PERGAMON

Nonlinear Analysis 42 (2000) 977–993

Nonlinear Analysis

www.elsevier.nl/locate/na

Bean's critical-state model as the $p \to \infty$ limit of an evolutionary p-Laplacian equation

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Received 17 December 1997; accepted 28 September 1998

Keywords: Superconductivity; Bean model; Variational inequalities; p-Laplacian

1. Introduction

The Bean critical-state model provides a description for the magnetization of type-II superconductors in a nonstationary external magnetic field. The model was first formulated for the simplest configuration of a cylindrical superconductor in a parallel field, see [2,8]. Since then more complicated cases have also been considered; in particular a very thin superconducting film in a perpendicular external field, see [3,18,20] and the references therein. Phenomenologically, the problem can be understood as a nonlinear eddy current problem. In accordance with the Faraday law of electromagnetic induction, the eddy currents in a conductor are driven by the electric fields induced by time variations of the magnetic flux. In an ordinary conductor, the vectors of the electric field and the current density are usually related by the linear Ohm law. Type-II superconductors are instead characterized in the Bean model by a highly nonlinear current–voltage relation. This nonlinearity gives rise to an interesting free boundary problem, which is considered here for the two specific geometrical configurations mentioned above: a long cylinder in a parallel magnetic field and a thin film in a perpendicular field.

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PII: S0362-546X(99)00147-9

In these cases the electric field, **e**, inside the isotropic superconductor has the same direction as the current density, **j**, and the superconducting material may be characterized by a scalar "current–voltage law". This nonlinear constitutive relation is given in the Bean model by a multivalued monotone graph:

$$|\mathbf{e}| \in \begin{cases} 0 & \text{if } |\mathbf{j}| < 1, \\ [0,\infty) & \text{if } |\mathbf{j}| = 1, \\ \emptyset & \text{if } |\mathbf{j}| > 1. \end{cases} \tag{1.1}$$

(Here we have adopted units in which the critical current density $j_c = 1$.) The magnetization model with this current–voltage law is equivalent to an evolutionary variational inequality, see [15], and such a formulation is convenient for both the numerical approximation and theoretical study of these magnetization problems [15–18]. In simple cases the solution to the Bean model can be found analytically; see, e.g., the two examples in the next section.

Physicists, however, usually approximate (1.1) by a smooth function in order to "simplify" the numerical discretization or to account for the thermally activated creep of the magnetic flux, see [3–5,19,20]. The power-law approximation

$$|\mathbf{e}| = |\mathbf{j}|^{p-1} \tag{1.2}$$

for a fixed large $p \in \mathbb{R}^+$, is the most often adopted. This approximation leads to evolutionary equations involving the p-Laplacian operator and it was assumed in the physical literature that their solutions converge to the Bean model solution as $p \to \infty$. In this paper, we study the behaviour of the solutions to these evolutionary equations for two geometrical configurations and prove rigorously that the convergence does indeed take place in each case to the unique solution of the corresponding evolutionary variational inequality, equivalent to the Bean critical-state model for that configuration.

For long cylinders in a parallel field, the variational inequality problem can be written in terms of the magnetic field and involves a gradient constraint. This problem is similar to that arising in another critical-state model: the sandpile growth model, see [13,14]. Recently, Aronsson et al. [1] have shown that the sandpile growth model can be obtained as the $p \to \infty$ limit of the Cauchy problem for an evolutionary p-Laplacian equation. We partially adopt their techniques in our consideration of the corresponding limits of the similar boundary value problems in superconductivity. It should be noted that the similarity between the magnetization of type-II superconductors and the growth of sandpiles is well known, see [5,8].

In the case of a thin superconducting film placed into a perpendicular magnetic field, a variational inequality with the same gradient constraint as in the cylindrical case can be derived for the stream function of a divergence-free two-dimensional (2D) sheet current density. This evolutionary variational inequality is "implicit" with respect to the time derivative and we prove that it is the $p \to \infty$ limit of an "implicit" evolutionary equation involving the p-Laplacian operator.

In the next section we derive variational formulations for the power law and Bean magnetization problems for these two specific geometrical configurations. Although the two configurations lead to different mathematical problems, these can be regarded as two special cases of a more general evolutionary problem involving the p-Laplacian.

Therefore, in Section 3 we analyse the well-posedness of this more general problem and study its limit as $p \to \infty$.

2. Variational formulation of the models

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain with a Lipschitz boundary $\partial\Omega$. If Ω is not simply connected, we allow it to have a finite number of "holes" Ω_i, $i = 1 \to I$, with Ω_i being a bounded domain with a connected boundary $\partial\Omega_i$. We set

$$\Omega^* := \Omega \cup \left(\bigcup_{i=1 \to I} \overline{\Omega}_i \right).$$

We define, for any $p \in [1, \infty]$, the function space

$$V_p := \{ v \in W_0^{1,p}(\Omega^*) \colon |\nabla v| = 0 \text{ a.e. in } \Omega_i,\ i = 1 \to I \}$$

and introduce the closed convex set

$$K := \{ v \in V_2 \colon |\nabla v| \leq 1 \text{ a.e. in } \Omega \},$$

where we have adopted the standard notation for Sobolev spaces with "a.e." denoting almost everywhere. Throughout $(\cdot,\cdot)_D$ denotes the standard $L^2(D)$ inner product for scalar and vector functions. For notational convenience, we drop the subscript in the case $D \equiv \Omega^*$. For later use, we note the following result:

Let X, Y and Z be Banach spaces with a compact embedding $X \hookrightarrow Y$ and a continuous embedding $Y \hookrightarrow Z$. Then the embeddings

$$\{ v \in L^2(0,T;X) \colon \partial_t v \in L^2(0,T;Z) \} \hookrightarrow L^2(0,T;Y) \tag{2.1a}$$

and

$$\{ v \in L^\infty(0,T;X) \colon \partial_t v \in L^2(0,T;Z) \} \hookrightarrow C([0,T];Y) \tag{2.1b}$$

are compact, see [21]. Here and throughout $\partial_t v \equiv \partial v/\partial t$ and $\partial_{x_i} v \equiv \partial v/\partial x_i$.

We now derive variational formulations for the Bean and power law magnetization problems in the cylindrical and thin-film configurations.

2.1. Long cylinder in a parallel field

Let a long cylindrical superconductor with cross-section Ω be placed into a non-stationary uniform external magnetic field, $\mathbf{h}_e(t)$, parallel to the cylindrical generators. Time variations of this field induce in the superconductor an electric current with density $\mathbf{j}(\mathbf{x}, t)$ parallel to the cross-section plane spanned by $\mathbf{x} \equiv (x_1, x_2)$. The magnetic field $\mathbf{h}(\mathbf{x}, t)$ produced by this current, and also the total field $\mathbf{h} + \mathbf{h}_e$, are parallel to $\mathbf{h}_e$ and orthogonal to $\mathbf{j}$. In this geometry, the magnetic field may be regarded as scalar and we denote the non-zero x_3-components of $\mathbf{h}$ and $\mathbf{h}_e$ by h and h_e, respectively. As the cylinder is long, the problem can be regarded as two-dimensional so that Maxwell's equations with the displacement current omitted can be written as follows:

$$\partial_t h + d_t h_{\rm e} + \operatorname{curl} \mathbf{e} = 0, \tag{2.2a}$$

$$\mathbf{curl}\, h = \mathbf{j}, \tag{2.2b}$$

where $d_t h_{\rm e} \equiv {\rm d}h_{\rm e}/{\rm d}t$. Here the permeability of the superconductor is assumed equal to the permeability of the vacuum, see [8], and scaled to be unity. We adopt throughout the standard notations $\operatorname{curl} \mathbf{v}(\mathbf{x}) = \partial_{x_1} v_2(\mathbf{x}) - \partial_{x_2} v_1(\mathbf{x})$ and $\mathbf{curl}\, v(\mathbf{x}) = (\partial_{x_2} v(\mathbf{x}), -\partial_{x_1} v(\mathbf{x}))$.

