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TOPOLOGICAL SERIES OF ISOLATED PLANE CURVE SINGULARITIES

by Robert SCHRAUWEN

ABSTRACT. For plane curve singularities, a topological definition of series of isolated singularities, based on the Milnor fibration, is given. Several topological invariants, including the spectrum, are computed.

1. INTRODUCTION

Let $f: (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be a plane curve singularity, in other words, let f be an element of the ring of convergent power series $\mathbf{C}\{x, y\}$. Assume $f \neq 0$. Because $\mathbf{C}\{x, y\}$ is factorial, we can write $f = f_1^{m_1} \cdots f_r^{m_r}$ with all f_i irreducible and whenever $i \neq j$, there is no unit u with $f_i = uf_j$. The *branches* of f are the curves $f_i(x, y) = 0$.

It is well-known that for $\varepsilon > 0$ small, the intersection $L = f^{-1}(0) \cap S_\varepsilon^3$ of the curve $X: f = 0$ and a small 3-sphere of radius ε is a *link*, consisting of r components corresponding to the branches of f, and that this link determines the topological type of f (or of X). Moreover, the map $f/|f|: S_\varepsilon^3 \backslash L \to S^1$ is a fibration, called the *Milnor fibration*.

It is natural to consider L as a *multilink*, i.e. a link with integral multiplicities assigned to each component. We use the notation $L = m_1 S_1 + \cdots + m_r S_r$, where $S_i = f_i^{-1}(0) \cap S_\varepsilon^3$. These multiplicities reflect in the behaviour of the Milnor fibre F (i.e. a typical fibre of the Milnor fibration, which is a Seifert surface bounded by L) near S_i: F approaches S_i from m_i directions (see [EN]).

The Milnor fibration is important in our discussion of *topological series of isolated singularities*. A striking feature of Arnol'd's series *A, D, E, J*, etc. (see [AGV]), is that they are somehow related to a non-isolated singularity. For example: $D_k: xy^2 + x^{k-1}$ is related to $D_\infty: xy^2$ and $Y_{r,s}: x^2y^2 + x^{r+4} + y^{s+4}$ to $Y_{\infty,\infty}: x^2y^2$. This relationship is still not completely understood.

In this paper we give (for plane curve singularities) a topological definition of series (definition 3.1), as follows. A singularity belongs to the topological series of a certain non-isolated singularity f, if its Milnor fibration arises from that of f by removing tubular neighbourhoods of the multiple components and putting something back in such a way that the result is the Milnor fibration of an isolated singularity.

With this definition in hand, we first investigate which isolated singularities belong to the series associated to a given non-isolated singularity. For example, it follows from theorem 3.4 that $D_k (k \geqslant 4)$, is the only possibility when we start with D_∞ (cf. [AGV], p. 243).

What interests us most is how the topology behaves within the series and with regard to the non-isolated singularity. We compute the Milnor number, the characteristic polynomial of the monodromy, and the spectrum of the series. For example, we will find in proposition 5.2, that the Milnor number of a series belonging to a singularity with transversal type A_1 increases linearly with steps of one, just as in the familiar case of the Arnol'd series. Many of these topological invariants have already been considered in the case of series of the form $f + \varepsilon l^k$, with l a general linear function. This was initiated by Iomdin (see [Lê]). But observe that in general such a series is a very small subseries of our topological series belonging to f.

In the last section we consider the question what we have to add to f to get a required element of the series. For instance, to $W^{\#}_{1,\infty}: (y^2 - x^3)^2$, one may add $x^{4+q}y$ and $x^{3+q}y^2$ for $q \geqslant 1$ to obtain the whole series $W^{\#}_{1,p}$. In the case that f has only transversal A_1 singularities, we obtain explicit conditions (theorem 6.5), mainly involving intersection properties.

We use the link L of f to describe the topological type. There is a nice notation for *algebraic links* (i.e. links arising as the link of a plane curve singularity) by means of graphs that we will call *EN-diagrams* after D. Eisenbud and W.D. Neumann, who developed these graphs in [EN].

The EN-diagrams and the underlying concept of *splicing*, which is due to Siebenmann and studied extensively in [EN], are used to state our results and proofs. For example, the definition of topological series is very clear in these terms: the corresponding non-isolated singularity is visible as a subdiagram of the diagram of the series. We will only recall the main points of splicing and EN-diagrams in the next section. For details we refer to [EN] and [Ne], where one can also find how to compute several familiar topological invariants from the EN-diagram. A method of computing the *spectrum* and a splice formula for spectra are of independent interest and they are given in section 4. We will show that the spectrum of a singularity is "almost additive" under splicing.

Our definition of topological series presents a natural idea behind couterexamples (found by J. Steenbrink and J. Stevens) to the spectrum conjecture (the spectrum determines the topology of a plane curve singularity) and the — equivalent — conjecture involving the real Seifert form (cf. [SSS]). Also, A. Neméthi used the idea of topological series to define his topological trivial series [Nm].

In the Appendix, we have included the EN-diagrams and some invariants of the Arnol'd series.

Acknowledgments. I wish to thank Dirk Siersma and Jan Stevens for their remarks and help.

2. Splicing and series

2.1. It is clear that singularities occur in series. The simplest series have been given names, such as *A, D, J,* etc., by Arnol'd. But how to define a series is unclear. One looked at deformation properties such as adjacencies, etc., because the goal is to define what a series means analytically. A proper analytical description can be given for series of the form $f + l^k$, where l is a sufficiently general linear form, see the work of Iomdin and Lê, [Lê]. But already in the case of Arnol'd's series, one finds that they are not of the 'Iomdin-type'. Some series are multi-indexed, such as

$$Y_{r,s}\colon x^2y^2 + x^{r+4} + y^{s+4}\ ,$$

and others, such as $W^\#$:

$$\begin{aligned} W^\#_{1,2q-1}&\colon\quad (y^2-x^3)^2 + x^{4+q}y \\ W^\#_{1,2q}&\colon\quad (y^2-x^3)^2 + x^{3+q}y^2 \end{aligned}$$

make smaller steps than a linear series.

However, the most apparent properties that hold a series together, are the topological invariants. For example, the Milnor number within Arnol'd's series, increases with steps of 1. Therefore it is worthwhile to go not as far as an analytical definition, but to look for a topological one.

Another property is that, as already mentioned in the Introduction, series of isolated singularities are clearly related to non-isolated singularities, and that the hierarchy of these non-isolated singularities reflects the hierarchy of the isolated singularities. This relationship is also not completely understood. Our topological definition, which works for plane curve singularities, makes clear which isolated singularities belong to the series of a given non-isolated singularity.

2.2. The motivation for our definition comes from the topology of the link exterior. We first need to recall some facts of splicing and EN-diagrams.

Let $f \in \mathbf{C}\{x, y\}$ be a plane curve singularity, and L the link of f embedded in S^3. L completely describes the topological type of f. There is a notation for L by means of a weighted graph, that we call an EN-diagram, introduced by Eisenbud and Neumann [EN]. The EN-diagram of f is closely related to the resolution graph of f (the dual graph of the good minimal resolution). In fact, as a graph, the EN-diagram is equal to the resolution graph with all linear chains contracted. We will call the vertices of valence 1 *dots,* and the vertices of valence at least 3 *nodes*. The arrows correspond to the components of L (or the irreducible components of f), and they have a multiplicity, equal to the multilink multiplicity. The nodes, dots and edges have topological meanings as well, we refer to [EN] for details. There are conversion rules from EN-diagram to resolution graph and back, see [EN], chapter V.

It is known that $M = S^3 \setminus N(L)$ (where $N(L)$ is a open tubular neighbourhood of L) is a *Waldhausen manifold,* see [EN] or [LMW]. This means that there is a decomposition of M in Seifert manifolds (the basic building blocks). The decomposition can be found in several ways, e.g. by means of the resolution of f or the polar decomposition of f. This is explained in detail in [LMW].

