RESIDUE COMPLEXES OVER NONCOMMUTATIVE RINGS

AMNON YEKUTIELI AND JAMES J. ZHANG

Abstract. Residue complexes were introduced by Grothendieck in algebraic geometry. These are canonical complexes of injective modules that enjoy remarkable functorial properties (traces).

In this paper we study residue complexes over noncommutative rings. These objects have a more intricate structure than in the commutative case, since they are complexes of bimodules. We develop methods to prove uniqueness, existence and functoriality of residue complexes.

For a polynomial identity algebra over a field (admitting a noetherian connected filtration) we prove existence of the residue complex and describe its structure in detail.

0. Introduction

0.1. **Motivation: A Realization of the Geometry of a Noncommutative Ring.** For a commutative ring A it is clear (since Grothendieck) what is the geometric object associated to A: the locally ringed space $\operatorname{Spec} A$. However if A is noncommutative this question becomes pretty elusive. One possibility is to consider the set $\operatorname{Spec} A$ of two-sided prime (or maybe primitive) ideals of A. Another possibility is to choose a side – say left – and to consider the category $\mathsf{Mod}\, A$ of left A-modules (or some related construction) as a kind of geometric object. Both these options are used very effectively in various contexts; but neither is completely satisfactory. A common (genuine) obstacle is the difficulty of localizing noncommutative rings. A "classical" account of the subject can be found in [MR]; recent developments are described in [SV] and its references.

In this paper we try another point of view. Taking our cue from commutative algebraic geometry, we try to construct a global algebraic object – the *residue complex* $\mathcal{K}_A$ – which encodes much of the geometric information of A.

Let us first examine an easy case which can explain where we are heading. Suppose $\mathbb{K}$ is a field and A is a finite $\mathbb{K}$-algebra. If A is commutative then $\operatorname{Spec} A$ is a finite set. The injective module $A^* := \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is a direct sum of indecomposable modules, each summand corresponding a point in $\operatorname{Spec} A$.

On the other hand if A is noncommutative the geometric object associated to it is a finite quiver $\vec{\Delta}$. The vertex set of $\vec{\Delta}$ is $\operatorname{Spec} A$, and the arrows (links) are

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determined by the bimodule decomposition of $\mathfrak{r}/\mathfrak{r}^2$, where $\mathfrak{r}$ is the Jacobson radical. The connected components of $\vec{\Delta}$ are called cliques. Here is the corresponding module-theoretic interpretation: the vertices of $\vec{\Delta}$ are the isomorphism classes of indecomposable summands of A^* as left module, the cliques are the indecomposable summands of A^* as bimodule, and the arrows in $\vec{\Delta}$ represent irreducible homomorphisms between vertices. Finally if $A \to B$ is a finite homomorphism then there is an A-bimodule homomorphism $\mathrm{Tr}_{B/A} : B^* \to A^*$.

The point of view we adopt in this paper is that for some infinite noncommutative $\mathbb{K}$-algebras A the module-theoretic interpretation of the geometry of A, as stated above, should also make sense. The generalization of the bimodule A^* is the residue complex $\mathcal{K}_A$. The additional data (not occurring in finite algebras) is that of specialization, which should be carried by the coboundary operator of $\mathcal{K}_A$.

There are certain cases in which we know this plan works. For commutative rings this is Grothendieck's theory of residual complexes, worked out in [RD], and reviewed in Subsection 0.2 below. If A is finite over its center $\mathrm{Z}(A)$ then the cliques of A biject to $\operatorname{Spec} \mathrm{Z}(A)$, and hence the geometry of A is understood; and the residue complex is $\mathcal{K}_A = \mathrm{Hom}_{\mathrm{Z}(A)}(A, \mathcal{K}_{\mathrm{Z}(A)})$.

If A is a twisted homogeneous coordinate ring of a projective variety X (with automorphism σ and σ-ample line bundle $\mathcal{L}$) we know the graded residue complex $\mathcal{K}_A$ exists (see [Ye1]). Here the indecomposable graded left module summands of $\mathcal{K}_A^{-q-1}$, $0 \leq q \leq \dim X$, are indexed by the points of X of dimension q; and the indecomposable graded bimodule summands are the σ-orbits of these points. A similar phenomenon (for $q = 0, 1$) occurs when A is a 3-dimensional Sklyanin algebra (see [Ye2]).

In Subsection 0.3 we give a brief explanation of the noncommutative residue complex, and state the main results of our paper.