In the domain Ω, the electric field $\mathbf{e}$ is parallel to the current density $\mathbf{j}$; their magnitudes are related by a constitutive relation characterizing the superconducting material, e.g. (1.1) or (1.2). Outside Ω, $\mathbf{j} = \mathbf{0}$ and it follows from (2.2b) that $\nabla h = 0$ there. Since the total field should be equal to $h_{\rm e}$ as $\mathbf{x} \to \infty$, it follows that the field h must be zero outside Ω^*. Because the surface current may usually be neglected, see [8], we assume that the magnetic field is continuous across $\partial\Omega$. To complete the model, we must specify the initial state:

$$h(\mathbf{x}, 0) = h^0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega^*.$$

Let us assume that the superconducting material is described by the Bean current-voltage relation (1.1). Then $|\mathbf{j}| = |\mathbf{curl}\, h| \le 1$. Since $|\mathbf{curl}\, h| = |\nabla h|$, we may regard K as the set of admissible magnetic fields $h(\cdot, t)$ on Ω^*. Multiplying (2.2a) by $v - h$, for any $v \in K$, and making use of (2.2b) and the Green's formula $(\operatorname{curl} \mathbf{e}, v) = (\mathbf{e}, \mathbf{curl}\, v)$, we obtain

$$(\partial_t h + d_t h_{\rm e}, v - h) = -(\mathbf{e}, \mathbf{curl}\{v - h\}) = (\mathbf{e}, \mathbf{j}) - (\mathbf{e}, \mathbf{curl}\, v). \tag{2.3}$$

Note that $\mathbf{e}$ and $\mathbf{j}$ have the same direction in Ω, $\mathbf{j} = \mathbf{0}$ in $\Omega^* \backslash \Omega$, and $|\mathbf{e}|$ is non-zero only if $|\mathbf{j}| = 1$. Since $|\mathbf{curl}\, v| = |\nabla v| \le 1$ almost everywhere, we have

$$(\mathbf{e}, \mathbf{j}) = (|\mathbf{e}|, |\mathbf{j}|) = (|\mathbf{e}|, 1) \ge (|\mathbf{e}|, |\mathbf{curl}\, v|) \ge (\mathbf{e}, \mathbf{curl}\, v), \tag{2.4}$$

and so $(\partial_t h + d_t h_{\rm e}, v - h) \ge 0$ for any $v \in K$. The problem can now be formulated as the evolutionary variational inequality:

Find $h \in L^\infty(0, T; K) \cap H^1(0, T; L^2(\Omega^*))$ such that for a.e. $t \in (0, T)$

$$(\partial_t h + d_t h_{\rm e}, v - h) \ge 0 \quad \forall v \in K \tag{2.5}$$

and

$$h|_{t=0} = h^0,$$

where the data $h^0 \in K$ and $h_{\rm e} \in H^1(0, T; \mathbb{R})$ are given.

We note that (2.5) may be considered as a non-stationary analogue of the completely plastic beam torsion problem [22]. If the domain Ω is simply connected, and hence $\Omega^* \equiv \Omega$, we have that

$$-\operatorname{dist}(\mathbf{x}, \partial\Omega) \le v(\mathbf{x}) \le \operatorname{dist}(\mathbf{x}, \partial\Omega) \quad \forall v \in K. \tag{2.6}$$

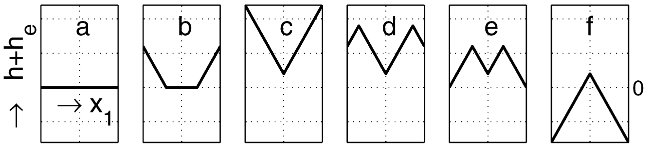

Fig. 1. Infinite slab in a parallel external field.

In this case the variational inequality (2.5) can be solved analytically:

If $h_e(t)$ *is non-decreasing,* $\quad h(\mathbf{x},t) = \max\{-\mathrm{dist}(\mathbf{x},\partial\Omega),\ h^0(\mathbf{x}) - h_e(t) + h_e(0)\}$;

If $h_e(t)$ *is non-increasing,* $\quad h(\mathbf{x},t) = \min\{\mathrm{dist}(\mathbf{x},\partial\Omega),\ h^0(\mathbf{x}) - h_e(t) + h_e(0)\}$.

It is easily checked using (2.6) that such $h \in L^\infty(0,T;K) \cap H^1(0,T;L^2(\Omega^*))$ solve (2.5) in the case of monotonic h_e. In addition, sequential application of these rules provides the solution if h_e is not monotonic. For example, in Fig. 1 we consider for simplicity the case when $\Omega^* \equiv \Omega$ is the infinite slab cross-section, $0 < x_1 < L$, $-\infty < x_2 < \infty$. Initially, the superconductor is in a zero-field state, $h^0(\cdot) = h_e(0) = 0$, (a). As the external field $h_e(t)$ grows, the magnetic field $h + h_e$ penetrates into the superconductor, (b) and (c). As the external field h_e decreases, (d), becomes zero, (e), and then changes its direction, (f), the superconductor does not pass by its previous states: the magnetization is hysteretic.

We now consider the above problem with the Bean current–voltage relation replaced by the power current–voltage law (1.2), which in vectorial form can be written as $\mathbf{e} = |\mathbf{j}|^{p-2}\mathbf{j}$ in Ω with $\mathbf{j} = \mathbf{curl}\,h$. Multiplying (2.2a) by a test function $v \in V_p$, using the Green's formula $(\mathrm{curl}\,\mathbf{e}, v) = (\mathbf{e}, \mathbf{curl}\,v) \equiv (\mathbf{e}, \mathbf{curl}\,v)_\Omega$ and the relation $\mathbf{curl}\,u \cdot \mathbf{curl}\,v \equiv \nabla u \cdot \nabla v$, we arrive at the following weak formulation of an evolutionary p-Laplacian equation for a given $p \in [2,\infty)$:

Find $h_p \in L^\infty(0,T;V_p) \cap H^1(0,T;L^2(\Omega^*))$ such that for a.e. $t \in (0,T)$,

$$(\partial_t h_p + d_t h_e, v) + (|\nabla h_p|^{p-2}\nabla h_p, \nabla v) = 0 \quad \forall v \in V_p \tag{2.7}$$

and

$$h_p|_{t=0} = h^0,$$

where the data h^0 and h_e are the same as those given for (2.5) above.