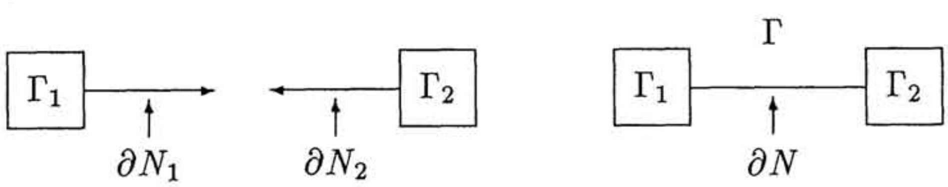
2.3. Glueing two pieces of this decomposition together uses the operation of *splicing,* due to L. Siebenmann and studied extensively in [EN]. Consider two (multi)links $L_1 = m_1 S_1 + L_1'$, $L_2 = m_2 S_2 + L_2'$, embedded in (separate copies of) S^3. Let N_1, N_2 be small tubular neighbourhoods of S_1, S_2. Then the *splice* L of L_1 and L_2 is the link

$$L = L_1' + L_2' ,$$

embedded in the homology sphere

$$\Sigma = (S^3 \setminus N_1) \cup_\partial (S^3 \setminus N_2) ,$$

the boundaries ∂N_1 and ∂N_2 of the tubular neighbourhoods glued meridian to longitude and vice versa. The EN-diagram Γ of L arises from the EN-diagrams of L_1 and L_2 by replacing the two arrows representing S_1 and S_2 by an edge (which represents the splice torus $\partial N_1 = \partial N_2$):



If we impose two conditions, described below, then L is again an algebraic link in S^3.

SPLICE CONDITION

$$m_1 = \mathrm{lk}(S_2, L_2') \quad \text{and} \quad m_2 = \mathrm{lk}(S_1, L_1')\ ,$$

i.e. m_1 has to be equal to the linking number of S_2 with the other components of L_2, counted with their multiplicities (and similarly for m_2).

Linking numbers can be computed easily from the EN-diagram, see [EN], section 10.

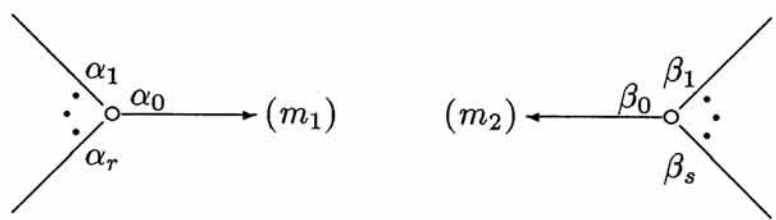
If the splice condition holds, then L is again a fibred link. In fact it forces that the Milnor fibres cut the splice torus in an (m_1, m_2)-torus link.

The second condition is a condition on the weights of the EN-diagram. It follows from [EN], Theorem 9.4, that we need the following condition in order that L is again algebraic:

ALGEBRAICITY CONDITION

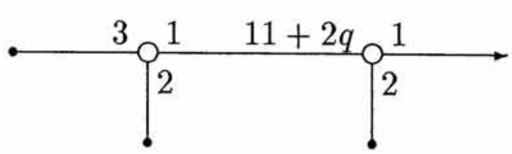
(a) The resulting link can be obtained by repeated cabling, and

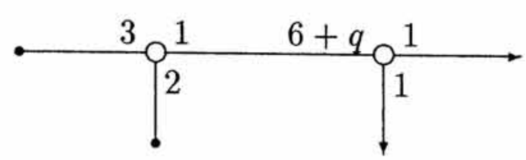
(b) If the EN-diagrams of both links near the splice arrows are as follows:



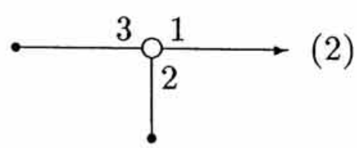
then the inequality $\alpha_0\beta_0 > \alpha_1 \cdots \alpha_r\beta_1 \cdots \beta_s$ must hold.

2.4. We now return to the ideas behind our definition of series. A typical series is the series consisting of $W^{\#}_{1,2q-1}$ and $W^{\#}_{1,2q}$, introduced earlier. Their EN-diagrams are:





It is clear that this is the result of splicing something to



which is precisely the EN-diagram of $W^{\#}_{1,\infty}$: $(y^2-x^3)^2$. In terms of the resolution: take a resolution of $f(x, y) = (y^2-x^3)^2$, and deform the double component slightly into an A_p, we then get a — partial — resolution of one of the $W^{\#}_{1,p}$. In terms of the splice decomposition: Consider the splice decomposition of a representative f_p of $W^{\#}_{1,p}$. It consists of two pieces, one of which is the complement of the link of f, wheras the other depends on the parameter p. This is equivalent to the statement that the Milnor fibration of f_p results from the Milnor fibration of f by removing a tubular neighbourhood of the link component, and replacing it by something else in such a way that the result is the Milnor fibration of f_p and leaving the rest unchanged. We will see later that this process will not give more than only the series $W^{\#}_{1,p}$. The link of f_p is a $(2, 6+p)$-cable on the link of the reduced singularity $f_R(x, y) = y^2 - x^3$, which is a $(2, 3)$-torus knot.

If we have a singularity with more than one double component, we can splice something to each of the components independently. We see this with our example $Y_{r,s}$, its EN-diagram is the result of splicing two pieces (one depending on r and one on s) to $(2) \leftrightarrow (2)$, the EN-diagram of $Y_{\infty,\infty}: x^2y^2$.

If we have a singularity f with a component of multiplicity greater than two, then we can get non-isolated singularities with lower multiplicities when we splice something to it. The simplest example is $f(x, y) = y^3$. In this case, the Milnor fibre consists of three discs. If we want to replace a small tubular neighbourhood of the knot with something else, in such a way that the result is again an algebraic link, we first of all have to take care that the fibres in the solid torus that we put back in, approach the boundary in a $(3, 0)$-torus link. It is intuitively clear that this is only possible with 3 components of multiplicity 1 or with 1 single and 1 double component. Indeed, in 3.8 we will see, that this gives the possibilities E_{6k}, E_{6k+1}, E_{6k+2} and $J_{k,\infty}$, and if we apply the same procedure again to $J_{k,\infty}$, we get the series $J_{k,p}$. In Arnol'd's list we find all these singularities in the series of y^3.

In the Appendix we have included the EN-diagrams of all Arnol'd series, and one sees that they all arise from splicing something to the link of the corresponding non-isolated singularity.

These examples motivate our definition of topological series, which will be presented next.

3. THE DEFINITION OF TOPOLOGICAL SERIES

3.1. *Definition.* Let $f \in \mathscr{O} = \mathbf{C}\{x, y\}$ have a non-isolated singularity. The *topological series* belonging to f consists of all topological types of isolated singularities whose link arise as the splice of the link of f with some other link.

So what we want is that the Milnor fibration of an element of the series differs from that of f only in small neighbourhoods of the components with higher multiplicities.

In terms of EN-diagrams: All arrows in the diagram of f with '(m)' ($m > 1$) in front of them, have to be replaced by subdiagrams with arrows with multiplicity 1 only, taking the splice and algebraicity conditions into consideration. The advantage of using EN-diagrams instead of resolution graphs can be observed here: it is not easy to describe the linear chains that arise in the resolution graphs of the isolated singularity.

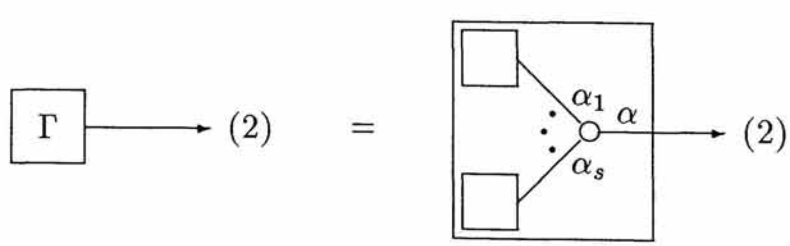
Below, we investigate what possibilities there are to replace an arrow '$\to (m)$' by something else, in the sense of the preceding remarks. It will follow that the topological series do not contain more singularities than we want them to. The method is purely combinatorial. We start with $m = 2$ and end with a formula giving the number of such possibilities.

3.2. *Notation.* If Γ is an EN-diagram, then we denote by $A(\Gamma)$ the set of arrow-heads of Γ, by $N(\Gamma)$ the set of non-arrow-heads (dots and nodes) and by $V(\Gamma) = A(\Gamma) \cup N(\Gamma)$ the set of all vertices.

The corresponding (multi)link is $L = L(\Gamma) = \sum_{i \in A(\Gamma)} m_i S_i$, and for $i \in N(\Gamma)$, S_i will denote the corresponding *virtual* component (cf. [EN]).

3.3. THE CASE OF A DOUBLE COMPONENT.

Suppose $f \in \mathscr{O}$ has link $L = \sum_{i \in A(\Gamma)} m_i S_i$. Suppose one of the components, $S_\diamond$, has multiplicity 2, i.e. $m_\diamond = 2$. Near the arrow $\diamond$, the EN-diagram Γ of L looks like this:



where the boxes may denote anything and the arrow is $\diamond \in A(\Gamma)$ (the second picture is only defined when $\Gamma \neq \bullet\!\!\to (2)$). Define the following numbers:

$$N_0 = \left[\frac{2\alpha_1 \cdots \alpha_s}{\alpha}\right], \text{ where } [\ .\ ] \text{ denotes integral part },$$

$$c = \sum_{j \in A(\Gamma), j \neq \diamond} m_j \operatorname{lk}(S_\diamond, S_j),$$

i.e. c is the linking number of $S_\diamond$ with the other components, counted with their multiplicities.