0.2. **Résumé: Residue Complexes in Algebraic Geometry.** Residue complexes in (commutative) algebraic geometry were introduced by Grothendieck [RD]. Suppose $\mathbb{K}$ is a field and X is a finite type $\mathbb{K}$-scheme. The residue complex of X is a bounded complex $\mathcal{K}_X$ of quasi-coherent sheaves with some remarkable properties. First each of the $\mathcal{O}_X$-modules $\mathcal{K}_X^{-q}$ is injective, and the functor $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{K}_X)$ is a duality of the bounded derived category with coherent cohomology $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$. Next, if $f : X \to Y$ is a proper morphism then there is a nondegenerate trace map (an actual homomorphism of complexes) $\mathrm{Tr}_f : f_* \mathcal{K}_X \to \mathcal{K}_Y$. Finally if X is smooth of dimension n over $\mathbb{K}$ then there is a canonical quasi-isomorphism $\Omega^n_{X/\mathbb{K}}[n] \to \mathcal{K}_X$.

In [RD] the residue complex $\mathcal{K}_X$ is closely related to the twisted inverse image functor. Indeed, if we denote by $\pi_X : X \to \operatorname{Spec} \mathbb{K}$ the structural morphism, then the twisted inverse image $\pi_X^! \mathbb{K} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ is a dualizing complex. There is a trace $\mathrm{Tr}_f : \mathrm{R} f_* \pi_X^! \mathbb{K} \to \pi_Y^! \mathbb{K}$ for a proper morphism $f : X \to Y$, and an isomorphism $\Omega^n_{X/\mathbb{K}}[n] \to \pi_X^! \mathbb{K}$ for X smooth.

The filtration of $\mathsf{Mod}\, \mathcal{O}_X$ by dimension of support (niveau filtration) gives rise to the *Cousin functor* E. For a complex $\mathcal{M}$ the Cousin complex $\mathrm{E}\mathcal{M}$ is the row $q = 0$ in the E_1 page of the niveau spectral sequence $E_1^{p,q} \Rightarrow \mathrm{H}^{p+q} \mathcal{M}$. In this way one obtains a functor $\mathrm{E} : \mathsf{D}^+(\mathsf{Mod}\, \mathcal{O}_X) \to \mathsf{C}^+(\mathsf{Mod}\, \mathcal{O}_X)$ where the latter is the (abelian) category of complexes. By definition the residue complex is $\mathcal{K}_X := \mathrm{E}\pi_X^! \mathbb{K}$, and there is a canonical isomorphism $\pi_X^! \mathbb{K} \cong \mathcal{K}_X$ in the derived category. Explicit constructions of the residue complex also exist; cf. [Ye3] and its references.

Here is what this means for affine schemes. If we consider a commutative finitely generated $\mathbb{K}$-algebra A, and $X := \operatorname{Spec} A$, then $\mathcal{K}_A := \Gamma(X, \mathcal{K}_X)$ is a bounded complex of injective A-modules. For any integer q there is a decomposition $\mathcal{K}_A^{-q} \cong \bigoplus J(\mathfrak{p})$, where $\mathfrak{p}$ runs over the prime ideals such that $\dim A/\mathfrak{p} = q$, and $J(\mathfrak{p})$ is the injective hull of $A/\mathfrak{p}$. The map $J(\mathfrak{p}) \hookrightarrow \mathcal{K}_A^{-q} \to \mathcal{K}_A^{-q+1} \twoheadrightarrow J(\mathfrak{q})$ is nonzero precisely when $\mathfrak{p} \subset \mathfrak{q}$. Moreover, $\mathcal{K}_A$ is dualizing, in the sense that the functor $\operatorname{Hom}_A(-, \mathcal{K}_A)$ is a duality of the bounded derived category with finite cohomologies $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$. If $A \to B$ is a finite homomorphism then there is a nondegenerate trace map $\operatorname{Tr}_{B/A} : \mathcal{K}_B \to \mathcal{K}_A$. And if A is smooth of relative dimension n then

$$0 \to \Omega^n_{A/\mathbb{K}} \to \mathcal{K}_A^{-n} \to \cdots \mathcal{K}_A^0 \to 0$$

is a minimal injective resolution.