In Section 3 we will show the well-posedness of (2.7) and (2.5), and that as $p \to \infty$ the unique solution of (2.7) converges to the unique solution of (2.5).

2.2. Thin film in a perpendicular field

We now derive the corresponding problems to (2.5) and (2.7) for the stream function of the sheet current density in the thin-film configuration. However, they are now somewhat more complicated. We assume that the film is flat and very thin. We denote by $\mathbf{j}(\mathbf{x},t)$ the 2D sheet current density related to the film mid-plane Ω, lying in the plane spanned by $\mathbf{x} \equiv (x_1, x_2)$. We once again assume that $\mathbf{e}$ and $\mathbf{j}$ have the same direction in Ω and that one of the current–voltage relations, (1.1) or (1.2), describe the superconducting material. The current in the superconductor is induced by the time variations of a uniform external magnetic field $\mathbf{h}_e(t)$ orthogonal to the film. We denote its x_3 component by $h_e(t)$. Since no current is fed into the film by an electric contact, we have that

$$\operatorname{div}\mathbf{j} = 0 \quad \text{in } \Omega, \quad \mathbf{j}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{2.8}$$

where $\mathbf{n} \equiv (n_1, n_2)$ is normal to the boundary $\partial\Omega$. These conditions should also be satisfied for a given initial distribution:

$$\mathbf{j}(\mathbf{x}, 0) = \mathbf{j}^0(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega.$$

To derive the variational formulation for thin-film magnetization problems, it is convenient to express the electric field via the vector and scalar potentials, $\mathbf{A}$ and ϕ:

$$\mathbf{e} + \partial_t\mathbf{A} + \nabla\phi = \mathbf{0}, \tag{2.9}$$

see e.g. [11, Section 6.4]. We define the space $H(\operatorname{div} 0, \Omega)$ as the closure of the set of smooth vector functions satisfying (2.8) in the norm $\|\mathbf{w}\|_{H(\mathrm{div},\Omega)} := (\|\mathbf{w}\|^2_{L^2(\Omega)} + \|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega)})^{1/2}$, see [9, pp. 26–29]. The scalar potential, ϕ, can be eliminated from (2.9) by multiplying it by an arbitrary function $\mathbf{w} \in H(\operatorname{div} 0, \Omega)$. Since $(\nabla\phi, \mathbf{w})_\Omega = 0$, we obtain the variational relation

$$(\mathbf{e} + \partial_t\mathbf{A}, \mathbf{w})_\Omega = 0 \quad \forall \mathbf{w} \in H(\operatorname{div} 0, \Omega). \tag{2.10}$$

The vector potential, $\mathbf{A}$, can be represented as the sum of a potential corresponding to the external field and the potential of the induced current: $\mathbf{A} = \mathbf{A}_e + \mathbf{A}_i$. Here $\mathbf{A}_e$ satisfies the equation

$$\operatorname{curl}\mathbf{A}_e = h_e, \tag{2.11}$$

and $\mathbf{A}_i$ is determined by the formula

$$\mathbf{A}_i(\mathbf{x}, t) = \int_\Omega \frac{\mathbf{j}(\mathbf{x}', t)}{4\pi|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{x}' \tag{2.12}$$

up to the gradient of a scalar function which is eliminated by $\mathbf{w}$ in (2.10).

For any $\mathbf{w} \in H(\operatorname{div} 0, \Omega)$, there exists a stream function $v \in W^{1,2}(\Omega)$ such that $\mathbf{w} = \mathbf{curl}\, v$. This function is determined up to an additive constant and is constant on each connected component of $\partial\Omega$. By setting it to zero on $\partial\Omega^*$, we determine this function in a unique way. We can also extend such functions continuously by a constant inside each hole Ω_i and regard them as elements of V_2. In fact we have that

$$\forall \mathbf{w} \in H(\operatorname{div} 0, \Omega), \ \exists! v \in V_2 \text{ such that } \mathbf{curl}\, v = \mathbf{w}, \tag{2.13a}$$

$$\forall v \in V_2, \text{then } \mathbf{w} = \mathbf{curl}\, v|_\Omega \in H(\operatorname{div} 0, \Omega); \tag{2.13b}$$

see e.g. [9, Chapter 1, Corollary 3.1].

Taking (2.11) and (2.13a) into account and applying a Green formula, we obtain that

$$(\mathbf{A}_e, \mathbf{w})_\Omega = (\mathbf{A}_e, \mathbf{curl}\, v) = (\mathbf{curl}\,\mathbf{A}_e, v) = (h_e, v). \tag{2.14}$$

Let $h(\cdot,t) \in V_2$ denote the stream function of the current density $\mathbf{j}$ (and *not* the induced magnetic field as in the cylindrical case). For any $\mathbf{w} \in H(\operatorname{div} 0, \Omega)$, it follows from (2.12) and (2.13a) that $(\mathbf{A}_{\mathrm{i}}, \mathbf{w}) = a(h, v)$, where

$$a(u,v) := \int_{\Omega^*} \int_{\Omega^*} \frac{\nabla u(\mathbf{x}) \cdot \nabla' v(\mathbf{x}')}{4\pi |\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{x}'. \tag{2.15}$$

Physically, $\frac{1}{2} a(h,h)$ is the energy of the magnetic field induced by the current $\mathbf{j} = \mathbf{curl}\, h$. Noting (2.13b), (2.14) and (2.15), we can now rewrite the variational relation (2.10) as

$$(\mathbf{e}, \mathbf{curl}\, v) + a(\partial_t h, v) + (d_t h_{\mathrm{e}}, v) = 0 \quad \forall v \in V_2. \tag{2.16}$$

Lemma 2.1. *The bilinear form* $a(\cdot,\cdot)$ *is symmetric, continuous and coercive on* $H_{00}^{1/2}(\Omega^*) \times H_{00}^{1/2}(\Omega^*)$, *where*

$$H_{00}^{1/2}(\Omega^*) := \{\chi \in H^{1/2}(\Omega^*)\colon \tilde{\chi} := \begin{cases} \chi & \text{in } \Omega^*, \\ 0 & \text{in } \mathbb{R}^2 \backslash \Omega^* \end{cases} \in H^{1/2}(\mathbb{R}^2)\}. \tag{2.17}$$