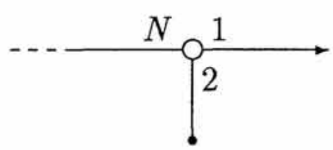
Note that we work with *minimal* EN-diagrams, which means that redundant dots (those attached to a node with weight 1) must be removed by using theorem 8.1 of [EN].

We now show what possibilities there are to replace the double component, in the sense of the remarks at the beginning of this section. Let

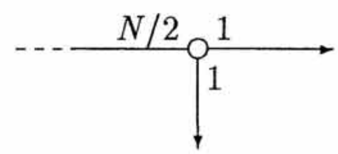
$$\Delta_k = \det(tI - h_*)$$

be the characteristic polynomial of the monodromy on $H_k(F)$, and let $\Delta_* = \Delta_1 / \Delta_0$. This function is related to the zeta function ζ_f of the monodromy (cf. [A'C]) by the relation $\zeta_f(f) = t^{-\chi(F)} \Delta_*(t^{-1})$ (where $\chi(F)$ is the Euler characteristic of F).

3.4. THEOREM. *The only two (classes of) possibilities to replace a double component, are:*



with $N > N_0$ odd



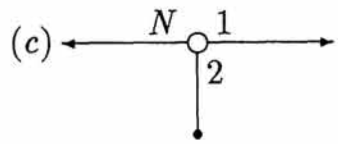
with $N > N_0$ even

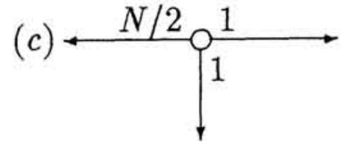
Furthermore, let Δ_*^∞ *be the* Δ_* *of* $L = L(\Gamma)$, *and* Δ_*^N *be the* Δ_* *of the new link. Then we have:*

$$\Delta_*^N(t) = \Delta_*^\infty(t) \cdot \left(t^{N+c} - (-1)^N\right)$$

In particular, the Milnor number is linear in N *with coefficient one.*

Proof. The EN-diagrams of the theorem can be regarded as being the results of splicing the links $L = L(\Gamma) = L(f)$ and those defined by the EN-diagrams Γ'_N in the next figure, along the components $S_\diamond$ and the one with multiplicity c, which we call S'_* with $* \in A(\Gamma'_N)$. (Note that c can be zero, in [EN] this has been given a natural interpretation).





That the multiplicity must be c follows from one half of the splice condition. The other half, $2 = \sum_{h \in A(\Gamma'_N), h \neq *} m_h \operatorname{lk}(S'_*, S'_h)$, implies that these two diagrams are the only two essentially different EN-diagrams with the required property, for we want $m_h = 1$. For the first link the splice condition reads '$2 = 2 \cdot 1$' and for the second '$2 = 1 \cdot 1 + 1 \cdot 1$'.

Finally, the algebraicity condition gives $N > N_0$.

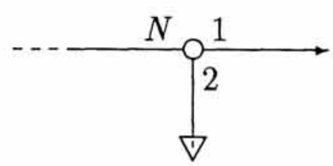
The Δ_* formula follows from [EN], theorem 4.3. □

The last statement of the theorem implies that if L is not the unknot, the Milnor numbers are related as follows:

$$\mu_N = \mu_\infty + N + c \qquad \text{if } F \text{ is connected},$$
$$\mu_N = \mu_\infty + N + c - 1 \quad \text{if } F \text{ is not connected}.$$

We see for example that in case f is of type A_∞, the series is precisely the whole A-series, and in case $f(x, y) = x^2y^2$, the series is the complete doubly indexed Y-series.

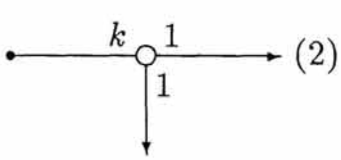
3.5. *Definition.* We combine the two possibilities in one graph, where, depending on whether N is odd or even, the first or the second graph of the theorem must be substituted.



Observe that for N even, this represents the graph with two arrows and edge weight $N/2$.

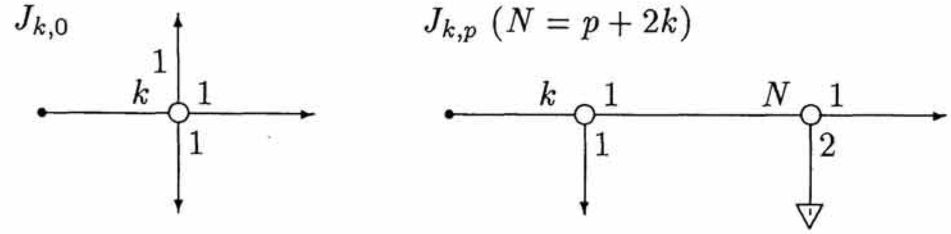
3.6. *Remark.* If $\alpha = 1$ or $\alpha = 2$ (see the figure at the beginning of this section), then the case $N = N_0$ is also allowed, although then the diagram has to be minimized by applying theorem 8.1 of [EN]. The monodromy formula still holds.

3.7. *Example.* $J_{k,\infty}$ has the equation $f(x, y) = y^2(y + x^k)$, its EN-diagram is pictured below:



$$\Delta_1^\infty(t) = (t-1)\frac{t^{3k}-1}{t^3-1}$$

We have $c = k$ and $N_0 = 2k$. The series is $J_{k,p}: y^3 + y^2x^k + x^{3k+p}, p \geqslant 0$. The case $p = 0$ is the special case with $N = N_0$.



We have

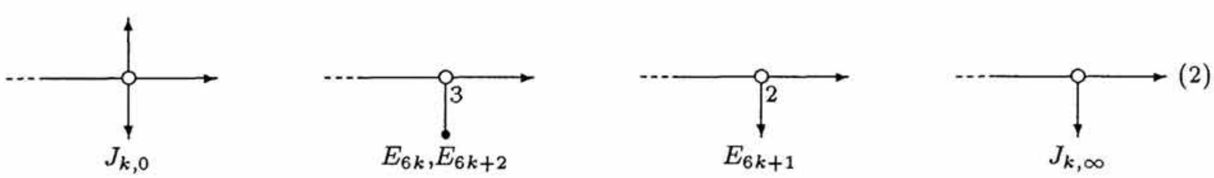
$$\Delta^N(t) = (t-1)\left(t^{N+k} - (-1)^N\right)\frac{t^{3k} - 1}{t^3 - 1}.$$

3.8. HIGHER MULTIPLICITIES.

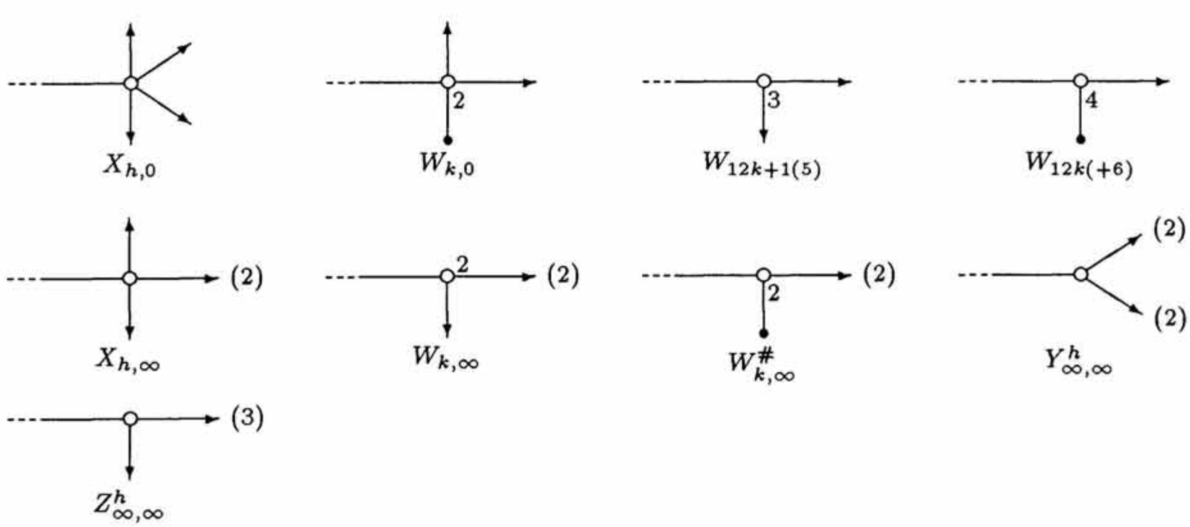
When we have higher multiplicities, exactly the same method can be used. The splice condition gives us always a finite number of links that can be spliced to the component with multiplicity m. We enumerate the possibilities when $m = 3$ and $m = 4$. The names refer to the simplest case when $f(x, y) = y^m$.

