \{f \in \widehat{H}_m \,|\, \mathrm{Supp}\widehat{f} \subset M_n^\pm\},$$

where $\mathcal{S}^*(\mathfrak{n})$ denotes the space of tempered distributions $\mathfrak{n}$. The $\widehat{H}_m$ and $\widehat{H}_m^\pm$ are Hilbert spaces with the scalar product

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathfrak{n},|x|^{-m}dx)}.$$

Define an action of Q on $\widehat{H}_m$ by

$$\delta_m(q)f(x) := L_m(q)f(a^{-1}(c + xd)), \text{ for } q = \begin{pmatrix} a & c \\ 0 & d \end{pmatrix},$$

or equivalently on the Fourier transform side by

$$\widehat{\delta_m(q)f}(\xi) = \exp(2\pi i\, \mathrm{Tr}(cd^{-1}\xi))L_{-m}^{-1}(q)\widehat{f}(d^{-1}\xi a).$$

Summarizing the main results of [SaSt90] we obtain

Theorem 2.1 ([SaSt90])**.** *Let $m \in \mathbb{Z}_{\geq 0}$. Then*

(i) The action of Q extends to a unitary representation δ_m of G on $\widehat{H}_m$.
(ii) $(G, \delta_m, \widehat{H}_m)$ is isomorphic to the Speh representation of G.
(iii) There exists an epimorphism $\pi_{-m} \to \delta_m$ and an embedding $\delta_m \subset \pi_m$. The latter is defined on the smooth vectors by the inclusion $\delta_m^\infty \subset \pi_m^\infty|_{\mathfrak{n}}$.
(iv) The restriction of δ_m to $\mathrm{SL}(2n, \mathbb{R})$ is a direct sum of two irreducible representations $\delta_m^\pm$, realized on the subspaces $\widehat{H}_m^\pm$.

Consider the determinant as a polynomial on $\mathfrak{n}$ and let $\Box$ denote the corresponding differential operator.

Theorem 2.2. *The operator $\Box^m$ defines a continuous G-equivariant map $\pi_{-m}^\infty \to \pi_m^\infty$ with image δ_m^∞.*

Proof. By [KV77, Proposition 2.3] (see also [Boe85]), the operator $\Box^m$ defines a continuous G-equivariant map $\pi_{-m}^\infty \to \pi_m^\infty$, which is non-zero by [SaSt90]. By [HL99, Theorems 3.4.2-3.4.4] π_{-m} has unique composition series in the strong sense, meaning that any quotient of π_{-m} has a unique irreducible subrepresentation, and all these irreducible subquotients are pairwise non-isomorphic. It is easy to see that π_m is dual to π_{-m} and thus their composition series are opposite. Hence the image of any nonzero intertwining operator from π_{-m} to π_m is the unique irreducible subrepresentation of π_m. Since δ_m^∞ is an irreducible subrepresentation of π_m, it is the image of $\Box^m$. $\Box$

Remark 2.3. *One can deduce Theorem 2.2 also from* [KS93]*, which computes the action of $\Box^m$ on every K-type, where $K = \mathrm{O}(2n, \mathbb{R})$. From the formula in* [KS93] *and the description of the K-types of the composition series of π_{-m} in* [HL99, Sah95] *one can see that $\Box^m$ does not vanish precisely on the K-types of δ_m.*

2.3. **Invariant distributions.**

Definition 2.4. *For an affine algebraic manifold M we denote by $\mathcal{S}(M)$ the space of Schwartz functions on M, that is smooth functions f such that df is bounded for any differential operator d on M with polynomial coefficients. We endow this space with a Fréchet topology using the sequence of seminorms $\mathcal{N}_d(f) := \sup_{x\in M} |df(x)|$, where d is a differential operator on M with polynomial coefficients. Also, for an algebraic vector bundle E over M we denote by $\mathcal{S}(M, E)$ the space of Schwartz sections of E. We denote by $\mathcal{S}^*(M, E)$ the space of continuous linear functionals on $\mathcal{S}(M, E)$ and call its elements tempered distributional sections. For a closed subvariety $Z \subset M$ we denote by $\mathcal{S}_M^*(Z, E) \subset \mathcal{S}^*(M, E)$ the subspace of tempered distributional sections supported in Z. For the theory of Schwartz functions and distributions on general semi-algebraic manifolds we refer the reader to* [AG08].

Theorem 2.5 ([AGS08, §B.2]). *Let a Nash group K act on a real algebraic manifold M. Let $Z \subset M$ be a Nash closed subset. Let $Z = \bigcup_{i=1}^{l} Z_i$ be a Nash K-invariant stratification of Z. Let χ be a character of K. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $1 \leq i \leq l$,*

$$\mathcal{S}^*(Z_i, \mathrm{Sym}^k(CN_{Z_i}^M))^{K,\chi} = 0.$$

Then $\mathcal{S}_M^*(Z)^{K,\chi} = 0$.