Proof. To analyse $a(\cdot,\cdot)$ we introduce the following space:

$$\begin{aligned} W^1(\mathbb{R}^3) := \{\eta\colon & [1 + |\mathbf{x}|^2 + x_3^2]^{-1/2} \eta \in L^2(\mathbb{R}^3) \text{ and } \partial_{x_m} \eta \in L^2(\mathbb{R}^3), \\ & m = 1 \to 3\} \end{aligned} \tag{2.18}$$

with the norm

$$\|\eta\|_{W^1(\mathbb{R}^3)} := \left[\sum_{m=1}^{3} \int_{\mathbb{R}^3} |\partial_{x_m} \eta|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}x_3 \right]^{1/2}, \tag{2.19}$$

where as before $\mathbf{x} \equiv (x_1, x_2)$; see [6, Chapter XIB, Section 1]. We introduce the trace space $H^{1/2}(\Omega^*)$ with the norm

$$\|\chi\|_{H^{1/2}(\Omega^*)} := \inf_{\substack{\eta \in W^1(\mathbb{R}^3) \\ \eta|_{\Omega^*} = \chi}} \|\eta\|_{W^1(\mathbb{R}^3)} \tag{2.20}$$

and its dual space, $H^{-1/2}(\Omega^*)$, with the norm

$$\|\xi\|_{H^{-1/2}(\Omega^*)} := \sup_{\substack{\chi \in H^{1/2}(\Omega^*) \\ \chi \neq 0}} \frac{\int_{\Omega^*} \xi \chi \, \mathrm{d}\mathbf{x}}{\|\chi\|_{H^{1/2}(\Omega^*)}} \equiv \sup_{\substack{\eta \in W^1(\mathbb{R}^3) \\ \eta \neq 0}} \frac{\int_{\Omega^*} \xi \eta \, \mathrm{d}\mathbf{x}}{\|\eta\|_{W^1(\mathbb{R}^3)}}. \tag{2.21}$$

In the above and throughout we have written the duality pairing between $H^{-1/2}(\Omega^*)$ and $H^{1/2}(\Omega^*)$ formally as an integral over Ω^*, as is common practice, for ease of notation. We then introduce the linear operator $\mathscr{G} : H^{-1/2}(\Omega^*) \to W^1(\mathbb{R}^3)$ defined by

$$(\mathscr{G}\xi)(\mathbf{x}, x_3) := \int_{\Omega^*} \frac{\xi(\mathbf{x}')}{4\pi [\,|\mathbf{x} - \mathbf{x}'|^2 + x_3^2]^{1/2}} \, \mathrm{d}\mathbf{x}' \quad \forall (\mathbf{x}, x_3) \in \mathbb{R}^3 \tag{2.22}$$

that is, $\mathscr{G}\xi \in W^1(\mathbb{R}^3)$ is the unique solution of

$$\sum_{m=1}^{3} \int_{\mathbb{R}^3} \partial_{x_m}(\mathscr{G}\xi)(\mathbf{x},x_3)\partial_{x_m}\eta(\mathbf{x},x_3)\,\mathrm{d}\mathbf{x}\,\mathrm{d}x_3 = \int_{\Omega^*} \xi(\mathbf{x})\eta(\mathbf{x})\,\mathrm{d}\mathbf{x} \qquad \forall \eta \in W^1(\mathbb{R}^3). \tag{2.23}$$

From (2.21) and (2.23) it follows that

$$\|\xi\|_{H^{-1/2}(\Omega^*)} \equiv \|\mathscr{G}\xi\|_{W^1(\mathbb{R}^3)}. \tag{2.24}$$

From (2.15), (2.22), (2.23), (2.19) and (2.24) it follows that

$$a(u,v) \equiv \sum_{k=1}^{2}\sum_{m=1}^{3} \int_{\mathbb{R}^3} \partial_{x_m}(\mathscr{G}\partial_{x_k}u)(\mathbf{x},x_3)\,\partial_{x_m}(\mathscr{G}\partial_{x_k}v)(\mathbf{x},x_3)\,\mathrm{d}\mathbf{x}\,\mathrm{d}x_3 \tag{2.25a}$$

and

$$a(u,u) \equiv \sum_{k=1}^{2} \|\mathscr{G}\partial_{x_k}u\|^2_{W^1(\mathbb{R}^3)} \equiv \sum_{k=1}^{2} \|\partial_{x_k}u\|^2_{H^{-1/2}(\Omega^*)}. \tag{2.25b}$$

We recall from [12, p. 85] if $\partial\Omega^* \in C^\infty$ and [10, Remark 1.4.4.7] if $\partial\Omega^* \in C^{0,1}$ that $\partial_{x_k}: H^{1/2}(\Omega^*) \to [H^{1/2}_{00}(\Omega^*)]' \supset H^{-1/2}(\Omega^*)$ is a continuous linear operator, where $H^{1/2}_{00}(\Omega^*)$ is defined by (2.17) with the norm

$$\|\chi\|_{H^{1/2}_{00}(\Omega^*)} := \left\{ \|\chi\|^2_{H^{1/2}(\Omega^*)} + \int_{\Omega^*} \frac{|\chi(\mathbf{x})|^2}{\mathrm{dist}(\mathbf{x},\partial\Omega^*)}\,\mathrm{d}\mathbf{x} \right\}^{1/2} \tag{2.26}$$

and $[H^{1/2}_{00}(\Omega^*)]'$ is its dual. From [10, (1,3,2,7) and (1,3,2,11)], we have that there exist positive constants C_1 and C_2 such that

$$C_1\|\tilde{\chi}\|_{H^{1/2}(\mathbb{R}^2)} \le \|\chi\|_{H^{1/2}_{00}(\Omega^*)} \le C_2\|\tilde{\chi}\|_{H^{1/2}(\mathbb{R}^2)} \quad \forall \chi \in H^{1/2}_{00}(\Omega^*). \tag{2.27}$$

In addition from (2.17), the facts that $\partial_{x_k}: H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2) := [H^{1/2}(\mathbb{R}^2)]'$ is a continuous linear operator and $C_0^\infty(\Omega^*)$ is dense in $H^{1/2}(\Omega^*)$, see [10, Lemma 1.4.1.3 and Theorem 1.4.2.4], it is easy to deduce that

$$\partial_{x_k}: H^{1/2}_{00}(\Omega^*) \to H^{-1/2}(\Omega^*) \tag{2.28}$$

is a continuous linear operator. Hence it follows from (2.25a–b) and (2.28) that $a(\cdot,\cdot)$ is a symmetric continuous bilinear form on $H^{1/2}_{00}(\Omega^*) \times H^{1/2}_{00}(\Omega^*)$.

Using the equivalent Fourier transform norm on $H^s(\mathbb{R})$, $s \in \mathbb{R}$, it follows, similarly to [23, Problem 21.9], that

$$\|\psi\|^2_{H^{1/2}(\mathbb{R}^2)} \le C\left[\|\psi\|^2_{H^{-1/2}(\mathbb{R}^2)} + \sum_{k=1}^{2}\|\partial_{x_k}\psi\|^2_{H^{-1/2}(\mathbb{R}^2)}\right] \quad \forall \psi \in H^{1/2}(\mathbb{R}^2). \tag{2.29}$$

Combining (2.27), (2.29) and noting (2.17), (2.28) and that $C_0^\infty(\Omega^*)$ is dense in $H^{1/2}_{00}(\Omega^*)$, see [10, Theorem 1.4.2.2], it is easily deduced that

$$\|\chi\|^2_{H^{1/2}_{00}(\Omega^*)} \le C\left[\|\chi\|^2_{H^{-1/2}(\Omega^*)} + \sum_{k=1}^{2}\|\partial_{x_k}\chi\|^2_{H^{-1/2}(\Omega^*)}\right] \quad \forall \chi \in H^{1/2}_{00}(\Omega^*). \tag{2.30}$$