In the diagrams, the splice edges have variable weight N, N having no common factor with the other weights. Further omitted edge weights are equal to 1. We only listed the diagrams with one node; some have an arrow of multiplicity greater than 1, which should be treated again.

The four possibilities for $m = 3$:



The nine possibilities for $m = 4$:



We now give a formula giving the number of essentially different diagrams with one node and only multiplicities less than m, that can be spliced to a component of multiplicity m.

PROPOSITION. *The number is:*

$$\sum_{q|m} \mathfrak{p}(m/q) + \sum_{1 \leqslant p \leqslant m-1} \; \sum_{q|(m-p),\, q>1} \mathfrak{p}((m-p)/q) - 1$$

where $\mathfrak{p}(n)$ *is the number of integer partitions of* n.

Proof. In such a diagram at most one dot appears, with at the node a weight $\geqslant 2$. The number of edges emerging from the node must be at least 3. There is at most one weight $\geqslant 1$. These are consequences of the algebraicity condition. The splice condition demands that the total linking number of the other components with the splice component equals m. The formula is now a matter of counting. □

For $m \leqslant 15$ we obtain:

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
number	0	2	4	9	12	22	27	42	54	76	91	134	159	211	263

This can be regarded as an upperbound on the number of symbols (such as A, $W^{\#}$, etc.) needed to give names to all singularities of corank m.

4. THE SPECTRUM OF A PLANE CURVE SINGULARITY

4.1. In this section we compute the spectrum of a plane curve singularity from the EN-diagram and we prove a splice formula for spectra. This will be needed in the next section, where we look at several invariants within a series. First we need to define a number of polynomials.

4.2. We denote by F the Milnor fibre of a plane curve singularity f.

Definition.

$$\Delta_0(t) = \text{char. pol. of } H_0(h): H_0(F) \to H_0(F)\,,$$
$$\Delta_1(t) = \text{char. pol. of } H_1(h): H_1(F) \to H_1(F)\,,$$
$$\Delta_*(t) = \Delta_1(t)/\Delta_0(t) \in \mathbf{Q}(t)$$

Recall that $H_0(F)$ and $H_1(F)$ have ranks d and μ, respectively, where d equals the number of connected components and μ the Milnor number.

We will also need the following polynomials. Let $h_*: H_1(F) \to H_1(F)$ be the algebraic monodromy.

Definition:

(a) Δ^1 is the characteristic polynomial of $h_* | \mathrm{Ker}(h_*^N - 1)$, where N is a common multiple of the order of the eigenvalues of h_*,

(b) Δ' is the characteristic polynomial of $h_* | \mathrm{Im}(H_1(\partial F) \to H_1(F))$.

The roots of Δ^1 are the eigenvalues of the 2×2-Jordan blocks of h_*.

Observe that all polynomials defined above can be obtained easily from the EN-diagram, cf. [EN], section 11 and [Ne].

4.3. The *spectrum* of a holomorphic function germ is a set of rational numbers with integral multiplicities, denoted as $\sum_{\alpha \in \mathbf{Q}} n_\alpha(\alpha)$ (an element of the free abelian group on $\mathbf{Q}$), which can be regarded as logarithms of the eigenvalues of the algebraic monodromy.

In the isolated singularity case we have that $\Delta_1(t) = \prod_\alpha (t - \exp(2\pi i \alpha))^{n_\alpha}$. In the case of plane curve singularities, the spectrum numbers α satisfy $-1 < \alpha < 1$, so for each eigenvalue $\lambda \neq 1$ there are two possible α's with $\lambda = \exp(2\pi i \alpha)$.

4.4. We follow [St] for a brief description of the spectrum. For details we refer to this source. Let $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be non-zero holomorphic function germ, and denote by F its Milnor fibre. The reduced cohomology groups $H^*(F) = H^*(F; \mathbf{C})$ carry a canonical mixed Hodge structure. The semi-simple part T_s of the monodromy acts as an automorphism of this mixed Hodge structure, and in particular it preserves the Hodge filtration $\mathscr{F}$. Write $\mathrm{Gr}^p_{\mathscr{F}} = \mathscr{F}^p / \mathscr{F}^{p+1}$, and let s_p be the dimension of $\mathrm{Gr}^p_{\mathscr{F}}$. There are rational numbers α_{pj} with $1 \leqslant j \leqslant s_p$, $n - p - 1 < \alpha_{pj} \leqslant n - p$ such that

$$\det(t \cdot \mathrm{Id} - T_s; \mathrm{Gr}^p_{\mathscr{F}}) = \prod_{j=1}^{s_p} (t - \exp(-2\pi i \alpha_{pj}))$$

Now we define $\mathrm{Sp}_n(H^k(F; \mathbf{C}), \mathscr{F}, T_s) = \sum_p \sum_j (\alpha_{pj})$ and:

$$\mathrm{Sp}(f) = \sum_{k=0}^{n} (-1)^{n-k} \mathrm{Sp}_n(H^k(F), \mathscr{F}, T_s)$$

It is clear that the spectrum is a finer invariant than the characteristic polynomial. Steenbrink has proved for instance that the spectrum distinguishes all quasi-homogeneous isolated singularities (not only curves). But already for plane curves the spectrum is not a complete invariant of the topological type. Details of these facts can be found in [SSS].

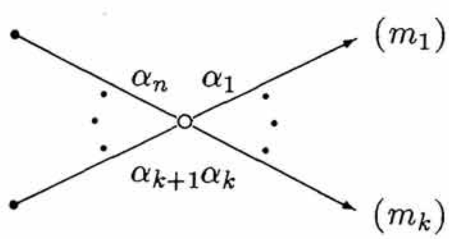
4.5. *Example.* Consider $f(x, y) = xy(y^2 - x^3)$ and $g(x, y) = xy(y - x^5)$. Then f and g have the same integral monodromy (see [MW]), their characteristic polynomial is $\Delta_1 = (t-1)(t^{11}-1)$. But

$$\mathrm{Sp}(f) = \sum_{i \in \{0,1,2,3,4,6\}} \left(-\frac{i}{11}\right) + \left(\frac{i}{11}\right)$$

$$\mathrm{Sp}(g) = \sum_{i \in \{0,1,2,3,4,5\}} \left(-\frac{i}{11}\right) + \left(\frac{i}{11}\right)$$

4.6. In [LS] a method is given to compute the spectrum of a reduced curve singularity from the resolution graph. However, the non-reduced case follows by the same methods. The results are closely related to those of Neumann on the equivariant signatures of the isometric structure on $H_1(F; \mathbf{C})$ given by the monodromy and the sesquilinearized Seifert form, see [Ne]. Below we combine the results of [LS] and [Ne] to obtain a purely topological method to compute the spectrum.

For a root of unity λ the signature σ_λ^- is defined in [Ne] and computed as the sum of the σ_λ^- of all the splice components. Consider a (very general) splice component:



For the moment, put $m_i = 0$ for $i \in \{k+1, \ldots, n\}$; so

$$m = \sum_j \alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n m_j$$

is the multiplicity of the central node. Choose integers $\beta_j (1 \leqslant j \leqslant n)$ with $\beta_j \alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n \equiv 1 \pmod{\alpha_j}$ and put $s_j = (m_j - \beta_j m)/\alpha_j$.

Remark. The numbers s_j are, modulo m, equal to the multiplicities of the neighbour vertices in the resolution graph.

For a real number x, let $\{x\}$ be the fractional part of x, and let

$$((x)) = \begin{cases} \frac{1}{2} - \{x\} & \text{if} \quad x \notin \mathbf{Z} \\ 0 & \text{if} \quad x \in \mathbf{Z} \end{cases}$$

4.7. PROPOSITION. *Write* $\lambda = \exp(2\pi ip/q)$ *with g.c.d.* $(p, q) = 1$. *Then we have (see Neumann [Ne]):*

$$\sigma_\lambda^- = \begin{cases} 0 & \textit{if} \quad q \textit{ does not divide } m, \\ 2\sum_{i=1}^n ((s_i p/q)) & \textit{if} \quad q \textit{ divides } m. \end{cases}$$ □

4.8. For λ a root of unity, let $b_{0,\lambda}$, b_λ, b_λ^1, b_λ' be the multiplicities of λ as a root of Δ_0, Δ_1, Δ^1, Δ', respectively (these polynomials have been defined in section 4.2) Let σ_λ^- be the signature as computed above. Write $e(\alpha) = \exp(2\pi i\alpha)$. $\mathrm{Sp}(f)$ denotes the spectrum of f.

