Rigid Dualizing Complexes via Differential Graded Algebras

Lecture Notes [1]

Amnon Yekutieli
Ben Gurion University, ISRAEL
http://www.math.bgu.ac.il/∼amyekut

Here is the plan of my lecture:

1. Dualizing Complexes: Overview
2. Rigid Complexes and DG Algebras
3. Properties of Rigid Complexes
4. Rigid Dualizing Complexes
5. Rigid Complexes and CM Homomorphisms

1. Dualizing Complexes: Overview

Let A be a noetherian commutative ring. Denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ the derived category of bounded complexes of A-modules with finitely generated cohomology modules.

Definition 1. (Grothendieck [RD]) A *dualizing complex* over A is a complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ satisfying the two conditions:

(i) R has finite injective dimension.
(ii) The canonical morphism $A \to \mathrm{RHom}_A(R, R)$ is an isomorphism.

Condition (i) means that there is an integer d such that $\mathrm{Ext}^i_A(M, R) = 0$ for all $i > d$ and all modules M.

Example 2. If $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension (say a field, or the ring of integers $\mathbb{Z}$) then

$$R := \mathbb{K} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, \mathbb{K})$$

is a dualizing complex.

Dualizing complexes over commutative rings are part of Grothendieck's duality theory in algebraic geometry, which was developed in [RD]. This duality theory deals with dualizing complexes on schemes and relations between them.

In this lecture I will explain a new approach to dualizing complexes over commutative rings, due to James Zhang and myself (see [YZ4] and [YZ5]). Specifically, I'll talk about existence and uniqueness of *rigid dualizing complexes*.

The purpose of rigidity is to eliminate automorphisms, and to make the dualizing complexes functorial.

[1] written: 14 Jan 2007

In a sequel paper [Ye2] we use the technique of *perverse coherent sheaves* to construct rigid dualizing complexes on schemes, and we reproduce almost all of the geometric Grothendieck duality theory. But that's a subject for a separate lecture.

Related work in noncommutative algebraic geometry (where rigid dualizing complexes were first introduced) can be found in [VdB, YZ1, YZ2, YZ3].

2. Rigid Complexes and DG Algebras

By default all rings considered in this talk are commutative.

Let me start with a discussion of rigidity for algebras over a field. Suppose $\mathbb{K}$ is a field, B is a $\mathbb{K}$-algebra, and $M \in \mathsf{D}(\mathsf{Mod}\, B)$.

According to Van den Bergh [VdB] a *rigidifying isomorphism* for M is an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{RHom}_{B \otimes_{\mathbb{K}} B}(B, M \otimes_{\mathbb{K}} M) \tag{1}$$

in $\mathsf{D}(\mathsf{Mod}\, B)$.

Now suppose A is any ring.

Trying to write A instead of $\mathbb{K}$ in formula (1) does not make sense: instead of $M \otimes_A M$ we must take the derived tensor product $M \otimes^{\mathrm{L}}_A M$; but then there is no obvious way to make $M \otimes^{\mathrm{L}}_A M$ into a complex of $B \otimes_A B$ - modules.

The problem is torsion: B might fail to be a flat A-algebra.

This is where *differential graded algebras* (DG algebras) enter the picture.

A DG algebra is a graded ring $\tilde{A} = \bigoplus_{i \in \mathbb{Z}} \tilde{A}^i$, together with a graded derivation $\mathrm{d} : \tilde{A} \to \tilde{A}$ of degree 1, satisfying $\mathrm{d} \circ \mathrm{d} = 0$.

A DG algebra quasi-isomorphism is a homomorphism $f : \tilde{A} \to \tilde{B}$ respecting degrees, multiplications and differentials, and such that $\mathrm{H}(f) : \mathrm{H}\tilde{A} \to \mathrm{H}\tilde{B}$ is an isomorphism (of graded algebras).

We shall only consider *super-commutative non-positive* DG algebras. Super-comm- utative means that $ab = (-1)^{ij} ba$ and $c^2 = 0$ for all $a \in \tilde{A}^i$, $b \in \tilde{A}^j$ and $c \in \tilde{A}^{2i+1}$. Non-positive means that $\tilde{A} = \bigoplus_{i \leq 0} \tilde{A}^i$.