Theorem 2.6 (Frobenius descent, see [AG09, Appendix B]). *Let a Nash group K act on a Nash manifold X. Let Z be a Nash manifold with a transitive action of K. Let $\phi : M \to Z$ be a K-equivariant map. Let $z \in Z$ be a point and $M_z := \phi^{-1}(z)$ be its fiber. Let K_z be the stabilizer of z in K. Let Δ_K and Δ_{K_z} be the modular characters of K and K_z. Let E be a K-equivariant Nash vector bundle over M.*

Then there exists a canonical isomorphism

$$\mathrm{Fr} : (\mathcal{S}^*(M_z, E|_{M_z}) \otimes \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1})^{K_z} \cong \mathcal{S}^*(M, E)^K.$$

From those two theorems we obtain

Corollary 2.7. *Let a Nash group K act on a real algebraic manifold M. Let $Z \subset M$ be a Nash closed subset. Suppose that Z has a finite number of orbits: $Z = \bigcup_{i=1}^{l} Kz_i$. Let χ be a character of K. Suppose that for any and $1 \leq i \leq l$,*

$$\mathrm{Sym}^*(N_{Kz_i,z_i}^M))^{K_z,\chi \cdot \Delta_K|_{K_z} \cdot \Delta_{K_z}^{-1}} = 0,$$

where Sym^* denotes the symmetric algebra. Then $\mathcal{S}_M^*(Z)^{K,\chi} = 0$.

Lemma 2.8. *Let G be a real algebraic group, and R be a (closed) algebraic subgroup. Consider the right action of R on G and suppose that G/R is compact. Let ξ be a character of R. Then we have a natural isomorphism of left G - representations*

$$(C^\infty(G, \xi)^R)^* \cong \mathcal{S}^*(G, \xi\Delta_R^{-1})^R \cong \mathcal{S}^*(G)^{(R,\xi^{-1}\Delta_R)}$$

Proof. Let $\mathfrak{Ind}(\xi)$ be the bundle on G/R corresponding to ξ. Consider the surjective submersion $\pi : G \to G/R$. It defines an isomorphism $C^\infty(G, \xi)^R \cong C^\infty(G/R, \mathfrak{Ind}(\xi))$.

Since G/R is compact, we have $C^\infty(G/R, \mathfrak{Ind}(\xi))^* \cong \mathcal{S}^*(G/R, \mathfrak{Ind}(\xi))$. Consider the diagonal action of G on $G \times G/R$ and the projections p_1, p_2 of $G \times G/R$ on both coordinates. From Theorem 2.6 we obtain

$$\mathcal{S}^*(G/R, \mathfrak{Ind}(\xi)) \cong \mathcal{S}^*(G \times G/R, p_1^*(\xi))^G \cong \mathcal{S}^*(G, \xi\Delta_R^{-1})^R$$

The isomorphism $\mathcal{S}^*(G, \xi\Delta_R^{-1})^R \cong \mathcal{S}^*(G)^{(R,\xi^{-1}\Delta_R)}$ is straightforward. □

3. Uniqueness of P-invariant functionals

In this section we assume that n is even. The goal of this section is to prove the following proposition.

Proposition 3.1. *For any integer m we have*

$$\dim((\pi_m^\infty)^*)^P \leq 1.$$

Since $\Delta_{\overline{Q}} = \gamma^{-n}$, we obtain from the definition of π_m and Lemma 2.8

$$(\pi_m^\infty)^* \cong \mathcal{S}^*(G)^{\overline{Q},\varepsilon^{m+1}\gamma^{(n-m)/2}} \tag{1}$$

and thus in order to prove Proposition 3.1 we have to show that for even n

$$\dim \mathcal{S}^*(G)^{P\times\overline{Q},1\times\varepsilon^{m+1}\gamma^{(n-m)/2}} \leq 1.$$

We will need the following proposition, which we will prove in section 5.

Proposition 3.2. *Denote $K := P \times \overline{Q}$, and let $x \notin N\overline{Q}$. Then*

$$\mathrm{Sym}^*(N_{Px\overline{Q},x}^G))^{K_x,\varepsilon^{m+1}\gamma^{(n-m)/2}\cdot\Delta_K|_{K_x}\Delta_{K_x}^{-1}} = 0.$$

From this proposition and Corollary 2.7 we obtain

Corollary 3.3.

$$\mathcal{S}^*_G(G - N\overline{Q})^{P\times\overline{Q},1\times\varepsilon^{m+1}\gamma^{(n-m)/2}} = 0.$$

By this corollary it is enough to analyze $\mathcal{S}^*(N\overline{Q})^{P\times\overline{Q},1\times\varepsilon^{m+1}\gamma^{(n-m)/2}}$. Let S denote the space of symmetric $n\times n$ matrices, and A denote the space of anti-symmetric $n\times n$ matrices. Identify $M \cong \mathrm{GL}_n(\mathbb{R})$ and let it act on S and on A by $x \mapsto gxg^t$.