Finally, using a similar proof to that for the standard Poincaré inequality on $H^1(\Omega^*)$, see e.g [6, Chapter IVB, Section 7, Proposition 2], we have that

$$\|\chi\|^2_{H^{1/2}_{00}(\Omega^*)} \leq C\sum_{k=1}^{2} \|\partial_{x_k}\chi\|^2_{H^{-1/2}(\Omega^*)} \quad \forall\chi\in H^{1/2}_{00}(\Omega^*). \tag{2.31}$$

In deriving (2.31), we have noted the compactness of the embedding $H^{1/2}(\Omega^*) \hookrightarrow H^{-1/2}(\Omega^*)$, see [10, Theorem 1.4.3.2], and from (2.26) that the only constant function in $H^{1/2}_{00}(\Omega^*)$ is the zero function. Hence it follows from (2.25a) and (2.31) that $a(\cdot,\cdot)$ is coercive on $H^{1/2}_{00}(\Omega^*)\times H^{1/2}_{00}(\Omega^*)$. $\square$

Let us now return to the variational relation (2.16). For the Bean current–voltage law (1.1) we have, as in the cylindrical case (2.4), $h(\cdot,t)\in K$ and $(\mathbf{e},\mathbf{curl}\,h)=(\mathbf{e},\mathbf{j})\geq(\mathbf{e},\mathbf{curl}\,v)$ for any $v\in K$. Therefore, the thin-film critical-state problem has the following variational formulation:

Find $h\in L^\infty(0,T;K)\cap H^1(0,T;H^{1/2}_{00}(\Omega^*))$ such that for a.e. $t\in(0,T)$

$$a(\partial_t h, v-h)+(d_t h_{\mathrm{e}}, v-h)\geq 0, \quad \forall v\in K \tag{2.32}$$

and

$$h|_{t=0}=h^0,$$

where we are given $h_{\mathrm{e}}\in H^1(0,T;\mathbb{R})$ and the stream function, $h^0\in K$, of the current density $\mathbf{j}^0$, which is assumed to satisfy $|\mathbf{j}^0|\leq 1$ as well as (2.8).

The current density in the Bean model cannot exceed the critical value and, as is known from various experiments and numerical simulations, the same critical state develops eventually for any initial current density distribution when the external field becomes sufficiently strong. The resulting saturated distribution of current density remains then unchanged under any further growth of the external field. If the domain Ω is simply connected, the corresponding stationary solutions of (2.32) are easily found analytically. Let us assume h_{e} varies monotonically. The stationary form of (2.32), and of (2.5), is then equivalent to the well known completely plastic beam torsion problem, see [22], and also to the maximal sandpile growth problem, see [13]. With, e.g., $d_t h_{\mathrm{e}}<0$, one has the following stationary formulation:

$$\text{Find } h_{\mathrm{S}}\in K \text{ such that } \int_\Omega (v-h_{\mathrm{S}})\leq 0 \quad \forall v\in K.$$

The unique solution to this problem, see [22,7], is $h_{\mathrm{S}}(\mathbf{x})=\mathrm{dist}(\mathbf{x},\partial\Omega)$, and this corresponds to the stationary current density distribution $\mathbf{j}_{\mathrm{S}}=\mathbf{curl}\,h_{\mathrm{S}}$ having $|\mathbf{j}_{\mathrm{S}}|=1$ a.e. in Ω. The direction of this current changes discontinuously at the ridges of the domain (a point $\mathbf{x}\in\Omega$ belongs to a ridge of Ω if there exist at least two different points on $\partial\Omega$, $\mathbf{x}_1$ and $\mathbf{x}_2$, such that $|\mathbf{x}-\mathbf{x}_1|=|\mathbf{x}-\mathbf{x}_2|=\mathrm{dist}(\mathbf{x},\partial\Omega)$). It is easy to see that the current contours are the level contours of h_{S}; these contours are shown for three different film shapes in Fig. 2.

The variational inequality (2.32) is similar to (2.5), but is “implicit” with respect to the time derivative of the solution. To show the existence of a (unique) solution

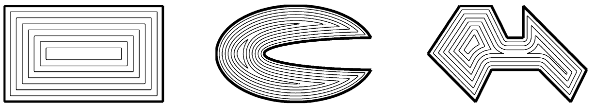

Fig. 2. Current contours in the critical state for various film shapes.

to (2.32), we will consider in the next section the $p \to \infty$ limit of solutions to this magnetization problem with the Bean critical-state law, (1.1), replaced by the power current–voltage law (1.2). It is easy to deduce from (2.16), using the same procedure as in deriving (2.7), the following variational formulation for this power-law problem with a given $p \in [2, \infty)$:

$$\text{Find } h_p \in L^\infty(0,T;V_p) \cap H^1(0,T;H^{1/2}_{00}(\Omega^*)) \text{ such that for a.e. } t \in (0,T)$$

$$a(\partial_t h_p, v) + (d_t h_e, v) + (|\nabla h_p|^{p-2}\nabla h_p, \nabla v) = 0 \quad \forall v \in V_p \tag{2.33}$$

and

$$h_p|_{t=0} = h^0,$$

where the data are the same as those given for (2.32) above.

Note that this evolutionary problem is also "implicit", but otherwise similar to the corresponding p-Laplacian equation in the cylindrical case (2.7).

3. Well-posedness and the limit as $p \to \infty$

We analyse the well-posedness of problems (2.7) and (2.33) and their limits as $p \to \infty$, by studying a more general problem.

Let B be a Banach space with norm $\|\cdot\|_B$ such that $H^1_0(\Omega^*) \hookrightarrow B \hookrightarrow L^2(\Omega^*)$ with the first embedding being compact. Then B', the dual of B, is such that $L^2(\Omega^*) \hookrightarrow B' \hookrightarrow H^{-1}(\Omega^*) \equiv [H^1_0(\Omega^*)]'$. Throughout we take $\|\nabla \cdot\|_{L^p(\Omega^*)}$ to be the norm on $W^{1,p}_0(\Omega^*)$. Let $b(\cdot,\cdot)$ be a symmetric coercive bilinear form on $B \times B$ and hence we have for some fixed constant $\sigma \in \mathbb{R}^+$ that

$$|\chi|_b := [b(\chi,\chi)]^{1/2} \geq \sigma \|\chi\|_B \quad \forall \chi \in B. \tag{3.1}$$

Then given the data

$$h^0 \in K \quad \text{and} \quad h_e \in H^1(0,T;B'), \tag{3.2}$$

we introduce for any $p \geq 2$ the problem:

(P_p) Find $h_p \in L^\infty(0,T;V_p) \cap H^1(0,T;B)$ such that $h_p(\cdot,0) = h^0(\cdot)$ and for a.e. $t \in (0,T)$

$$b(\partial_t h_p, v) + (|\nabla h_p|^{p-2}\nabla h_p, \nabla v) = -(\partial_t h_e, v) \quad \forall v \in V_p. \tag{3.3}$$

We remark that the initial condition makes sense from the definition of B above and the embedding (2.1b) with $X \equiv W^{1,p}_0(\Omega^*)$ and $Y \equiv Z \equiv B$.