We view a ring A as a DG algebra concentrated in degree 0. Given a DG algebra homomorphism $A \to \tilde{A}$ we say that $\tilde{A}$ is a DG A-algebra.

Let A be a ring. A *semi-free* DG A-algebra is a DG A-algebra $\tilde{A}$, such that after forgetting the differential $\tilde{A}$ is isomorphic, as graded A-algebra, to a super-polynomial algebra on some graded set of variables.

Definition 3. Let A be a ring and B an A-algebra. A *semi-free DG algebra resolution of B relative to A* is a quasi-isomorphism $\tilde{B} \to B$ of DG A-algebras, where $\tilde{B}$ is a semi-free DG A-algebra.

Such resolutions always exist, and they are unique up to quasi-isomorphism.

Example 4. Take $A := \mathbb{Z}$ and $B = \mathbb{Z}/(6)$. Define $\tilde{B}$ to be the super-polynomial algebra $\mathbb{Z}[\xi]$ on the variable ξ of degree -1. So $\tilde{B} = \mathbb{Z} \oplus \mathbb{Z}\xi$ as free $\mathbb{Z}$-module, and

$\xi^2 = 0$. Let $\mathrm{d}(\xi) := 6$. Then $\tilde{B} \to \mathbb{Z}/(6)$ is a semi-free DG algebra resolution of $\mathbb{Z}/(6)$ relative to $\mathbb{Z}$.

For a DG algebra A one has the category $\mathsf{DGMod}\,\tilde{A}$ of DG $\tilde{A}$-modules. It is analogous to the category of complexes of modules over a ring, and by a similar process of inverting quasi-isomorphisms we obtain the derived category $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})$; see [Ke], [Hi].

For a ring A (a DG algebra concentrated in degree 0) we have

$$\tilde{\mathsf{D}}(\mathsf{DGMod}\,A) = \mathsf{D}(\mathsf{Mod}\,A),$$

the usual derived category.

It is possible to derive functors of DG modules, again in analogy to $\mathsf{D}(\mathsf{Mod}\,A)$.

An added feature is that for a quasi- isomorphism $\tilde{A} \to \tilde{B}$ the restriction of scalars functor

$$\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B}) \to \tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})$$

is an equivalence.

Getting back to our original problem, suppose A is a ring and B is an A-algebra. Choose a semi-free DG algebra resolution $\tilde{B} \to B$ relative to A. For $M \in \mathsf{D}(\mathsf{Mod}\,B)$ define

$$\mathrm{Sq}_{B/A}\,M := \mathrm{RHom}_{\tilde{B}\otimes_A \tilde{B}}(B, M \otimes^{\mathrm{L}}_A M)$$

in $\mathsf{D}(\mathsf{Mod}\,B)$.

Theorem 5. ([YZ4]) *The functor*

$$\mathrm{Sq}_{B/A} : \mathsf{D}(\mathsf{Mod}\,B) \to \mathsf{D}(\mathsf{Mod}\,B)$$

is independent of the resolution $\tilde{B} \to B$.

The functor $\mathrm{Sq}_{B/A}$, called the *squaring operation*, is nonlinear. In fact, given a morphism $\phi : M \to M$ in $\mathsf{D}(\mathsf{Mod}\,B)$ and an element $b \in B$ one has

$$\mathrm{Sq}_{B/A}(b\phi) = b^2\,\mathrm{Sq}_{B/A}(\phi) \tag{2}$$

in

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,B)}(\mathrm{Sq}_{B/A}\,M, \mathrm{Sq}_{B/A}\,M).$$

Definition 6. Let B be a noetherian A-algebra, and let M be a complex in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,B)$ that has finite flat dimension over A. Assume

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{B/A}\,M$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,B)$. Then the pair (M, ρ) is called a *rigid complex over* B *relative to* A.