Lemma 3.4.

$$\mathcal{S}^*(N\overline{Q})^{P\times\overline{Q},1\times\varepsilon^{m+1}\gamma^{(n-m)/2}} \cong \mathcal{S}^*(A)^{\mathrm{GL}_n(\mathbb{R}),\det^{1-m}} \cong \mathcal{S}^*(A)^{\mathrm{GL}_n(\mathbb{R}),\mathrm{sgn}^{m+1}|\cdot|^{m-n}}$$

Proof. Identify $U \cong S$ and let it act on itself by translations. Then $N\overline{Q}$ is isomorphic as a $P \times \overline{Q}$-space to $A \times S \times \overline{Q}$, where $\overline{Q}$ acts on the third coordinate (by right translations), U acts on the second coordinate and M acts on the first and the second coordinates. Note that the action on $S \times \overline{Q}$ is transitive and that $\Delta_{\overline{Q}} = \gamma^{-n}$ and $\Delta_P \begin{pmatrix} g & 0 \\ 0 & g^t \end{pmatrix} = |g|^{n+1}$. The first isomorphism follows now from Frobenius descent.

The second isomorphism is given by Fourier transform on A defined using the trace form. □

Let $O \subset A$ denote the open dense subset of non-degenerate matrices and Z denote its complement. The following lemma is a straightforward computation.

Lemma 3.5.

(i) Every orbit in Z includes an element of the form $x = \begin{pmatrix} 0_{k\times k} & 0 \\ 0 & \omega_{n-k} \end{pmatrix}$.

(ii) $N^A_{\mathrm{GL}_n(\mathbb{R})x,x} \cong \left\{ \begin{pmatrix} 0_{k\times k} & b \\ 0 & 0 \end{pmatrix} \right\}$ *and* $\mathrm{GL}_n(\mathbb{R})_x = \left\{ \begin{pmatrix} a_{k\times k} & 0 \\ c & d \end{pmatrix} \text{ such that } d \text{ is symplectic} \right\}$

(iii) $\Delta_{GL_n(\mathbb{R})_x} = |\cdot|^{-(n-k)}$

Corollary 3.6.

$$\mathrm{Sym}^*(N^A_{\mathrm{GL}_n(\mathbb{R})x,x})^{GL_n(\mathbb{R})_x,\mathrm{sgn}^{m+1}|\cdot|^{m-n}\cdot\Delta^{-1}_{GL_n(\mathbb{R})_x}} = 0$$

Proof. From the previous lemma $\mathrm{sgn}^{m+1}|\cdot|^{m-n}\cdot\Delta^{-1}_{GL_n(\mathbb{R})_x} = \mathrm{sgn}^{k+1}\det^{m-k}$. If n is even then so is k and thus this is not an algebraic character of $GL_n(\mathbb{R})_x$ and thus there are no tensors that change under this character. □

Corollary 3.7.

$$\dim \mathcal{S}^*(A)^{\mathrm{GL}_n(\mathbb{R}),\mathrm{sgn}^{m+1}|\cdot|^{m-n}} \leq 1$$

Proof. By Corollary 3.6 and Corollary 2.7,

$$\mathcal{S}^*_A(Z)^{\mathrm{GL}_n(\mathbb{R}),\mathrm{sgn}^{m+1}|\cdot|^{m-n}} = 0. \tag{2}$$

Therefore, the restriction of equivariant distributions to O is an embedding. Now,

$$\dim \mathcal{S}^*(O)^{\mathrm{GL}_n(\mathbb{R}),\mathrm{sgn}^{m+1}|\cdot|^{m-n}} \leq 1$$

since O is a single orbit. □

Proposition 3.1 follows now from Corollary 3.7, Lemma 3.4, Corollary 3.3 and (1).

Remark 3.8. *For odd n Corollary 3.3 does not hold. For example, the smallest orbit does support an equivariant distribution.*

4. Construction of the H-invariant functional

Let n be even. In this section we construct an H-invariant functional ϕ on π_m^∞ for any $m \in \mathbb{Z}_{\geq 0}$ and show that its restriction to δ_m^∞ is non-zero. Define a polynomial p on $\mathrm{Mat}(2n \times 2n, \mathbb{R})$ by

$$p\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \det(D^t B - B^t D) \tag{3}$$