0.3. **Statement of Main Results.** In the present paper we study a noncommutative version of the above. Now A is an associative, unital, noetherian, affine (i.e. finitely generated) $\mathbb{K}$-algebra, not necessarily commutative. We denote by $\mathsf{Mod}\, A$ the category of left A-modules and by A^{op} the opposite algebra.

A dualizing complex over the algebra A is, roughly speaking, a complex R of bimodules, such that the two derived functors $\operatorname{RHom}_A(-, R)$ and $\operatorname{RHom}_{A^{\mathrm{op}}}(-, R)$ induce a duality between $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})$. The full definition of this, as well as of other important notions, are included in the body of the paper. Dualizing complexes over noncommutative rings have various applications, for instance in ring theory (see [YZ2]), representation theory (see [Ye5], [EG] and [BGK]), and even theoretical physics (see [KKO]).

The twisted inverse image $\pi^!_X \mathbb{K}$ of the commutative picture is generalized to the *rigid dualizing complex* R, as defined by Van den Bergh [VdB1]. Indeed if A is commutative and $X = \operatorname{Spec} A$ then $R := \mathrm{R}\Gamma(X, \pi^!_X \mathbb{K})$ is a rigid dualizing complex. For noncommutative A we know that a rigid dualizing complex R_A (if exists) is unique, and for a finite homomorphism $A \to B$ there is at most one rigid trace $\operatorname{Tr}_{B/A} : R_B \to R_A$.

It is known [YZ2] that if R is an *Auslander dualizing complex* then the canonical dimension associated to R, namely $\operatorname{Cdim} M := -\inf\{q \mid \operatorname{Ext}^q_A(M, R) \neq 0\}$ for a finite module M, is an exact dimension function.

The residue complex of A is by definition a rigid Auslander dualizing complex $\mathcal{K}_A$, consisting of bimodules $\mathcal{K}_A^{-q}$ which are injective, and pure of dimension q with respect to Cdim, on both sides. Again, if A is commutative then this definition is equivalent to that of [RD].

The Cousin functor is available in the noncommutative situation too. Assume we have a rigid Auslander dualizing complex R_A. The canonical dimension Cdim gives a filtration of $\mathsf{Mod}\, A$ by "dimension of support", and just like in the commutative case we obtain a Cousin functor $\mathrm{E} : \mathsf{D}^+(\mathsf{Mod}\, A \otimes A^{\mathrm{op}}) \to \mathsf{C}^+(\mathsf{Mod}\, A \otimes A^{\mathrm{op}})$. However usually $\mathrm{E} R_A$ will not be a residue complex!

The first main result gives a sufficient condition for the existence of a residue complex (it is not hard to see that this condition is also necessary). We say R_A has a pure minimal injective resolution on the left if in the minimal injective resolution $R_A \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, A)$ each I^{-q} is pure of $\operatorname{Cdim} = q$; likewise on the right.

Theorem 0.1. *Suppose A is a noetherian $\mathbb{K}$-algebra and R_A is an Auslander rigid dualizing complex over A. Assume R_A has pure minimal injective resolutions on both sides. Then $\mathcal{K}_A := \mathrm{E} R_A$ is a residue complex.*

This result included in Theorem 4.8 in the body of the paper. We also have a result guaranteeing the existence of a trace between residue complexes (it is part of Theorem 5.4). One calls a ring homomorphism $A \to B$ a finite centralizing homomorphism if $B = \sum A b_i$ where the b_i are finitely many elements of B that commute with every $a \in A$.

Theorem 0.2. *Let $A \to B$ be a finite centralizing homomorphism between noetherian $\mathbb{K}$-algebras. Suppose the two conditions below hold.*

(i) *There are rigid dualizing complexes R_A and R_B and the rigid trace morphism $\operatorname{Tr}_{B/A} : R_B \to R_A$ exists.*
(ii) *R_A is an Auslander dualizing complex and it has pure minimal injective resolutions on both sides.*

Let $\mathcal{K}_A := \mathrm{E} R_A$ be the residue complex of A (cf. Theorem 0.1). Then $\mathcal{K}_B := \mathrm{E} R_B$ is the residue complex of B. The homomorphism of complexes $\mathrm{E}(\operatorname{Tr}_{B/A}) : \mathcal{K}_B \to \mathcal{K}_A$ is a rigid trace, and it induces an isomorphism of complexes of A-bimodules

$$\mathcal{K}_B \cong \operatorname{Hom}_A(B, \mathcal{K}_A) \cong \operatorname{Hom}_{A^{\mathrm{op}}}(B, \mathcal{K}_A).$$

In Subsection 0.1 we listed a few classes of algebras for which residue complexes were previously known to exist. More examples appear in Section 5 of the paper (e.g. the first Weyl algebra, the universal enveloping algebra of a nilpotent 3-dimensional Lie algebra). In the remainder of this subsection we discuss our results for the class of polynomial identity (PI) algebras.