Definition 7. Say (M, ρ) and (N, σ) are rigid complexes over B relative to A. A morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\,B)$ is called a *rigid morphism relative to* A if the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \operatorname{Sq}_{B/A} M \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Sq}_{B/A}(\phi)} \\ N & \xrightarrow{\sigma} & \operatorname{Sq}_{B/A} N \end{array}$$

is commutative.

We denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$ the category of rigid complexes over B relative to A.

Example 8. Take $M = B := A$. Then

$$\operatorname{Sq}_{A/A} A = \operatorname{RHom}_{A\otimes_A A}(A, A \otimes_A A) = A,$$

and we interpret this as the tautological rigidifying isomorphism

$$\rho^{\mathrm{tau}} : A \xrightarrow{\simeq} \operatorname{Sq}_{A/A} A.$$

The *tautological rigid complex* is

$$(A, \rho^{\mathrm{tau}}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)_{\mathrm{rig}/A}.$$

3. Properties of Rigid Complexes

The first property of rigid complexes explains their name.

Theorem 9. ([YZ4]) *Let A be a ring, B a noetherian A-algebra, and*

$$(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}.$$

Assume the canonical homomorphism

$$B \to \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(M, M)$$

is bijective. Then the only automorphism of (M, ρ) in

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$$

is the identity $\mathbf{1}_M$.

The proof is very easy: an automorphism ϕ of M has to be of the form $\phi = b\,\mathbf{1}_M$ for some invertible element $b \in B$. If ϕ is rigid then $b = b^2$ (cf. formula (2)), and hence $b = 1$.

We find it convenient to denote ring homomorphisms by f^* etc. Thus a ring homomorphism $f^* : A \to B$ corresponds to the morphism of schemes

$$f : \operatorname{Spec} B \to \operatorname{Spec} A.$$

Let A be a noetherian ring. Recall that an A-algebra B is called essentially finite type if it is a localization of some finitely generated A-algebra.

We say that B is *essentially smooth* (resp. *essentially étale*) over A if it is essentially finite type and formally smooth (resp. formally étale).

Example 10. If A' is a localization of A then $A \to A'$ is essentially étale. If $B = A[t_1, \ldots, t_n]$ is a polynomial algebra then $A \to B$ is smooth, and hence also essentially smooth.

Let A be a noetherian ring and $f^* : A \to B$ an essentially smooth homomorphism. Then $\Omega^1_{B/A}$ is a finitely generated projective B-module.

Let

$$\operatorname{Spec} B = \coprod_i \operatorname{Spec} B_i$$

be the decomposition into connected components, and for every i let n_i be the rank of $\Omega^1_{B_i/A}$. We define a functor

$$f^\sharp : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$$

by

$$f^\sharp M := \bigoplus_i \Omega^{n_i}_{B_i/A}[n_i] \otimes_A M.$$

Recall that a ring homomorphism $f^* : A \to B$ is called finite if B is a finitely generated A-module. Given such a finite homomorphism we define a functor

$$f^\flat : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$$

by

$$f^\flat M := \mathrm{RHom}_A(B, M).$$

Theorem 11. ([YZ4]) *Let A be a noetherian ring, let B, C be essentially finite type A-algebras, let $f^* : B \to C$ be an A-algebra homomorphism, and let*

$$(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}.$$

(1) *If f^* is finite and $f^\flat M$ has finite flat dimension over A, then $f^\flat M$ has an induced rigidifying isomorphism*

$$f^\flat(\rho) : f^\flat M \xrightarrow{\simeq} \mathrm{Sq}_{C/A}\, f^\flat M.$$

(2) *If f^* is essentially smooth then $f^\sharp M$ has an induced rigidifying isomorphism*

$$f^\sharp(\rho) : f^\sharp M \xrightarrow{\simeq} \mathrm{Sq}_{C/A}\, f^\sharp M.$$

4. Rigid Dualizing Complexes

Let $\mathbb{K}$ be a noetherian regular ring of finite Krull dimension. We denote by $\mathsf{EFTAlg}\,/\mathbb{K}$ the category of essentially finite type $\mathbb{K}$-algebras.