Note that p is non-negative, H-invariant on the left and changes under the right multiplication by $\overline{Q}$ by the character $|\cdot|\gamma^{-1}$. Consider the meromorphic family of distributions on $\mathrm{Mat}(2n \times 2n, \mathbb{R})$ given by

$$\xi_\lambda^m := p^\lambda |\cdot|^{-\lambda} \varepsilon^{m+1}. \tag{4}$$

This family is defined by [Ber72]. For $\mathrm{Re}\,\lambda > 0$, the restriction of this distribution to $G = \mathrm{GL}(2n, \mathbb{R})$ is a non-zero smooth function, and thus the restriction η_λ^m of the family is not an identical zero. Note that

$$\eta_\lambda^m \in \mathcal{S}^*(G)^{(H \times \overline{Q}, 1 \times \varepsilon^{m+1}\gamma^\lambda)}.$$

Let $\alpha \in \mathcal{S}^*(G)$ be the principal part of this family at $\lambda = \frac{n-m}{2}$. By (1) α defines a non-zero H-invariant functional ϕ on π_m^∞.

Lemma 4.1. $\phi|_{\delta_m^\infty} \neq 0$.

Proof. By Theorem 2.2 it is enough to show that $\Box^m \phi \neq 0$. By Corollary 3.3, $\alpha|_{N\overline{Q}} \neq 0$. It is enough to show that $(\Box^m \alpha)|_{N\overline{Q}} \neq 0$. As in §3, let $A \subset N$ denote the subspace of anti-symmetric matrices and $O \subset A$ the open subset of non-degenerate matrices. Note that $\alpha|_{N\overline{Q}} \neq 0$ is $P \times \overline{Q}$-equivariant and let $\beta \in \mathcal{S}^*(A)^{\mathrm{GL}_n(\mathbb{R}), \det^{1-m}}$ be the distribution on A corresponding to α by the Frobenius descent (see Lemma 3.4). Note that $\mathcal{F}(\Box^m \beta)$ is $\mathcal{F}(\beta)$ multiplied by a polynomial. Thus it is enough to show that $\mathcal{F}(\beta)$ has full support, i.e. $\mathcal{F}(\beta)|_O \neq 0$. This follows from the equivariance properties of $\mathcal{F}(\beta)$ by (2). $\Box$

This argument in fact proves slightly more.

Lemma 4.2. $\phi|_{(\delta_m^+)^\infty} \neq 0$.

Proof. If g is a Schwartz function on $M_n^+ \subset N$ then its Fourier transform $\widehat{g}$ determines a vector in $(\delta_m^+)^\infty$ by Theorem 2.1. Thus it is enough to find such a g for which $\zeta(\widehat{g}) \neq 0$, where ζ denotes the P-invariant distribution on N corresponding to α.

Let f be a compactly supported smooth function on O such that $\beta(\mathcal{F}(f)) \neq 0$. Since the determinant is positive on O, there exists a compact neighborhood Z of zero in the space S of symmetric n by n matrices such that $\mathrm{Supp}(f) + Z \subset M_n^+$. Let h be a smooth function on S which is supported on Z and s.t. $h(0) = 1$. Let $g := f \boxtimes h$ be the function on N defined by $g(X + Y) := f(X)h(Y)$ where $X \in A$ and $Y \in S$. Let $\mathcal{F}_S$ denote the Fourier transform on S. Then we have

$$\zeta(\widehat{g}) = \zeta(\mathcal{F}(f) \boxtimes \mathcal{F}_S(h)) = \beta(\mathcal{F}(f)) \neq 0.$$

$\Box$

Remark 4.3. *(i) For odd n, the polynomial p is identically zero, since the matrix $D^t B - B^t D$ is an anti-symmetric matrix of size n.*

(ii) The polynomial p defines the open orbit of H on $G/\overline{Q}$. In general, one can show that if a linear complex algebraic group G acts on a complex affine algebraic manifold M, both defined over $\mathbb{R}$, W is a basic open subset of M defined by a G-equivariant polynomial p with real coefficients, χ is a character of the group of real points $G(\mathbb{R})$ and there exists a non-zero $(G(\mathbb{R}), \chi)$-equivariant holonomic tempered distribution ξ on $W(\mathbb{R})$ then there exists a non-zero $(G(\mathbb{R}), \chi)$-equivariant holonomic tempered distribution on $M(\mathbb{R})$.

To prove that consider the analytic family of distributions $|p|^\lambda \xi$ on W. For $\mathrm{Re}\,\lambda$ big enough, it can be extended to a family η_λ on $M(\mathbb{R})$. By [Ber72] *the family η_λ has a meromorphic continuation to the entire complex plane. Note that the distributions in this family are equivariant with a character that depends analytically on λ. Thus taking the principal part at $\lambda = 0$ we obtain a non-zero $(G(\mathbb{R}), \chi)$-equivariant holonomic tempered distribution on $M(\mathbb{R})$.*

Note that since this construction involves taking principal part, the obtained distribution is not necessary an extension of the original ξ. This can already be seen in the case when M is the affine line, W is the complement to 0 and G is the multiplicative group.