THEOREM. $\mathrm{Sp}(f) = \sum n_\alpha(\alpha)$ *with:*

$$n_\alpha = \begin{cases} (b_{e(\alpha)} + b'_{e(\alpha)} - \sigma^-_{e(\alpha)})/2 & \textit{if} \quad -1 < \alpha < 0 \\ r - 1 \ (r = \#\ \textit{branches}) & \textit{if} \quad \alpha = 0 \\ (b_{e(\alpha)} - b'_{e(\alpha)} + \sigma^-_{e(\alpha)})/2 - b_{0,e(\alpha)} & \textit{if} \quad 0 < \alpha < 1 \end{cases}$$

Proof. The proposition is a translation of the results of [LS], extended to the case of non-reduced singularities. The difference with [LS] is, that the roots of Δ', coming from the boundary, must be added to the weight one part, and the roots of Δ_0 must be subtracted from the weight zero part. In the language of [Ne]: The Γ_λ and the $-\Lambda_\lambda^1$ part contribute to the negative (weight 1) spectrum numbers, the Λ_λ^1 part contributes to the positive (weight 0) spectrum numbers. The pairs of eigenvalues in the 2×2-Jordan blocks are evenly distributed among the positive and negative parts. The roots of Δ_0 give only weight 0 spectrum numbers and they have negative multiplicity. □

4.9. A point which may cause confusion is the fact that in the definition of spectrum *reduced* (co)homology is used. Therefore we define $\mathrm{Sp}_*(f) = \mathrm{Sp}(f) - (0)$. It is now possible to compare Sp_* with Δ_*: If $\mathrm{Sp}_*(f) = \sum_\alpha n_\alpha(\alpha)$, then $\Delta_*(t) = \prod_{\alpha \in \mathbf{Q}} (t - e(\alpha))^{n_\alpha}$.

Example. The A_∞ singularity has $\mathrm{Sp}_* = -\left(\frac{1}{2}\right) - (0)$. Recall that its Δ_* equals $(t^2 - 1)^{-1}$. D_∞ has spectrum $\mathrm{Sp} = (0)$, so $\mathrm{Sp}_* = 0$ ('empty'). Let $f(x, y) = (y^2 - x^3)(y^3 - x^2)$ be the A'Campo singularity. Then:

$$\mathrm{Sp}_*(f) = \left(-\frac{1}{2}\right) + 2\left(-\frac{3}{10}\right) + 2\left(-\frac{1}{10}\right) + 2\left(\frac{1}{10}\right) + 2\left(\frac{3}{10}\right) + \left(\frac{1}{2}\right).$$

As with all isolated singularities, this spectrum is symmetrical (i.e. if (α) is in the spectrum, then so is $(-\alpha)$). This is not the case with non-isolated singularities. The asymmetry comes from the fact that the Milnor fibre can have more than one connected component and from the fact that the monodromy possibly acts non-trivially on the boundary of F. Both can be seen in:

$$\mathrm{Sp}_*(x^2 y^2) = \left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right).$$

Observe that the Δ_* of $x^2 y^2$ is just 1, as with D_∞.

4.10. The Δ_* behaves well under splicing: it is the product of the Δ_* of the splice components. Our topological way of looking at spectra asks for a formula of splicing spectra. It appears that $\mathrm{Sp}_* = \mathrm{Sp} - (0)$ is *almost* additive.

Example. In the example above we computed the spectrum of the A'Campo singularity. Both splice components are isomorphic to that of the non-isolated singularity $x^2(y^2 - x^3)$, which has spectrum:

$$\mathrm{Sp}_* = \left(-\frac{1}{2}\right) + \left(-\frac{3}{10}\right) + \left(-\frac{1}{10}\right) + \left(\frac{1}{10}\right) + \left(\frac{3}{10}\right).$$

So we have to add both spectra, but instead of $2\left(-\frac{1}{2}\right)$ we have $\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)$. This is the result of the new edge in the EN-diagram, giving a new 2×2-block.

4.11. THEOREM. *Let L be the result of splicing L' and L'' along components S' and S'', respectively. Let $m'(m'')$ be the multilink multiplicity of $S'(S'')$ and put $q = \mathrm{g.c.d.}(m', m'')$. Then*

$$\mathrm{Sp}_*(L) = \mathrm{Sp}_*(L') + \mathrm{Sp}_*(L'') + \sum_{i=1}^{q=1} (i/q) - (-i/q).$$

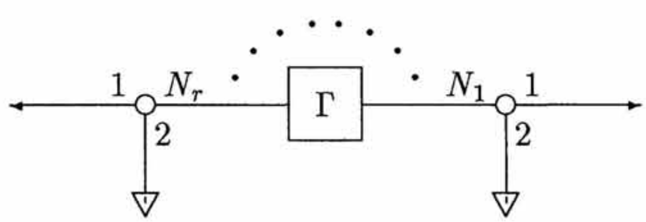
Proof. If $q = 1$ the theorem is clear. Now suppose $q > 1$. Consider the behaviour of the polynomials Δ_0, Δ^1 and Δ' under this splice operation. Splicing introduces a new edge E which contributes to Δ^1 with a factor $t^q - 1$. This introduces new 2×2-Jordan blocks. Both splice components have $\sum_{i=1}^{q-1} \left(-\frac{i}{q} \right)$ in their spectrum (coming from Δ'). But, as both eigenvalues in a 2×2-block are of different weight, L has $\sum_{i=1}^{q-1} \left(-\frac{i}{q} \right) + \left(\frac{i}{q} \right)$ instead of the sum of both parts. It is clear from theorem 4.8 that all other parts of the spectra of L' and L'' have to be added. □

5. Invariants in the case that f has only transversal A_1 singularities

In this section we describe the topology and equation of a topological series that belongs to a non-isolated singularity with only transversal A_1 singularities.

Throughout this section, $f \in \mathscr{O}$ is of the form $f = f_1^2 \cdots f_r^2 g$, with $f_1, \ldots, f_r$ irreducible and g reduced. The critical set of f is $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$, and the transverse type of f along Σ_i is A_1. For all $i \in \{1, \ldots, r\}$, we have numbers N_{0i} and c_i as defined in section 3.3. Let $N_i > N_{0i}$ $(1 \leqslant i \leqslant r)$. According to theorem 3.4, a typical element of the series belonging to f has the topological type (EN-diagram) Γ^*:

$\Gamma^* =$



That is: each arrow of the EN-diagram Γ of f belonging to a double component, is replaced in the way described in theorem 3.4. So varying the N_i will give us the complete series belonging to f.

The following two propositions are easy consequences of theorem 3.4. Let $N = (N_1, \ldots, N_r)$ and let f_N have topological type Γ^*.

5.1. Proposition. *Let* $\Delta_*[f]$ *and* $\Delta_*[f_N]$ *be the* Δ_* *of* f *and* f_N *respectively. Then:*

$$\Delta_*[f_N](t) = \Delta_*[f](t) \cdot \prod_{i=1}^{r} \left(t^{N_i + c_i} - (-1)^{N_i} \right) . \qquad \square$$

5.2. PROPOSITION. *Let* μ_∞ *be the Milnor number of* f *and* μ_N *that of* f_N. *Let* $\mu_0 \in \{1, 2\}$ *be the number of connected components of the Milnor fibre of* f. *Then:*

$$\mu_N = \mu_\infty - \mu_0 + 1 + \sum_{i=1}^{r} (N_i + c_i) .$$

The numbers μ_∞, μ_0 *and* c_i $(1 \leqslant i \leqslant r)$ *depend only on* f. □

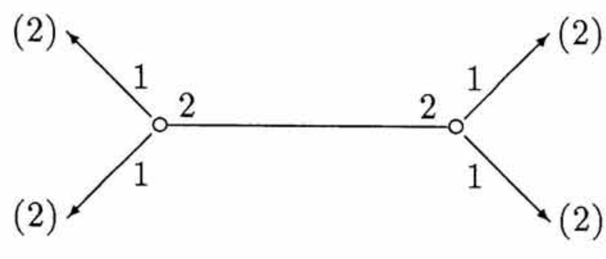
5.3. We conclude this list of topological invariants with the formula of the spectrum of a series (see section 4).