We obtain problems (2.7) and (2.33) with a more general choice of data h_{e} by choosing $B\equiv L^2(\Omega^*)$, $b(\cdot,\cdot)\equiv(\cdot,\cdot)$ and $B\equiv H^{1/2}_{00}(\Omega^*)$, $b(\cdot,\cdot)\equiv a(\cdot,\cdot)$ in (P_p), respectively. We note that $H^1_0(\Omega^*)$ is compactly embedded in $H^{1/2}_{00}(\Omega^*)$, by recalling the equivalent interpolation definition $H^{1/2}_{00}(\Omega^*):=[H^1_0(\Omega^*),L^2(\Omega^*)]_{1/2}$, see [12, p. 66].

In order to establish existence of a solution h_p to (P_p) and a number of stability bounds, we consider a discretization of (P_p) in time. Let $N\tau=T$ and $t_n:=n\tau$, $n=0\rightarrow N$. Then for $p\geq 2$ we introduce the problem

$(\mathrm{P}_{p,\tau})$ For $n=1\rightarrow N$, find $h^n_{p,\tau}\in V_p$ such that

$$b\left(\frac{h^n_{p,\tau}-h^{n-1}_{p,\tau}}{\tau},v\right)+(|\nabla h^n_{p,\tau}|^{p-2}\nabla h^n_{p,\tau},\nabla v)=-\left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\partial_t h_{\mathrm{e}}(\cdot,t)\,\mathrm{d}t,v\right) \qquad \forall v\in V_p, \tag{3.4}$$

where $h^0_{p,\tau}=h^0$.

We remark that the right-hand side is well defined on noting the continuous embedding

$$H^1(0,T;B')\hookrightarrow C([0,T];H^{-1}(\Omega^*)). \tag{3.5}$$

Lemma 3.1. *There exists a unique solution* $\{h^n_{p,\tau}\}_{n=0}^N$ *to* $(P_{p,\tau})$. *Moreover, the following stability bounds hold*:

$$\max_{n=0\rightarrow N}\|h^n_{p,\tau}\|^2_B+\tau\sum_{n=1}^N\|\nabla h^n_{p,\tau}\|^p_{L^p(\Omega^*)}+\tau\sum_{n=1}^N\left\|\frac{h^n_{p,\tau}-h^{n-1}_{p,\tau}}{\tau}\right\|^2_B$$
$$+\frac{1}{p}\max_{n=0\rightarrow N}\|\nabla h^n_{p,\tau}\|^p_{L^p(\Omega^*)}\leq C, \tag{3.6}$$

where $C\in\mathbb{R}^+$ *is independent of* τ and p.

Proof. For a fixed integer $n\in[1,N]$ and given $h^{n-1}_{p,\tau}$ there exists a solution to (3.4); since this is Euler–Lagrange equation of the minimization problem

$$\inf_{v\in V_p}\left\{\frac{1}{2}\tau\left|\frac{v-h^{n-1}_{p,\tau}}{\tau}\right|^2_b+\frac{1}{p}\int_{\Omega^*}|\nabla v|^p\,\mathrm{d}\mathbf{x}-\left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\partial_t h_{\mathrm{e}}(\cdot,t)\,\mathrm{d}t,v\right)\right\}. \tag{3.7}$$

If there exist two solutions, $h^{n,i}_{p,\tau}$ $i=1,2$, to (3.4); then $\bar{h}^n_{p,\tau}:=h^{n,1}_{p,\tau}-h^{n,2}_{p,\tau}$ is such that $\bar{h}^n_{p,\tau}\in V_p$ and

$$|\bar{h}^n_{p,\tau}|^2_b+\tau(|\nabla h^{n,1}_{p,\tau}|^{p-2}\nabla h^{n,1}_{p,\tau}-|\nabla h^{n,2}_{p,\tau}|^{p-2}\nabla h^{n,2}_{p,\tau},\nabla\bar{h}^n_{p,\tau})=0. \tag{3.8}$$

It follows from the convexity of $|\nabla\cdot|^p$ and a Poincaré inequality that $\bar{h}^n_{p,\tau}\equiv 0$. Hence we have existence and uniqueness of the solution to $(\mathrm{P}_{p,\tau})$.

Choosing $v \equiv h^n_{p,\tau}$ in (3.4) and applying Young and Holder inequalities yields for $n = 1 \rightarrow N$ that

$$
\begin{aligned}
& b\left(\frac{h^n_{p,\tau} - h^{n-1}_{p,\tau}}{\tau}, h^n_{p,\tau}\right) + \|\nabla h^n_{p,\tau}\|^p_{L^p(\Omega^*)} = -\left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \partial_t h_{\mathrm{e}}(\cdot,t)\,\mathrm{d}t, h^n_{p,\tau}\right) \\
&\le \tau^{-1}\left\| \int_{t_{n-1}}^{t_n} \partial_t h_{\mathrm{e}}(\cdot,t)\,\mathrm{d}t\right\|_{H^{-1}(\Omega^*)} \|\nabla h^n_{p,\tau}\|_{L^2(\Omega^*)} \\
&\le |\Omega^*|^{(p-2)/2p}\,\tau^{-1}\left(\int_{t_{n-1}}^{t_n} \|\partial_t h_{\mathrm{e}}(\cdot,t)\|_{H^{-1}(\Omega^*)}\,\mathrm{d}t\right) \|\nabla h^n_{p,\tau}\|_{L^p(\Omega^*)} \\
&\le \frac{p-1}{p}(|\Omega^*|^{(p-2)/2p}\,\tau^{-1})^{p/(p-1)}\left(\int_{t_{n-1}}^{t_n} \|\partial_t h_{\mathrm{e}}(\cdot,t)\|_{H^{-1}(\Omega^*)}\,\mathrm{d}t\right)^{p/(p-1)} \\
&\quad + \frac{1}{p}\|\nabla h^n_{p,\tau}\|^p_{L^p(\Omega^*)} \\
&\le \frac{p-1}{p}|\Omega^*|^{(p-2)/2(p-1)}\tau^{-1}\int_{t_{n-1}}^{t_n} \|\partial_t h_{\mathrm{e}}(\cdot,t)\|^{p/(p-1)}_{H^{-1}(\Omega^*)}\,\mathrm{d}t \\
&\quad + \frac{1}{p}\|\nabla h^n_{p,\tau}\|^p_{L^p(\Omega^*)}.
\end{aligned} \tag{3.9}
$$

On noting the identity

$$
2(s-r)s = s^2 - r^2 + (s-r)^2 \qquad \forall\ r,s \in \mathbb{R}; \tag{3.10}
$$

and summing (3.10) from $n = 1 \rightarrow m$ for any integer $m \in [1,N]$ yields that

$$
\begin{aligned}
& \frac{1}{2}\max_{n=1\rightarrow N} |h^n_{p,\tau}|^2_b + \frac{1}{2}\tau^2\sum_{n=1}^{N}\left|\frac{h^n_{p,\tau} - h^{n-1}_{p,\tau}}{\tau}\right|^2_b + \frac{(p-1)}{p}\tau\sum_{n=1}^{N}\|\nabla h^n_{p,\tau}\|^p_{L^p(\Omega^*)} \\
&\le |h^0|^2_b + \frac{2(p-1)}{p}|\Omega^*|^{\frac{p-2}{2(p-1)}}\|\partial_t h_{\mathrm{e}}\|^{p/(p-1)}_{L^{p/(p-1)}(0,T;H^{-1}(\Omega^*))}.
\end{aligned} \tag{3.11}
$$