If G has finitely many orbits on M then any $G(\mathbb{R})$-equivariant distribution on $M(\mathbb{R})$ is holonomic.

5. Proof of Proposition 3.2

We start from the description of the double cosets $P \setminus G/\overline{Q}$. Let r_1, r_2, s, t be non-negative integers such that $r_1 + r_2 + 2s + 2t = n$. We will view $2n \times 2n$ matrices as 10×10 block matrices in the following way. First of all, we view them as 2×2 block matrices with each block of size $n \times n$. Now, we divide each block to 5×5 blocks of sizes $r_1, r_2, s, s, 2t$ in correspondence. Denote by σ_{16} the permutation matrix that permutes blocks 1 and 6, by σ_{39} the permutation matrix that permutes blocks 3 and 9, and by $\tau_{5,10}$ the matrix which has $\begin{pmatrix} \mathrm{Id}_{2t} & \omega_{2t} \\ 0 & \mathrm{Id}_{2t} \end{pmatrix}$ in blocks 5 and 10 and is equal to the identity matrix in other blocks. Recall the notation $\omega_{2t} := \begin{pmatrix} 0 & \mathrm{Id}_t \\ -\mathrm{Id}_t & 0 \end{pmatrix}$. Denote

$$x_{r_1,r_2,s,t} := \sigma_{16}\sigma_{39}\tau_{5,10}.$$

Lemma 5.1. *Each double coset in $P \setminus \mathrm{GL}_{2n}(\mathbb{R})/\overline{Q}$ includes a unique element of the form $x_{r_1,r_2,s,t}$. The orbits in $N\overline{Q}$ correspond to $r_2 = s = 0$.*

Proof. Recall that $G/\overline{Q}$ is the Grassmannian of n-dimensional subspaces of $\mathbb{R}^{2n}$. Let $L := \mathrm{Span}\{e_{n+1}, \dots e_{2n}\} \subset \mathbb{R}^{2n}$ be the standard Lagrangian subspace. To an n-dimensional subspace $W \subset \mathbb{R}^{2n}$ we associate the following invariants:

$$r_2 := \dim L \cap W \cap W^{\perp},\ r_1 := \dim W^{\perp} \cap W - r_2,\ s := \dim L \cap W - r_2,\ t := (n - r_1 - r_2)/2 - s$$

Note that $n - r_1 - r_2$ is even since it is the rank of $\omega|_W$. Clearly, $W \in N\overline{Q}$ if and only if $r_2 = s = 0$.

Note the equality of vectors

$$(v_1, 0, v_2, 0, \omega_{2t}u \,|\, 0, w_2, w_1, 0, u)^t = x_{r_1,r_2,s,t}(0, 0, 0, 0, 0 \,|\, v_1, w_2, w_1, v_2, u)^t.$$

It is enough to show that W can be brought, using the action of P, to a space of vectors of the form $(v_1, 0, v_2, 0, \omega_{2t}u \,|\, 0, w_2, w_1, 0, u)^t$.

Clearly, W can be brought to a space of vectors of the form $(v, Aw + Bv \,|\, Cw, w, Dw)^t$, where $\mathrm{size}(v) + \mathrm{size}(w) = n$ and A is a square matrix. Let us write this in more detailed form, with the same block sizes in the first n coordinates and last n coordinates:

$$(v_1,\ v_2,\ A_{11}w_1 + A_{12}w_2 + B_{11}v_1 + B_{12}v_2,\ A_{21}w_1 + A_{22}w_2 + B_{21}v_1 + B_{22}v_2 \,|C_1w_1 + C_2w_2,\ w_1,\ w_2,\ D_1w_1 + D_2w_2)^t$$

Denote the first four blocks by e_i and the last by f_i. For any i and any $j \neq i$, M allows us to do the following operations:

$$(1)_i \quad e_i \mapsto ge_i, \quad f_i \mapsto g^{-t}f_i,$$

$$(2)_{ij} \quad e_i \mapsto e_i + ae_j, \quad f_i \mapsto f_j - A^t f_i.$$

Similarly, U allows us to do two more operations:

$$(3)_{ij} \quad e_i \mapsto e_i + bf_j, \quad e_j \mapsto e_j + b^t f_i$$

$$(4)_i \quad e_i \mapsto e_i + (c + c^t)f_i$$

Using $(2)_{31}$ and $(2)_{41}$, and redefining C and D we get $B = 0$. Using $(2)_{21}$ and $(2)_{21}$, and redefining A we get $C = 0$ and $D = 0$.