We remind that a PI ring A is one that satisfies some polynomial identity $f(x_1, \ldots, x_n) = 0$, and hence is close to being commutative (a commutative ring satisfies the identity $x_1 x_2 - x_2 x_1 = 0$). We have a quite detailed knowledge of the residue complex of a PI algebra A, assuming it admits some noetherian connected filtration. A a noetherian connected filtration on the algebra A is a filtration $\{F_n A\}$ such that the graded algebra $\operatorname{gr}^F A$ is a noetherian connected graded $\mathbb{K}$-algebra. Most known examples of noetherian affine PI algebras admit noetherian connected filtrations, but there are counterexamples (see [SZ]).

Theorem 0.3. *Let A be an affine noetherian PI algebra admitting a noetherian connected filtration.*

(1) *A has a residue complex $\mathcal{K}_A$.*
(2) *Let $B = A/\mathfrak{a}$ be a quotient algebra. Then B has a residue complex $\mathcal{K}_B$, there is a rigid trace $\operatorname{Tr}_{B/A} : \mathcal{K}_B \to \mathcal{K}_A$ that is an actual homomorphism of complexes of bimodules, and $\operatorname{Tr}_{B/A}$ induces an isomorphism*
$$\mathcal{K}_B \cong \operatorname{Hom}_A(B, \mathcal{K}_A) = \operatorname{Hom}_{A^{\mathrm{op}}}(B, \mathcal{K}_A) \subset \mathcal{K}_A.$$

This is Theorem 6.6 in the body of the paper.

The next theorem describes the structure of the residue complex $\mathcal{K}_A$ of a PI algebra.

Recall that the prime spectrum $\operatorname{Spec} A$ is a disjoint union of cliques. For any clique Z we denote by $A_{S(Z)}$ the localization at Z, and for a module M we let $\Gamma_Z M$ be the submodule supported on Z. We say a clique Z_1 is a specialization of a clique Z_0 if there are prime ideals $\mathfrak{p}_i \in Z_i$ with $\mathfrak{p}_0 \subset \mathfrak{p}_1$.

The q-skeleton of $\operatorname{Spec} A$ is the set of prime ideals $\mathfrak{p}$ such that $\operatorname{Cdim} A/\mathfrak{p} = q$. It is a union of cliques.

For any prime ideal $\mathfrak{p}$ we let $J(\mathfrak{p})$ be the indecomposable injective A-module with associated prime $\mathfrak{p}$, and $\mathrm{r}(\mathfrak{p})$ is the Goldie rank of $A/\mathfrak{p}$.

Theorem 0.4. *Let A be a PI $\mathbb{K}$-algebra admitting a noetherian connected filtration, and let $\mathcal{K}_A$ be its residue complex.*

(1) *For every q there is a canonical A-bimodule decomposition*
$$\mathcal{K}_A^{-q} = \bigoplus_Z \Gamma_Z \mathcal{K}_A^{-q}$$
where Z runs over the cliques in the q-skeleton of $\operatorname{Spec} A$.

(2) *Fix one clique Z in the q-skeleton of* $\operatorname{Spec} A$. *Then $\Gamma_Z \mathcal{K}_A^{-q}$ is an indecomposable A-bimodule.*

(3) *$\Gamma_Z \mathcal{K}_A^{-q}$ is an injective left $A_{S(Z)}$-module, and its socle is the essential submodule*
$$\bigoplus_{\mathfrak{p} \in Z} \mathcal{K}_{A/\mathfrak{p}}^{-q} \cong \bigoplus_{\mathfrak{p} \in Z} \operatorname{Hom}_A(A/\mathfrak{p}, \mathcal{K}_A^{-q}) \subset \Gamma_Z \mathcal{K}_A^{-q}.$$

(4) *There is a (noncanonical) decomposition of left $A_{S(Z)}$-modules*
$$\Gamma_Z \mathcal{K}_A^{-q} \cong \bigoplus_{\mathfrak{p} \in Z} J_A(\mathfrak{p})^{\mathrm{r}(\mathfrak{p})}.$$

(5) *Suppose Z_i is a clique in the $(q-i)$-skeleton of* $\operatorname{Spec} A$, *for $i = 0, 1$. Then Z_1 is a specialization of Z_0 iff the composed homomorphism*
$$\Gamma_{Z_0} \mathcal{K}_A^{-q} \hookrightarrow \mathcal{K}_A^{-q} \to \mathcal{K}_A^{-q+1} \twoheadrightarrow \Gamma_{Z_1} \mathcal{K}_A^{-q+1}$$
is nonzero.