PROPOSITION. *Define for* $1 \leqslant i \leqslant r$: $\gamma_i = 0$ *if* c_i *is even and* $\gamma_i = 1/2$ *when* c_i *is odd. Write* $\nu_i = N_i + c_i$. *Then:*

$$\mathrm{Sp}(f_N) = \mathrm{Sp}(f) + \sum_{i=1}^{r} \sum_{j=0}^{\nu_i - 1} \left(\frac{1}{2} - \frac{\gamma_i + j}{\nu_i} \right)$$

Proof. One can use the proof of [St], theorem 4.5, but it is also possible to work out the various cases using the method of section 4. For the proof of [St], the following observation is needed. Let F_i' be a transversal slice transverse to Σ_i (in this case F_i' consists of two points — the transverse type is A_1). Let $T_i: H_0(F_i') \to H_0(F_i')$ be the monodromy of the local system over the punctured disc $\Sigma_i \backslash \{0\}$. Then it is well-known that T_i is the identity if c_i is even and $-$ identity if c_i is odd. In fact, even if the transversal type is not A_1, the following holds. Let $t_i: H_0(F_i') \to H_0(F_i')$ be the Milnor fibration monodromy of f restricted to a transversal slice through $x \in \Sigma_i$. Then t_i is a cyclic permutation of the finite number of points in F_i', and $T_i = t_i^{-c_i}$. □

Example. Let $f(x, y) = (y^2 - x^4)^2 (x^2 - y^4)^2$; its EN-diagram is:



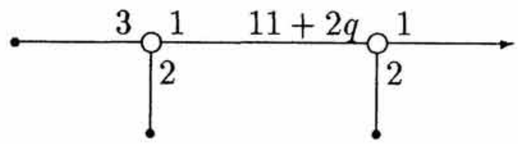
Observe that according to the proposition, the spectrum of f_N is independent of the order of $N_1, \ldots, N_4$. If we take $N = (5, 5, 6, 6)$ and $N = (5, 6, 5, 6)$ we get the same spectrum but different topological types, because the EN-diagrams are not equivalent. This is the counterexample to the spectrum conjecture found by Steenbrink and Stevens, cf. [SSS].

6. Equations

In this section we discuss the equations of series: what do we have to add to f to obtain a required element of its series?

In the example $W^{\#}$ at the beginning of section 4, we had:

$W^{\#}_{1,2q-1} : (y^2 - x^3)^2 + x^{4+q}y$



The Puiseux expansion of $W^{\#}_{1,\infty}$: $f(x, y) = (y^2 - x^3)^2$, is $x = t^2, y = t^3$. When we substitute this in $x^{4+q}y$, we get t^{11+2q}, which is just the number N in the EN-diagram.

More generally, it appears that adding $\varphi \in \mathcal{O}$ with $\varphi(t^2, t^3)$ of order $11 + 2q$, gives the same result, although there are various kinds of exceptions.

In theorem 6.5 below, we give conditions on φ such that $f + \varepsilon\varphi$ has the required type, where ε is introduced in order to fulfil transversality properties. This avoids exceptional cases such as when $f(x, y) = y^2$ and $\varphi(x, y) = 2x^k y + x^{2k}$, the sum is then a non-isolated singularity.

Again, f has only transversal A_1 singularities; but the following lemma is valid in greater generality.

6.1. Lemma. *Let $f, \psi \in \mathcal{O}$ and assume f has a non-isolated singularity. If for all small $\varepsilon > 0$, $f + \varepsilon\psi$ has a singularity topologically equivalent to f, then for almost all ε the zero sets of f and $f + \varepsilon\psi$ are equal.*

Proof. First take $f(x, y) = y^n$ with $n > 1$. Assume that for no ε, f and $f + \varepsilon\psi$ have the same zero set. Then we may assume $f + \varepsilon\psi = \big(y + F(x, \varepsilon)\big)^n$ where $F(x, \varepsilon) \not\equiv 0$, regarded also as a function, of ε, can be written as

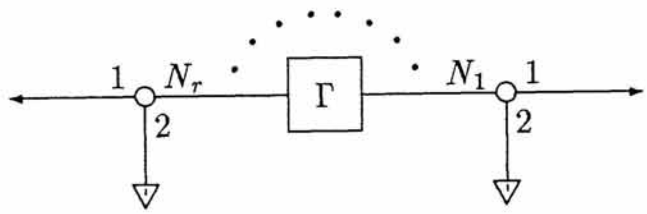
$$F(x, \varepsilon) = \sum_{i>0} a_i(\varepsilon) x^i .$$

Here $a_i(\varepsilon)$ may have positive fractional powers of ε. $f + \varepsilon\psi$ is linear in ε. By writing out the equation

$$\frac{\partial}{\partial \varepsilon}(y + F(x, \varepsilon))^n = 0 ,$$

one immediately sees that this impossible. If f is not of the form y^n then there always is a small neighbourhood away from the origin where it is. There we can apply the above argument. □

6.2. Let $f = f_1^2 \cdots f_r^2 g$ have EN-diagram Γ, and let $N_i > N_{0i}$ and c_i be defined as usual. We are looking for φ with the property that $f + \varepsilon\varphi$ has the topological type of EN-diagram $\Gamma^* = \Gamma^*(N_1, \ldots, N_r)$:

$\Gamma^* =$ 

By Puiseux's Theorem [Ph], we can choose coordinates x, y of $\mathbf{C}^2$ in such a way that the Puiseux expansions of the $\Sigma_i, (1 \leqslant i \leqslant r)$ have the form:

$$\begin{cases} x = t^{n_i} \\ y = \eta_i(t) = \sum_{k \geqslant 1} c_{ik} t^k \end{cases}$$

For each i we have the valuation function υ_i: $\mathscr{O} \to \mathbf{N} \cup \{\infty\}$ given by

$$\upsilon_i(\varphi) = \operatorname{ord}_t \varphi\big(t^{n_i}, \eta_i(t)\big) = \dim_{\mathbf{C}} \mathscr{O}/(f_i, \varphi) .$$

After considering various examples, one is tempted to think that whenever for all i, $\upsilon_i(\varphi) = N_i + c_i$, $f + \varepsilon\varphi$ has, for general ε, the required topological type given by EN-diagram $\Gamma^*(N_1, \ldots, N_r)$. The following example shows that this is not true. Take $f(x, y) = y^2$, and $\varphi(x, y) = x^k y + x^N$. Although $\upsilon(\varphi) = N$, the topological type is determined by k and not by N when $2k < N$. So we have to take care of low order multiples of f. We will do this by considering υ and an extra valuation $\upsilon^{(2)}$.

6.3. *Definition.* Consider $h = h_{\text{red}}^2$, where $h_{\text{red}} \in \mathscr{O}$ is irreducible with Puiseux expansion $x = t^n, y = \sum a_i t^i$. Let β be the largest characteristic exponent. For $\alpha \in \mathbf{C}, n \in \mathbf{N}$, define $w_{\alpha,N}$: $\mathscr{O} \to \mathbf{N} \cup \{\infty\}$ by:

$$w_{\alpha,N}(\varphi) = \operatorname{ord}_\tau\big(\varphi(\tau^{2n}, \sum a_i \tau^{2i} + \alpha\tau^{2\beta + N - N_0})\big) .$$

Finally, define $\upsilon^{(2)}$: $\mathscr{O} \to \mathbf{N} \cup \{\infty\}$ by:

$$\upsilon^{(2)}(\varphi) = \begin{cases} \min_{\alpha \neq 0} w_{\alpha,\upsilon(\varphi)}(\varphi) & \text{if} \quad \upsilon(\varphi) < \infty\ , \\ \min_{\alpha \neq 0} w_{\alpha,2\upsilon(\varphi/h)}(\varphi) & \text{if} \quad \upsilon(\varphi) = \infty\ . \end{cases}$$

Notice that $\upsilon^{(2)}(\varphi) = \infty \Leftrightarrow \varphi \in (h)$. If $N - N_0$ is odd, the number $\upsilon^{(2)}(\varphi)$ is equal to the intersection number of φ with some curve which has as its Puiseux pairs the Puiseux pairs of h_{red} with one extra pair, $(N-N_0, 2)$, added.

6.4. *Example.* Take $f(x, y) = y^2$ and $\varphi(x, y) = x^k y + x^N$. Then $\upsilon(\varphi) = N$ and $\upsilon^{(2)}(\varphi) = \min\{2k + N, 2N\}$. Observe that the type of f is A_{m-1} with $m = \upsilon^{(2)}(\varphi) - \upsilon(\varphi)$.

6.5. Let $f = f_1^2 \cdots f_r^2 g$ be as above. For $1 \leqslant i \leqslant r$ we now have valuations $\upsilon_i^{(2)}$ as in the preceding definition. Recall $\Gamma^*(N_1, \ldots, N_r)$ is obtained from the EN-diagram Γ of f by replacing all multiple arrows as in the last picture.