Using $(3)_{32}$ and $(3)_{42}$ and $(3)_{43}$ we get $A_{11} = A_{21} = A_{22} = 0$. Using $(3)_{33}$ we make A_{12} anti-symmetric. Now, using $(1)_3$ we can replace A_{12} by $gA_{12}g^t$ and thus we can bring it to the form $A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & \omega_{2t} \end{pmatrix}$. $\square$

Lemma 5.2. *Let $K := P \times \overline{Q}$ and $x := x_{r_1,r_2,s,t}$. Then*
(i) If $s > 0$ then

$$\mathrm{Sym}^*(N^G_{Px\overline{Q},x}))^{K_x,\varepsilon^{m+1}\gamma^{(n-m)/2}\cdot\Delta_K|_{K_x}\Delta^{-1}_{K_x}} = 0.$$

(ii) If $s = 0$ then

$$\mathrm{Sym}^*(N^G_{Px\overline{Q},x}))^{K_x,\varepsilon^{m+1}\gamma^{(n-m)/2}\cdot\Delta_K|_{K_x}\Delta^{-1}_{K_x}} \cong \mathrm{Sym}^*(gl_{r_1})^{GL_{r_1},|\cdot|^{-m-r_1}\,\mathrm{sgn}^{m+1}} \otimes \mathrm{Sym}^*(o_{r_2})^{GL_{r_2},det^{2t-m+1}}$$

where o_{r_2} denotes the space of antisymmetric matrices and GL_i act by $a \mapsto gag^t$.

For the proof of this lemma see §5.1.

Lemma 5.3. *Let $k, l \in \mathbb{Z}_{\geq 0},\ r \in \mathbb{Z}_{>0}$.*
(i) If $k \neq l \pmod 2$ then

$$\mathrm{Sym}^*(gl_r)^{GL_r,|\cdot|^k\,\mathrm{sgn}^l} = 0.$$

(ii) If $k > 0$ and r is odd then

$$\mathrm{Sym}^*(o_r)^{GL_r,det^k} = 0.$$

Proof.
(i)The only algebraic characters of GL_r are powers of the determinant.
(ii) The stabilizer in GL_r of every matrix in o_r has an element with determinant bigger than 1. □

Proof of Proposition 3.2. By Lemmas 5.1 and 5.2 it is enough to show that

$$\mathrm{Sym}^*(gl_{r_1})^{GL_{r_1},|det|^{-m-r_1}\,\mathrm{sign}(det)^{m+1}} \otimes \mathrm{Sym}^*(o_{r_2})^{GL_{r_2},det^{2t-m+1}} = 0 \tag{5}$$

Note that since n is even, r_1 and r_2 are of the same parity. If they are even then (5) follows from Lemma 5.3(i), and otherwise from Lemma 5.3(ii). □

5.1. **Proof of Lemma 5.2.** Let $x = x_{r_1,r_2,s,t}$ be as in the lemma. We need to compute the space $N^G_{x,Px\overline{Q}}$, the stabilizer K_x and its modular function. In order to do that we compute the conjugates of P and its Lie algebra $\mathfrak{p}$ under x.

Lemma 5.4. *Let $q = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in Q$. Then $x^{-1}qx = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where*

$$A = \begin{pmatrix} d_{11} & 0 & d_{14} & 0 & 0 \\ b_{21} & a_{22} & b_{24} & a_{24} & a_{25} \\ d_{41} & 0 & d_{44} & 0 & 0 \\ b_{41} & a_{42} & b_{44} & a_{44} & a_{45} \\ b_{51} - \omega d_{51} & a_{52} & b_{54} - \omega d_{54} & a_{54} & a_{55} \end{pmatrix} \quad B = \begin{pmatrix} 0 & d_{12} & d_{13} & 0 & d_{15} \\ a_{21} & b_{22} & b_{23} & a_{23} & b_{25} + a_{25}\omega \\ 0 & d_{42} & d_{43} & 0 & d_{45} \\ a_{41} & b_{42} & b_{43} & a_{43} & b_{45} + a_{45}\omega \\ a_{51} & b_{52} - \omega d_{52} & b_{53} - \omega d_{53} & a_{53} & b_{55} + a_{55}\omega - \omega d_{55} \end{pmatrix}$$

$$C = \begin{pmatrix} b_{11} & a_{12} & b_{14} & a_{14} & a_{15} \\ d_{21} & 0 & d_{24} & 0 & 0 \\ d_{31} & 0 & d_{34} & 0 & 0 \\ b_{31} & a_{32} & b_{34} & a_{34} & a_{35} \\ d_{51} & 0 & d_{54} & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} a_{11} & b_{12} & b_{13} & a_{13} & b_{15} + a_{15}\omega \\ 0 & d_{22} & d_{23} & 0 & d_{25} \\ 0 & d_{32} & d_{33} & 0 & d_{35} \\ a_{31} & b_{32} & b_{33} & a_{33} & b_{35} + a_{35}\omega \\ 0 & d_{52} & d_{53} & 0 & d_{55} \end{pmatrix}$$