Definition 12. A *rigid dualizing complex* over A relative to $\mathbb{K}$ is a rigid complex (R_A, ρ_A) such that R_A is a dualizing complex.

Theorem 13. ([YZ5]) *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let A be an essentially finite type $\mathbb{K}$-algebra.*

(1) *The algebra A has a rigid dualizing complex (R_A, ρ_A), which is unique up to a unique rigid isomorphism.*

(2) *Given a finite homomorphism* $f^* : A \to B$*, there is a unique rigid isomorphism* $f^\flat(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B)$.
(3) *Given an essentially smooth homomorphism* $f^* : A \to B$ *, there is a unique rigid isomorphism* $f^\sharp(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B)$.

Here is how the rigid dualizing complex (R_A, ρ_A) is obtained. We begin with the tautological rigid complex

$$(\mathbb{K}, \rho^{\mathrm{tau}}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,\mathbb{K})_{\mathrm{rig}/\mathbb{K}},$$

which is dualizing. Now the structural homomorphism $\mathbb{K} \to A$ can be factored into

$$\mathbb{K} \xrightarrow{f^*} B \xrightarrow{g^*} C \xrightarrow{h^*} A,$$

where f^* is smooth (B is a polynomial algebra over $\mathbb{K}$); g^* is finite (a surjection); and h^* is also smooth (a localization). Then

$$(R_A, \rho_A) := h^\sharp\, g^\flat\, f^\sharp(\mathbb{K}, \rho^{\mathrm{tau}}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)_{\mathrm{rig}/\mathbb{K}}.$$

Definition 14. Given a homomorphism $f^* : A \to B$ in $\mathsf{EFTAlg}\,/\mathbb{K}$, define the *twisted inverse image functor*

$$f^! : \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\,A) \to \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\,B)$$

by the formula

$$f^! M := \mathrm{RHom}_B\big(B \otimes^{\mathrm{L}}_A \mathrm{RHom}_A(M, R_A), R_B\big).$$

It is not hard to show that the assignment $f^* \mapsto f^!$ is a 2-functor from the category $\mathsf{EFTAlg}\,/\mathbb{K}$ to the 2-category Cat of all categories.

One can show, using Theorem 13, that this operation has very good properties. For instance, when f^* is finite, then there is a functorial, nondegenerate trace morphism

$$\mathrm{Tr}_f : f^! M \to M.$$

5. Rigid Complexes and CM Homomorphisms

In this final section I'll talk about the relation between rigid complexes and Cohen-Macaulay homomorphisms.

Definition 15. A ring A is called *tractable* if there is an essentially finite type homomorphism $\mathbb{K} \to A$, for some regular noetherian ring of finite Krull dimension $\mathbb{K}$.

The homomorphism $\mathbb{K} \to A$ is *not* part of the structure – there is no preferred $\mathbb{K}$. "Most commutative noetherian rings we know" are tractable...

Theorem 16. ([YZ5], [Ye2]) *Let* A *be a tractable ring, and let* B *be an essentially finite type* A*-algebra of finite flat dimension (e.g.* B *is flat over* A*).*

(1) *There exists a unique (up to unique rigid isomorphism) rigid complex* $R_{B/A}$ *over* B *relative to* A*, which is nonzero on each connected component of* $\operatorname{Spec} B$.

(2) *If A is a Gorentein ring (e.g. a regular ring) then $R_{B/A}$ is a dualizing complex over B*

Let $f^* : A \to B$ be a finite type flat homomorphism of relative dimension n; namely the fibers of $f : \operatorname{Spec} B \to \operatorname{Spec} A$ are all equidimensional of dimension n.

Recall that f^* is called a *Cohen-Macaulay* homomorphism if the fibers of f are all n-dimensional Cohen-Macaulay schemes.

Theorem 17. ([Ye2]) *Let A be a tractable ring, and let $f^* : A \to B$ be a finite type flat homomorphism of relative dimension n. Then the following conditions are equivalent:*

(i) *f^* is a Cohen-Macaulay homomorphism.*
(ii) *$\mathrm{H}^i R_{B/A} = 0$ for all $i \neq -n$, and the B-module*

$$\boldsymbol{\omega}_{B/A} := \mathrm{H}^{-n} R_{B/A}$$

is flat over A.