Choosing $v \equiv h^n_{p,\tau} - h^{n-1}_{p,\tau}$ in (3.4), noting the convexity of $|\cdot|^p$, (3.1) and applying Young and Holder inequalities yields that

$$
\begin{aligned}
& \tau\left|\frac{h^n_{p,\tau} - h^{n-1}_{p,\tau}}{\tau}\right|^2_b + \frac{1}{p}\|\nabla h^n_{p,\tau}\|^p_{L^p(\Omega^*)} \\
&\le \tau\left|\frac{h^n_{p,\tau} - h^{n-1}_{p,\tau}}{\tau}\right|^2_b + (|\nabla h^n_{p,\tau}|^{p-2}\nabla h^n_{p,\tau}, \nabla(h^n_{p,\tau} - h^{n-1}_{p,\tau})) + \frac{1}{p}\|\nabla h^{n-1}_{p,\tau}\|^p_{L^p(\Omega^*)} \\
&\le \tau\left|\left(\frac{h^n_{p,\tau} - h^{n-1}_{p,\tau}}{\tau}, \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \partial_t h_{\mathrm{e}}(\cdot,t)\,\mathrm{d}t\right)\right| + \frac{1}{p}\|\nabla h^{n-1}_{p,\tau}\|^p_{L^p(\Omega^*)}
\end{aligned}
$$

$$\le \frac{1}{2}\tau\sigma^2 \left\| \frac{h^n_{p,\tau} - h^{n-1}_{p,\tau}}{\tau} \right\|^2_B + \frac{1}{2}\tau^{-1}\sigma^{-2} \left\| \int_{t_{n-1}}^{t_n} \partial_t h_{\mathrm{e}}(\cdot,t)\,\mathrm{d}t \right\|^2_{B'}$$

$$+\frac{1}{p}\,\|\nabla h^{n-1}_{p,\tau}\|^p_{L^p(\Omega^*)}$$

$$\le \frac{1}{2}\tau \left\| \frac{h^n_{p,\tau} - h^{n-1}_{p,\tau}}{\tau} \right\|^2_b + \frac{1}{2}\sigma^{-2} \int_{t_{n-1}}^{t_n} \|\partial_t h_{\mathrm{e}}(\cdot,t)\|^2_{B'}\,\mathrm{d}t$$

$$+\frac{1}{p}\,\|\nabla h^{n-1}_{p,\tau}\|^p_{L^p(\Omega^*)}. \tag{3.12}$$

Summing (3.12) from $n=1\to m$ for any integer $m\in[1,N]$ yields that

$$\frac{1}{2}\tau\sum_{n=1}^{N} \left| \frac{h^n_{p,\tau} - h^{n-1}_{p,\tau}}{\tau} \right|^2_b + \frac{1}{p}\max_{n=1\to N}\,\|\nabla h^n_{p,\tau}\|^p_{L^p(\Omega^*)} \le \sigma^{-2}\,\|\partial_t h_{\mathrm{e}}\|^2_{L^2(0,T;B')}$$

$$+\frac{2}{p}\,\|\nabla h^0\|^p_{L^p(\Omega^*)}. \tag{3.13}$$

Combining (3.11) and (3.13), and noting (3.1) and (3.2) yields the desired result (3.6). □

Theorem 3.1. *There exists a unique solution h_p to* (P_p). *Moreover, the following stability bounds hold*:

$$\|h_p\|^2_{L^\infty(0,T;B)} + \|\nabla h_p\|^p_{L^p(0,T;L^p(\Omega^*))} + \|\partial_t h_p\|^2_{L^2(0,T;B)}$$

$$+\frac{1}{p}\,\|\nabla h_p\|^p_{L^\infty(0,T;L^p(\Omega^*))} \le C, \tag{3.14}$$

where $C\in\mathbb{R}^+$ is independent of p.

Proof. Let

$$h_{p,\tau}(\cdot,t) := \frac{t-t_{n-1}}{\tau}h^n_{p,\tau}(\cdot) + \frac{t_n-t}{\tau}h^{n-1}_{p,\tau}(\cdot), \quad t\in[t_{n-1},t_n],\ n\ge 1, \tag{3.15a}$$

$$\hat{h}_{p,\tau}(\cdot,t) := h^n_{p,\tau}(\cdot), \quad t\in(t_{n-1},t_n],\ n\ge 1, \tag{3.15b}$$

$$h_{\mathrm{e},\tau}(\cdot,t) := \frac{t-t_{n-1}}{\tau}h_{\mathrm{e}}(\cdot,t_n) + \frac{t_n-t}{\tau}h_{\mathrm{e}}(\cdot,t_{n-1}), \quad t\in[t_{n-1},t_n],\ n\ge 1. \tag{3.15c}$$

It follows from (3.15a–c), that (3.4) for $n=1\to N$, can be restated as for a.e. $t\in(0,T)$

$$b(\partial_t h_{p,\tau},v) + (|\nabla\hat{h}_{p,\tau}|^{p-2}\nabla\hat{h}_{p,\tau},\nabla v) = -(\partial_t h_{\mathrm{e},\tau},v) \qquad \forall v\in V_p. \tag{3.16}$$

It follows from (3.15a–b) and (3.6) that

$$\|h_{p,\tau}\|^2_{L^\infty(0,T;B)} + \|\nabla\hat{h}_{p,\tau}\|^p_{L^p(0,T;L^p(\Omega^*))} + \|\partial_t h_{p,\tau}\|^2_{L^2(0,T;B)}$$
$$+\frac{1}{p}\|\nabla h_{p,\tau}\|^p_{L^\infty(0,T;L^p(\Omega^*))} + \frac{1}{p}\|\nabla\hat{h}_{p,\tau}\|^p_{L^\infty(0,T;L^p(\Omega^*))} \leq C. \tag{3.17}$$

It follows from (3.15a–b) and (3.17) that

$$\|h_{p,\tau} - \hat{h}_{p,\tau}\|^2_{L^2(0,T;B)} \leq \tau^2\|\partial_t h_{p,\tau}\|^2_{L^2(0,T;B)} \leq C\tau^2. \tag{3.18}$$

It follows from (3.17) that for fixed p we can extract a subsequence $\tau_i \to 0$ and a limit h_p such that

$$h_{p,\tau_i} \to h_p \qquad \text{weak-}* \text{ in } L^\infty(0,T;V_p), \tag{3.19a}$$

$$\partial_t h_{p,\tau_i} \to \partial_t h_p \qquad \text{weakly in } L^2(0,T;B). \tag{3.19b}$$

As the embedding $H^1_0(\Omega^*) \hookrightarrow B$ is compact; it follows from (3.19a–b) and (2.1a) that

$$h_{p,\tau_i} \to h_p \qquad \text{strongly in } L^2(0,T;B). \tag{3.20}$$

It follows from (3.17), (3.20) and (3.18) that for a suitable subsequence $\tau_{i,j}$ of the subsequence τ_i

$$\hat{h}_{p,\tau_{i,j}} \to h_p \qquad \text{weak-}* \text{ in } L^\infty(0,T;V_p), \quad \text{strongly in } L^2(0,T;B). \tag{3.21}$$