This lemma is a straightforward computation, which can be done using a computer.
We can identify $T_xG \cong \mathfrak{gl}_{2n}$. Under this identification $T_xPx\overline{Q} \cong x^{-1}\mathfrak{p}x + \overline{\mathfrak{q}}$ and

$$N^G_{x,Px\overline{Q}} \cong \mathfrak{gl}_{2n}/(x^{-1}\mathfrak{p}x + \overline{\mathfrak{q}}) \cong \mathfrak{n}/(\mathfrak{n} \cap (x^{-1}\mathfrak{p}x + \overline{\mathfrak{q}})).$$

From the previous lemma we obtain

Corollary 5.5. *Let $V \subset \mathfrak{n}$ denote the subspace consisting of matrices of the form*

$$\begin{pmatrix} n_{11} & n_{12} & 0 & n_{14} & n_{15} \\ n_{12}^t & n_{22} & 0 & 0 & 0 \\ n_{31} & 0 & 0 & n_{34} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ n_{15}^t & 0 & 0 & 0 & 0 \end{pmatrix},$$

such that $n_{22} = -n_{22}^t$.

Then V projects isomorphically onto $\mathfrak{n}/(\mathfrak{n} \cap (x^{-1}\mathfrak{p}x + \overline{\mathfrak{q}}))$.

Now let us analyze the stabilizer K_x. From Lemma 5.4 we obtain

Corollary 5.6.

(i) Using the projection on the first coordinate

$$K_x \cong P \cap x\overline{Q}x^{-1} \cong \{\begin{pmatrix} A & B \\ 0 & A^{-t} \end{pmatrix} \in P \; s.t. A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{22} & 0 & 0 & 0 \\ 0 & A_{32} & A_{33} & 0 & 0 \\ 0 & A_{42} & 0 & A_{44} & 0 \\ 0 & A_{52} & 0 & 0 & A_{55} \end{pmatrix},$$

where A_{55} is symplectic and B is a symmetric matrix of the form $B = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ B_{12}^t & 0 & 0 & 0 & 0 \\ B_{13}^t & 0 & B_{33} & 0 & B_{35} \\ B_{14}^t & 0 & 0 & B_{44} & B_{45} \\ B_{15}^t & 0 & B_{35}^t & B_{45}^t & 0 \end{pmatrix}\}.$

(ii) The modular function of K_x is given by

$$\Delta_{K_x}(\begin{pmatrix} A & B \\ 0 & A^{-t} \end{pmatrix}) = |A_{11}|^{2n-r_1+1}|A_{22}|^{-n+r_1+r_2}|A_{33}|^{n-r_1-s+1}|A_{44}|^{n-r_1-s+1}.$$

(iii) Let $q = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \overline{Q} \cap x^{-1}Px$. Let $k = (xqx^{-1}, q) \in K_x$. Then k acts on V by

$$k \cdot n = pr_V(AnD^{-1}),$$

where $pr_V : \mathfrak{n} \to V$ denotes the projection.

Corollary 5.7. *Denote*

$$\chi := \varepsilon^{m+1}\gamma^{(n-m)/2} \cdot \Delta_K|_{K_x}\Delta_{K_x}^{-1}.$$

Let

$$q = \mathrm{diag}(a, b, c, c^{-t}, \mathrm{Id}, a^{-t}, b^{-t}, d, d^{-t}, \mathrm{Id}).$$

Let $k := (xqx^{-1}, q) \in K_x$. Then

$$\chi(k) = (\mathrm{sgn}(a)\,\mathrm{sgn}(b)\,\mathrm{sgn}(c)\,\mathrm{sgn}(d))^{m+1}|a|^{-m-r_1}|b|^{2s+2t-m+1}|c|^{-r_1-s}|d|^{-r_1-s}.$$

Proof.

$$\gamma(q) = |a|^2|b|^2 \quad \text{and} \quad \Delta_{\overline{Q}}(q) = |a|^{-2n}|b|^{-2n}$$

$$xqx^{-1} = \mathrm{diag}(a^{-t}, b, d^{-t}, c^{-t}, \mathrm{Id}, a, b^{-t}, d, c, \mathrm{Id})$$

$$\Delta_K(k) = |a|^{-3n-1}|b|^{-n+1}|c|^{-n-1}|d|^{-n-1}$$

$$\Delta_{K_x}(k) = |a|^{-2n+r_1-1}|b|^{-n+r_1+r_2}|c|^{-n+r_1+s-1}|d|^{-n+r_1+s-1}$$

□

Now we are ready to prove Lemma 5.2.

Proof of Lemma 5.2. If $s > 0$ then $\mathrm{Sym}^*(V)^{K_x,\chi} = 0$, since tensors cannot have negative homogeneity degrees. Otherwise, V involves only 3 blocks - the ones numbered $1, 2$ and 5.