The module $\boldsymbol{\omega}_{B/A}$ is the *dualizing module* of B relative to A.

Note that the complex $\boldsymbol{\omega}_{B/A}[n]$ is rigid relative to A, but in general it is not a dualizing complex (cf. part (2) of previous theorem.) Still the fibers of $\boldsymbol{\omega}_{B/A}[n]$ are dualizing complexes – this can be seen by taking A' to be a field in the next result.

Let me end with a "rigid" version of Conrad's base change theorem [Co]:

Theorem 18. ([Ye2]) *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a cartesian diagram of rings, i.e.

$$B' \cong A' \otimes_A B,$$

with A and A' tractable rings. Assume $A \to B$ is a Cohen-Macaulay homomorphism. (There isn't any restriction on the homomorphism $A \to A'$.) Then:

(1) *$A' \to B'$ is a Cohen-Macaulay homomorphism.*
(2) *There is a unique isomorphism of B'-modules*

$$\boldsymbol{\omega}_{B'/A'} \cong A' \otimes_A \boldsymbol{\omega}_{B/A}$$

which respects rigidity.

References

[AK] A. Altman and S. Kleiman, "Introduction to Grothendieck Duality," Lecture Notes in Math. **20**, Springer, 1970.

[AJL] L. Alonso, A. Jeremías and J. Lipman, Duality and flat base change on formal schemes, in "Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes," Contemp. Math. **244**, Amer. Math. Soc., 1999, 3-90.

[Be] K. Behrend, Differential Graded Schemes I, preprint.

[Co] B. Conrad, "Grothendieck Duality and Base Change," Lecture Notes in Math. **1750**, Springer, 2000.

[Hi] V. Hinich, Homological algebra of homotopy algebras, Comm. Algebra **25** (1997), no. 10, 3291-3323.

[HS] R. Hübl and P. Sastry, Regular differential forms and relative duality, Amer. J. Math. **115** (1993), no. 4, 749-787.

[HK] R. Hübl and E. Kunz, Regular differential forms and duality for projective morphisms, J. Reine Angew. Math. **410** (1990), 84-108.

[Hu] R. Hübl, "Traces of Differential Forms and Hochschild Homology," Lecture Notes in Math. **1368**, Springer, 1989.

[Ke] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63-102.

[Li] J. Lipman, "Residues and Traces of Differential Forms via Hochschild Homology," Contemporary Mathematics **61**, Amer. Math. Soc., Providence, RI, 1987.

[Ne] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, J. Amer. Math. Soc. **9** (1996), no. 1, 205-236.

[RD] R. Hartshorne, "Residues and Duality," Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[VdB] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered ring, J. Algebra **195** (1997), no. 2, 662-679.

[Ye1] A. Yekutieli, "An Explicit Construction of the Grothendieck Residue Complex" (with an appendix by P. Sastry), Astérisque **208** (1992).

[Ye2] A. Yekutieli, Rigidity, Residues, and Grothendieck Duality for Schemes, in preparation.

[YZ1] A. Yekutieli and J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra **213** (1999), no. 1, 1-51.

[YZ2] A. Yekutieli and J.J. Zhang, Rigid Dualizing Complexes and Perverse Sheaves over Differential Algebras, Compositio Math. **141** (2005), 620-654.

[YZ3] A. Yekutieli and J.J. Zhang, Dualizing Complexes and Perverse Sheaves on Noncommutative Ringed Schemes, Selecta Math. **12** (2006), 137-177.

[YZ4] A. Yekutieli and J.J. Zhang, Rigid Complexes via DG Algebras, to appear in Trans. Amer. Math. Soc. Eprint math.AC/0603733 at http://arXiv.org.

[YZ5] A. Yekutieli and J.J. Zhang, Rigid Dualizing Complexes over Commutative Rings, eprint math.AG/0601654 http://arXiv.org.