This theorem is repeated as Theorem 6.14 in the body of the paper.

Observe that part (4) of the theorem says that the prime spectrum $\operatorname{Spec} A$ is encoded in the left module decomposition of the complex $\mathcal{K}_A$. By the left-right symmetry (replacing A with A^{op}) the same is true for the right module decomposition of $\mathcal{K}_A$. Parts (1) and (2) imply that the cliques in $\operatorname{Spec} A$ are encoded in the bimodule decomposition of $\mathcal{K}_A$. Part (5) says that specializations are encoded in the coboundary operator of $\mathcal{K}_A$.

We end this Subsection with a disclaimer. If the algebra A is "too noncommutative" then it will not have a residue complex (for instance $A = \mathrm{U}(\mathfrak{sl}_2)$). Thus the scope of the theory of residue complexes is necessarily limited. Our upcoming paper [YZ3] presents an alternative approach to address precisely this issue.

0.4. **Outline of the Paper.**

Section 1. We begin by recalling the notions of localizing subcategories and torsion functors in the module category $\mathsf{Mod}\, A$ of a ring A. Given a localizing subcategory $\mathsf{M} \subset \mathsf{Mod}\, A$ we consider the derived functor $\mathrm{R}\Gamma_{\mathsf{M}}$, and the cohomology with support in M, namely $\mathrm{H}^q_{\mathsf{M}} := \mathrm{H}^q \mathrm{R}\Gamma_{\mathsf{M}}$. We recall what is an exact dimension function, and relate it to localizing subcategories. M-flasque modules are defined. The main result here is Theorem 1.23, dealing with cohomology with supports for bimodules.

Section 2. Given a filtration $\mathsf{M} = \{\mathsf{M}^p\}$ of $\mathsf{Mod}\, A$ by localizing subcategories we can define the Cousin functor $\mathrm{E}_{\mathsf{M}} : \mathsf{D}^+(\mathsf{Mod}\, A) \to \mathsf{C}^+(\mathsf{Mod}\, A)$. The main result here is Theorem 2.11 which provides a sufficient condition for a complex M to be isomorphic to its Cousin complex $\mathrm{E}_{\mathsf{M}} M$ in $\mathsf{D}^+(\mathsf{Mod}\, A)$.

Section 3. The definitions of rigid dualizing complex and rigid trace are recalled. We show that the rigid dualizing complex is compatible with central localization.

Section 4. Here we look at residual complexes, which are Auslander dualizing complexes consisting of bimodules that are injective and pure on both sides. We prove Theorem 4.8, which is the essential ingredient of Theorem 0.1 above. Also we prove Theorem 4.10, asserting the existence of a residual complex over an FBN algebra A with an Auslander dualizing complex satisfying a certain symmetry condition.

Section 5. A residue complex is a residual complex that is also rigid. The main result in this section is Theorem 5.4 which relates the rigid trace to residue complexes. As a corollary we deduce that a residue complex $\mathcal{K}_A$ over an algebra A is unique up to a unique isomorphism of complexes (Corollary 5.5). We present examples of algebras with residue complexes. Also we explain what our results mean for commutative algebras.

Section 6. Besides proving Theorems 0.3 and 0.4, we also prove that for a prime PI algebra A of dimension n the generic component $\mathcal{K}_A^{-n}$ is untwisted; in fact it is isomorphic as bimodule to the ring of fractions Q. Several examples are studied.

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1. Cohomology with Support in a Localizing Subcategory

In algebraic geometry, given a scheme X and a closed subset $Z \subset X$, the functor Γ_Z is defined: for any sheaf $\mathcal{M}$, $\Gamma_Z \mathcal{M} \subset \mathcal{M}$ is the subsheaf of sections supported on Z. The derived functors $\mathrm{H}^q \mathrm{R}\Gamma_Z \mathcal{M} = \mathcal{H}^q_Z \mathcal{M}$ are called the sheaves of cohomologies of $\mathcal{M}$ with support in Z. More generally one can take a family of supports $\mathcal{Z}$, which is a family of closed sets satisfying suitable conditions (e.g. $\mathcal{Z}_q = \{Z \text{ closed}, \dim Z \leq q\}$).