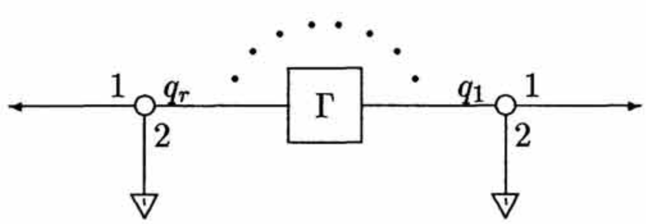
THEOREM. *Suppose* $\varphi \in \mathscr{O}$ *satisfies* $\upsilon^{(2)}(\varphi) = \upsilon_1^{(2)}(\varphi) \cdots \upsilon_r^{(2)}(\varphi) < \infty$. *Then* $f + \varepsilon\varphi$ *has, for almost all* $\varepsilon \neq 0$, *the topological type given by EN-diagram* $\Gamma^*(N_1, \ldots, N_r)$, *with for* $1 \leqslant i \leqslant r$*:*

(a) $N_i = \upsilon_i^{(2)}(\varphi) - \upsilon_i(\varphi) - c_i$ *if* $\upsilon_i(\varphi) < \infty$, *or*

(b) $N_i = 2\upsilon_i(\varphi/f_i) - c_i$ *if* $\upsilon_i(\varphi) = \infty$,

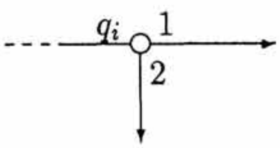
provided that $N_i > N_{i0}$ *for all* i.

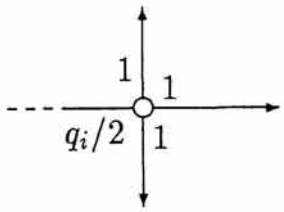
Proof. Since the order of $\varphi|\Sigma_i$ is $> N_{0i} + c_i$ and ε is general (use lemma 6.1), the good minimal resolution of $f + \varepsilon\varphi$ also resolves the singularities of f. So the EN-diagram of f is a subdiagram of that of $f + \varepsilon\varphi$. Hence, according to theorem 3.4, $f + \varepsilon\varphi$ has the EN-diagram:

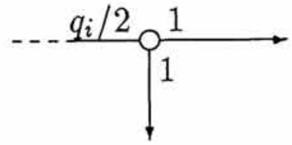


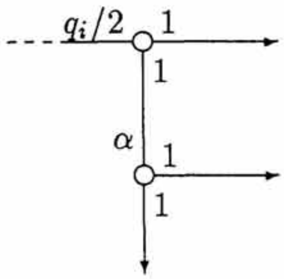
for certain numbers q_i, $(1 \leqslant i \leqslant r)$. It remains to prove that q_i equals the number N_i stated in the theorem.

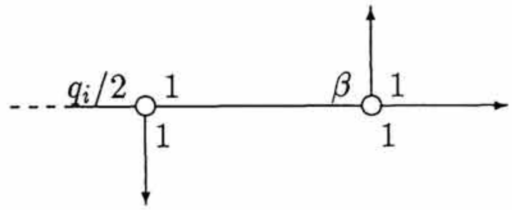
For this purpose we consider one specific i at the time, and draw f_i in the same picture as $f + \varepsilon\varphi$. That is, we draw the EN-diagram of their product, unless f_i happens to be a branch of $f + \varepsilon\varphi$. Using an argument analogous to the one presented in section 3.3 (which provided the two possible extensions to the EN-diagram), we conclude that the situation near f_i is as in one of the five following cases:











In each picture, the arrow pointing downwards represents f_i. Observe that when f_i is removed (replaced by a dot) we get back the situation of the original picture as it should. We now compute q_i in each case, using the interpretation of the valuations as intersection numbers with f_i. Recall that they can be computed by walking from arrow to arrow in the EN-diagram, see [EN], section 10. To clarify matters, we explain in each case the local situation as follows. In the resolution of f we take suitable local coordinates u, v near the strict transform of the branch f_i in such a way that $f_i^2 = v^2$ and that the branches of $f + \varepsilon\varphi$ near f_i have the form mentioned.

Pictures # 1, # 2 and # 5: One computes $v_i(\varphi) = q_i + c_i$ and $v_i^{(2)} = 2q_i + 2c_i$. Therefore $q_i = N_i$. In picture # 1, q_i is odd and in picture # 2 even. In both cases the local situation is $v(v^2 + u^s)$ with $s = q_i - N_{i0}$. In picture # 5, the two branches have intersection number $\beta > q_i/2$ with each other.

Picture # 3: One computes $v_i(\varphi) = \infty$ and $v_i(\varphi/f_i) = q_i/2 + c_i$. Therefore $q_i = N_i$. The local situation is $v(v + u^{s/2})$ with s as before.

Picture # 4: One computes $v_i(\varphi) = q_i/2 + \alpha + c_i$ and $v_i^{(2)}(\varphi) = 3q_i/2 + \alpha + 2c_i$. Again we obtain that q_i equals the number N_i of the theorem. The local situation is $v(v^2 + u^{s/2}v + u^{\alpha + s/2})$. □

6.6. *Remark.* We want to point out at this point that it is easy to find a φ satisfying the condition. One can use the method of [EN], pages 57-58. An interesting observation is, that in general the monomials of φ themselves will have a smaller order in t than φ.

6.7. THE CASE THAT f IS ARBITRARY.

If $f = f_1^{m_1} \cdots f_r^{m_r} g$ with f_i irreducible, $m_i \geqslant 2$ and g reduced, we still have that $f + \varepsilon\varphi$ has the diagram of f with the multiple arrows replaced. We know exactly which replacements are possible (see section 3.8). To find out what is the type of $f + \varepsilon\varphi$, it again suffices to investigate linking behaviour. Some possibilities that only become apparent when f_i and $f + \varepsilon\varphi$ are drawn in one diagram (that is the diagram of their product), have to be opted out by considering linking with cables which are known to be correct, using such valuations as $\upsilon^{(2)}$.

Although the tests become increasingly difficult, this gives a way to generalize theorem 6.5.

6.8. IOMDIN TYPE SERIES.

We end with a remark on series of the form $f + \varepsilon l^k$, where l is a linear form not tangent to any branch of f and $k \geqslant k_0$, the largest polar ratio of f. These series have been studied by Iomdin and Lê, see [Lê], not only in the curve case but for general dimensions. Siersma [Si] has given a formula for the Δ_* of these series. In the curve case this is just a special case of our results. Notice that:

$$\upsilon_i(l) = d_i k \quad \text{where} \quad d_i = e_l(\Sigma_i) = \Sigma_i \cdot l \,,$$
$$\upsilon_i^{(2)}(l) = 2d_i k \,.$$

We would like to stress again that these Iomdin type series are generally much coarser than our topological series: they are single indexed and for example the Milnor number increases with steps of $d = d_1 + \cdots + d_r$ within the series.

APPENDIX

In this appendix the EN-diagrams of the series of plane curve singularities listed in [AGV] are drawn.

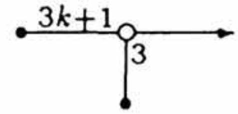
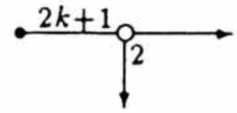
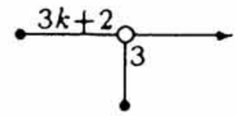
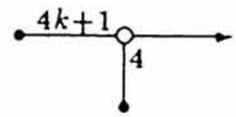
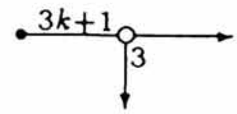
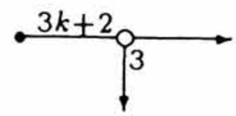
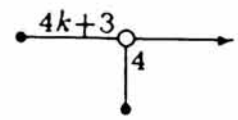
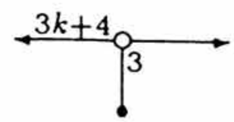
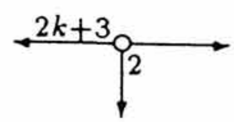
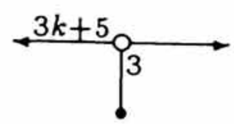
The first part consists of the *exceptional families E, W* and *Z.*