Let $p \in \mathrm{Sym}^*(V)^{K_x,\chi}$. Identify K_x with a subgroup of $\overline{Q}$ using the second coordinate.

Consider the action of the block A_{21}. It can map any non-zero vector in the block n_{11} to any vector in the block n_{12}. This action does not change any element in any other block of V (it does effect n_{22}, but not its anti-symmetric part). Also, the character χ does not depend on A_{21}. Therefore p does not depend on the variables in the block n_{12}.

In the same way, using the action of A_{52}, we can show that p does not depend on the variables in the block n_{15}. Therefore, p depends only on n_{11} and n_{22}. Hence

$$\mathrm{Sym}^*(V)^{K_x,\chi} \cong \mathrm{Sym}^*(\mathfrak{gl}_{r_1})^{\mathrm{GL}_{r_1},|\cdot|^{-m-r_1}\mathrm{sgn}^{m+1}} \otimes \mathrm{Sym}^*(o_{r_2})^{\mathrm{GL}_{r_2},|\cdot|^{2t-m+1}\mathrm{sgn}^{m+1}}.$$

□

6. Non-existence of an H-invariant functional for odd n

In this section we prove that if n is odd then there are no $\mathrm{U}(n)$-invariant functionals on the Speh representation and therefore there are no H-invariant functionals. We do that using K-type analysis. The maximal compact subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$ is $K := \mathrm{O}(2n, \mathbb{R})$, and $\mathrm{U}(n) = K \cap H$ is a symmetric subgroup of K. We show that no K-type of δ_m has a $\mathrm{U}(n)$-invariant vector.

The root system of K is of type D_n, and we make the usual choice of positive roots

$$\{\varepsilon_i \pm \varepsilon_j : i < j\}$$

where ε_i is the i-th unit vector in $\mathbb{R}^n$. With this choice, the highest weights of K-modules are given by integer sequences $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ such that

$$\mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n \geq 0. \tag{6}$$

Remark 6.1. *From the definition of π_m we see that if n is odd and m is even then the central element $-\mathrm{Id} \in G$ acts by scalar -1, and there are neither P-invariant nor $\mathrm{U}(n)$ -invariant functionals on δ_m^∞.*

Since δ_m^∞ is the irreducible quotient of π_{-m}, the following theorem follows from [HL99, Theorems 3.4.2 - 3.4.4] (see also [Sah95]).

Theorem 6.2. *The K-types of $\pi_{\pm m}$ are given by sequences as in (6) with $\mu_i \equiv m+1 \pmod 2$, while the K-types of the Speh representation δ_m satisfy the additional condition $\mu_n \geq m+1$.*

Lemma 6.3. *If n is odd then no K-type $(\mu_1, \ldots, \mu_n)$ with $\mu_n \neq 0$ has $\mathrm{U}(n)$-invariant vectors.*

Proof. Let ρ be an irreducible representation of K with $\mu_n \neq 0$. Suppose that ρ has a non-zero $\mathrm{U}(n)$-invariant vector. Then $\rho = \rho_1 \oplus \rho_2$, where ρ_i are irreducible non-zero representations of $K^0 = \mathrm{SO}(2n, \mathbb{R})$. The pair $(K, \mathrm{U}(n))$ is a symmetric pair of compact groups and therefore a Gelfand pair. Hence the $\mathrm{U}(n)$-invariant vector is unique up to a scalar and belongs to one of the ρ_i. Denote it by v and say $v \in \rho_1$.

Consider $g := \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \in K$. Since n is odd, $g \notin K^0$. Hence $\rho(g)v \notin \rho_1$, since otherwise ρ would be reducible. However, g normalizes $\mathrm{U}(n)$ and hence $\rho(g)v$ is $\mathrm{U}(n)$-invariant and therefore proportional to v. Contradiction. □

Corollary 6.4. *If n is odd then there are no $\mathrm{U}(n)$-invariant functionals on δ_m^∞.*

Proof. By Remark 6.1 we can assume that m is odd. Then by Lemma 6.3 and Theorem 6.2, no K-type of δ_m has a $\mathrm{U}(n)$-invariant vector. Therefore, the space of K-finite vectors, which decomposes to a direct sum of K-types, does not have a $\mathrm{U}(n)$-invariant functional. This space is dense in δ_m^∞, hence there are no $\mathrm{U}(n)$-invariant functionals on δ_m^∞ either. □

Remark 6.5. *Using the Cartan-Helgason theorem and the table in* [Kna85, Appendix C, §2], *it can be shown that the K-types that have U_n-invariant vectors are of the form $\mu_{2i-1} = \mu_{2i}$ for $1 \leq i \leq n/2$ and if n is odd then $\mu_n = 0$, which gives an alternative proof of Lemma 6.3.*

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