For any given $p \geq 2$ let $\mathscr{L}_p : V_p \to V'_p$, the dual of V_p, be such that

$$\langle \mathscr{L}_p w, v\rangle_{V_p} \equiv (|\nabla w|^{p-2}\nabla w, \nabla v) \qquad \forall w, v \in V_p, \tag{3.22}$$

where $\langle\cdot,\cdot\rangle_{V_p}$ denotes the duality pairing $V'_p \times V_p$. It is easily established that $\mathscr{L}_p$ is a bounded, continuous and uniformly monotone operator. Applying the standard monotonicity trick, see e.g. [23, p. 474], we can now pass to the limit $\tau_{i,j} \to 0$ in (3.16) using (3.19b), (3.21) and noting (3.2). It follows that

$$\mathscr{L}_p \hat{h}_{p,\tau_{i,j}} \to \mathscr{L}_p h_p \qquad \text{weakly in } L^2(0,T;V'_p) \tag{3.23}$$

and the limit h_p solves (P$_p$). In addition we have from (3.17) and (3.19a–b) that the bounds (3.14) hold.

Finally if there exist two solutions to (P$_p$), h^i_p $i = 1, 2$; then $\bar{h}_p := h^1_p - h^2_p$ is such that $\bar{h}_p \in L^\infty(0,T;V_p) \cap H^1(0,T;B)$, $\bar{h}_p(\cdot,0) = 0$ and for a.e. $t \in (0,T)$

$$b(\partial_t\bar{h}_p, \bar{h}_p) + (|\nabla h^1_p|^{p-2}\nabla h^1_p - |\nabla h^2_p|^{p-2}\nabla h^2_p, \nabla\bar{h}_p) = 0. \tag{3.24}$$

It follows that $\bar{h}_p \equiv 0$. Therefore, we have that the limit h_p in (3.20) is the unique solution of (P$_p$). Hence the whole sequence $\{h_{p,\tau}\}$ converges to h_p. □

It follows from (3.14) that we can extract a subsequence $p_i \to \infty$ and a limit h such that

$$h_{p_i} \to h \qquad \text{weak-}* \text{ in } L^\infty(0,T;V_2), \tag{3.25a}$$

$$\partial_t h_{p_i} \to \partial_t h \qquad \text{weakly in } L^2(0,T;B). \tag{3.25b}$$

As the embedding $H_0^1(\Omega^*) \hookrightarrow B$ is compact; it follows from (3.25a–b) and (2.1a) that

$$h_{p_i} \to h \qquad \text{strongly in } L^2(0,T;B). \tag{3.26}$$

Given the data (3.2), we introduce the problem:

(P) Find $h \in L^\infty(0,T;K) \cap H^1(0,T;B)$ such that $h(\cdot,0) = h^0(\cdot)$ and for a.e. $t \in (0,T)$

$$b(\partial_t h, v-h) \geq -(\partial_t h_{\mathrm{e}}, v-h) \qquad \forall v \in K. \tag{3.27}$$

Once again the initial data makes sense from noting the embedding (2.1b) with $X \equiv W_0^{1,p}(\Omega^*)$ and $Y \equiv Z \equiv B$. We have the following result.

Theorem 3.2. *The limit h in* (3.26) *is the unique solution of* (P). *Hence* $h_p \to h$ *as* $p \to \infty$. *Moreover, we have that*

$$\|h - h_p\|^2_{L^\infty(0,T;B)} \leq C\left[\frac{1}{p} + \inf_{v \in L^2(0,T;K)} \|h_p - v\|_{L^2(0,T;B)}\right]. \tag{3.28}$$

Proof. Firstly, we prove that the limit $h \in L^\infty(0,T;K)$. This proof is essentially the same as that of Lemma 3.1 in [1]. We include it for completeness. For any fixed $\delta > 0$, let

$$D_\delta := \{(\mathbf{x},t) \in \Omega_T^* := \Omega^* \times (0,T) : \nabla h(\mathbf{x},t) \text{ is defined and } |\nabla h(\mathbf{x},t)| \geq 1+\delta\}. \tag{3.29}$$

Then it follows from the second bound in (3.14) that

$$\begin{aligned}(1+\delta)|D_\delta| &\leq \int\int_{D_\delta} |\nabla h| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leq \liminf_{p_i \to \infty} \int\int_{D_\delta} |\nabla h_{p_i}| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\leq \liminf_{p_i \to \infty} \left(\int_0^T \int_\Omega |\nabla h_{p_i}|^{p_i} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right)^{1/p_i} |D_\delta|^{1-1/p_i} \leq |D_\delta|. \end{aligned} \tag{3.30}$$

Therefore, $|D_\delta| = 0$ and hence $|\nabla h| \leq 1$ a.e. in Ω_T^*. Additionally $|\nabla h| = 0$ in $\Omega_i \times (0,T)$, $i = 1 \to I$. Hence we have that $h \in L^\infty(0,T;K)$.

Secondly, it follows from (3.3) and the convexity of $|\cdot|^p$ that for a.e. $t \in (0,T)$

$$\begin{aligned} &b(\partial_t h_p, h_p - v) + (\partial_t h_{\mathrm{e}}, h_p - v) \\ &\quad\leq b(\partial_t h_p, h_p - v) + (\partial_t h_{\mathrm{e}}, h_p - v) + \frac{1}{p}\int_\Omega |\nabla h_p|^p \,\mathrm{d}\mathbf{x} \\ &\quad= (|\nabla h_p|^{p-2}\nabla h_p, \nabla(v - h_p)) + \frac{1}{p}\int_\Omega |\nabla h_p|^p \,\mathrm{d}\mathbf{x} \\ &\quad\leq \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}\mathbf{x} \leq \frac{1}{p}|\Omega| \qquad \forall v \in K. \end{aligned} \tag{3.31}$$

Letting $p = p_i \to \infty$ in (3.31) and noting (3.25b) and (3.26) yields that h solves (P).

If there exist two solutions to (P), h^i $i = 1,2$; then $\bar{h} := h^1 - h^2$ is such that $\bar{h} \in L^\infty(0,T;V_2) \cap H^1(0,T;B)$, $\bar{h}(\cdot,0) = 0$ and for a.e. $t \in (0,T)$

$$b(\partial_t \bar{h}, \bar{h}) \leq 0. \tag{3.32}$$

It follows that $\bar{h} \equiv 0$. Therefore, we have that the limit h in (3.26) is the unique solution of (P). Hence the whole sequence $\{h_p\}$ converges to h.

Finally it follows from (3.31) with $v \equiv h$ and (3.27) that for a.e. $t \in (0,T)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|h-h_p|_b^2 &\equiv b(\partial_t(h-h_p),h-h_p)\\ &\leq \frac{1}{p}|\Omega| + [b(\partial_t h, v-h_p) + (\partial_t h_{\rm e}, v-h_p)] \qquad \forall v \in K. \qquad (3.33)\end{aligned}$$

Integrating (3.33) in time and noting (3.2),(3.25b) and (3.1) yields the desired result (3.28). □

Acknowledgements

J.W. Barrett acknowledges travel support from Marks and Spencer Ltd.

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