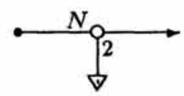
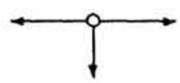
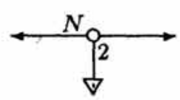
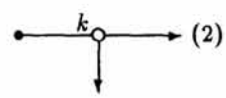
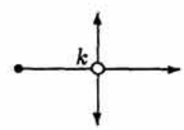
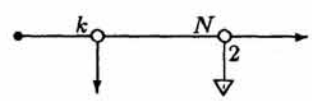
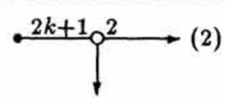
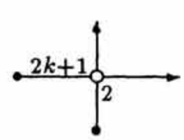
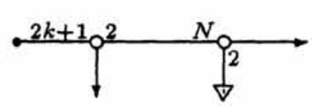
The second part contains the *infinite series A, D, J, W, $W^{\#}$, X, Y* and *Z.* All variants are given. In the tables, we have that:

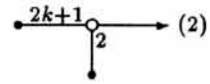
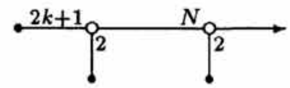
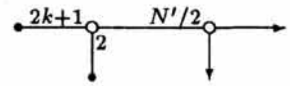
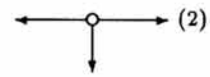
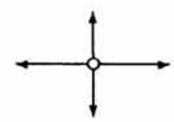
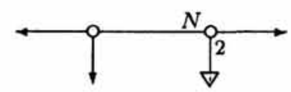
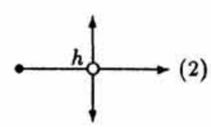
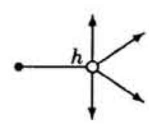
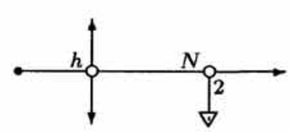
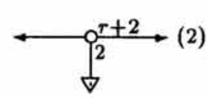
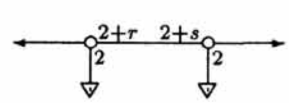
(a) μ = the Milnor number;

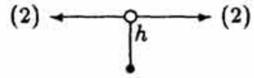
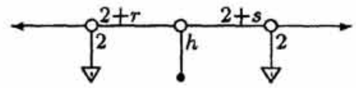
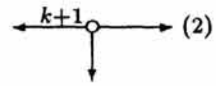
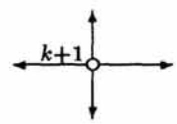
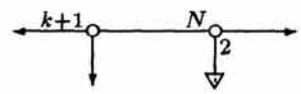
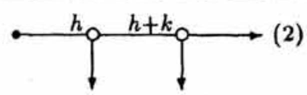
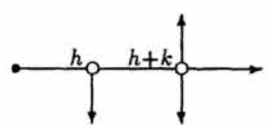
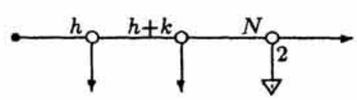
(b) N_0 and the graph constant c are as in theorem 3.4;

(c) Δ_*^∞ is the Δ_* of the non-isolated singularity, the Δ_* of an element of the series can be obtained by multiplying with $t^{N \cdot c} - (-1)^N$.

Name	Formula	μ	EN-diagram
E_{6k}	$y^3 + x^{3k+1}$	$6k$	
E_{6k+1}	$y^3 + x^{2k+1}y$	$6k+1$	
E_{6k+2}	$y^3 + x^{3k+2}$	$6k+2$	
W_{12k}	$y^4 + x^{4k+1}$	$12k$	
W_{12k+1}	$y^4 + yx^{3k+2}$	$12k+1$	
W_{12k+5}	$y^4 + yx^{3k+2}$	$12k+5$	
W_{12k+6}	$y^4 + yx^{3k+3}$	$12k+6$	
Z_{6k+11}	$x(y^3 + yx^{2k+3} + x^{3k+4})$	$6k+11$	
Z_{6k+12}	$x(y^3 + yx^{2k+3} + x^{3k+5})$	$6k+12$	
Z_{6k+13}	$x(y^3 + yx^{2k+4} + x^{3k+5})$	$6k+13$	

Name	Formula	μ	EN-diagram	
A_∞	y^2	0		$\Delta_*^\infty = \dfrac{1}{t^2-1}$
A_0	y	0		$p \geq 2$, $N = p+1$,
A_1	$y^2 + x^2$	1		$N_0 = 1$, $c = 0$
A_p	$y^2 + x^{p+1}$	p		
D_∞	xy^2	1		$\Delta_*^\infty = 1$
D_4	$xy^2 + x^3$	4		$p \geq 5, N = p-2,$ $N_0 = 2, c = 1$
D_p	$xy^3 + x^{p-1}$	p		
$J_{k,\infty}$	$y^3 + x^k y^2$	$3k-2$		$\Delta_*^\infty = \dfrac{t^{3k}-1}{t^3-1}$
$J_{k,0}$	$y^3 + x^k y + x^{3k}$	$6k-2$		$k \geq 2, p \geq 1, c = k$ $N = p+2k, N_0 = 2k$
$J_{k,p}$	$y^3 + x^k y^2 + x^{3k+p}$	$6k-2+p$		
$W_{k,\infty}$	$y^4 + y^2 x^{2k+1}$	$8k+1$		$\Delta_*^\infty = \dfrac{t^{8k+4}-1}{t^4-1}$
$W_{k,0}$	$y^4 + y^2 x^{2k+1} + x^{4k+2}$	$12k+3$		$k \geq 1, p \geq 1,$ $N = p + 2k + 1$
$W_{k,p}$	$y^4 + y^2 x^{2k+1} + x^{4k+2+p}$	$12k+3+p$		$N_0 = 2k+1,$ $c = 2k+1$

Name	Formula	μ	EN-diagram	
$W^{\#}_{k,\infty}$	$(y^2+x^{2k+1})^2$	$4k$		$\Delta_*^\infty = \dfrac{t^{4k+2}+1}{t^4-1}$
$W^{\#}_{k,2q-1}$	$(y^2+x^{2k+1})^2+yx^{3k+1+q}$	$12k+2q+2$		$k\geq 1, q\geq 1,$ $c=0$
$W^{\#}_{k,2q}$	$(y^2+x^{2k+1})^2+y^2x^{2k+1+q}$	$12k+2q+3$		$N=8k+2q+3$ $N'=8k+2q+4$
X_∞	$y^4+x^2y^2$	5		$\Delta_*^\infty = t^4-1$
X_9	$y^4+x^2y^2+x^4$	9		$p\geq 10, N=p-7,$ $N_0=2, c=2$
X_p	$y^4+x^2y^2+x^{4+p-9}$	p		
$X_{h,\infty}$	$y^4+x^hy^3+x^{2h}y^2$	$8h-3$		$\Delta_*^\infty = \dfrac{(t^{4h}-1)^2}{t^4-1}$
$X_{h,0}$	$y^4+x^hy^3+x^{2h}y^2+x^{3h}y$	$12h-3$		$h\geq 2, p\geq 1,$ $N=p+2h,$ $N_0=2h, c=2h$
$X_{h,p}$	$y^4+x^hy^3+x^{2h}y^2+x^{4h+p}$	$12h-3+p$		
$Y_{\infty,\infty}$	x^2y^2	4		$\Delta_*^{\infty,\infty}=1$
$Y_{r,\infty}$	$y^{4+r}+x^2y^2$	$r+5$		$r,s\geq 1,$ $c_1=c_2=2$
$Y_{r,s}$	$y^{4+r}+x^2y^2+x^{4+s}$	$9+r+s$		

Name	Formula	μ	EN-diagram	
$Y^h_{\infty,\infty}$		$4h-2$		$h \geq 2,\ r, s \geq 1$
$Y^h_{r,s}$	See [AGV], p. 248	$12h+r+s-3$		
$Z_{k,\infty}$	$xy^3 + x^{k+2}y^2$	$3k+5$		$\Delta_*^\infty = t^{3k+4} - 1$
$Z_{k,0}$	$xy^3 + x^{k+2}y^2 + x^{3k+4}$	$6k+9$		$k, p \geq 1, c = k+2$ $N = p + 2k + 2,$ $N_0 = 2k + 2$
$Z_{k,p}$	$xy^3 + x^{k+2}y^2 + x^{3k+4+p}$	$6k+9+p$		
$Z^h_{k,\infty}$		$8h+3k-3$		$\Delta_*^\infty = \dfrac{(t^{4h}-1)(t^{4h+3k}-1)}{t^4 - 1}$
$Z^h_{k,0}$	See [AGV], p. 249	$12h+6k-3$		$h \geq 2,\ k, p \geq 1,$ $N = p + 2h + 2k,$ $N_0 = 2h + 2k,$ $c = 2h + k$
$Z^h_{k,p}$	See [AGV], p. 249	$12h+6k-3+p$		

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