In this section we consider an analogous construction replacing the scheme X with the category $\mathsf{Mod}\, A$ of left modules over a ring A. The role of family of supports is played by a localizing subcategory $\mathsf{M} \subset \mathsf{Mod}\, A$. This idea already appeared in [Ye2], but here we expand the method significantly. With minor modifications the contents of this section and the next one will apply to any noetherian quasi-scheme X (in the sense of [VdB2]).

We begin with a quick review of Gabriel's theory of torsion, following [Ste, Chapter VI], but using notation suitable for our purposes. Fix a ring A. A *left exact radical* (or *torsion functor*) is an additive functor $\Gamma : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$, which is a subfunctor of the identity functor $\mathbf{1}_{\mathsf{Mod}\, A}$, left exact, and $\Gamma(M/\Gamma M) = 0$ for any $M \in \mathsf{Mod}\, A$. It follows that $\Gamma\Gamma M = \Gamma M$, and if $N \subset M$ then $\Gamma N = N \cap \Gamma M$.

A *hereditary torsion class* is a class of objects $\mathsf{M} \subset \mathsf{Mod}\, A$ closed under subobjects, quotients, extensions and infinite direct sums. The full subcategory M is a *localizing subcategory.* Given a left exact radical Γ, the subcategory

$$\mathsf{M}_\Gamma := \{M \mid \Gamma M = M\}$$

is localizing. Conversely, given a localizing subcategory M, the functor

$$M \mapsto \Gamma_{\mathsf{M}} M := \{m \in M \mid A \cdot m \in \mathsf{M}\}$$

is a left exact radical. One has $\Gamma_{\mathsf{M}_\Gamma} = \Gamma$ and $\mathsf{M}_{\Gamma_{\mathsf{M}}} = \mathsf{M}$.

A third equivalent notion is that of *left Gabriel topology* (or *filter*) in A, which is a set $\mathfrak{F}$ of left ideals of A satisfying some axioms (that we shall not list here). Given a localizing subcategory $\mathsf{M} \subset \mathsf{Mod}\, A$, the set of left ideals

$$\mathfrak{F}_{\mathsf{M}} := \{\mathfrak{a} \subset A \text{ left ideal } \mid A/\mathfrak{a} \in \mathsf{M}\}$$

is a left Gabriel topology, and any left Gabriel topology arises this way. On the other hand given a left Gabriel topology $\mathfrak{F}$ the functor

$$\Gamma_{\mathfrak{F}} := \varinjlim_{\mathfrak{a} \in \mathfrak{F}} \operatorname{Hom}_A(A/\mathfrak{a}, -), \tag{1.1}$$

where $\mathfrak{F}$ is partially ordered inclusion, is a left exact radical.

Below are some examples of localizing subcategories.

Example 1.2. Let Z be a set of two-sided ideals of a ring A, each of which is finitely generated as left ideal. Then the set

$$\mathfrak{F}_Z := \{ \text{ left ideals } \mathfrak{a} \subset A \mid \mathfrak{m}_1 \cdots \mathfrak{m}_n \subset \mathfrak{a} \text{ for some } \mathfrak{m}_1, \ldots, \mathfrak{m}_n \in Z\} \tag{1.3}$$

is a left Gabriel topology (cf. [Ste, Proposition VI.6.10]). The corresponding torsion functor is denoted Γ_Z and the localizing subcategory is M_Z. If $Z = \{\mathfrak{m}\}$ we also write $\Gamma_{\mathfrak{m}}$ and $\mathsf{M}_{\mathfrak{m}}$. If A is commutative then $\mathfrak{F}_{\mathfrak{m}}$ is the usual $\mathfrak{m}$-adic topology, and $\Gamma_{\mathfrak{m}} M$ is the submodule of elements supported on $\operatorname{Spec} A/\mathfrak{m}$.

Keeping this example in mind, in the general situation of a localizing subcategory M we call $\Gamma_{\mathsf{M}} M$ the submodule of elements supported on M.

A localizing subcategory M is called *stable* if whenever $M \in \mathsf{M}$ and $M \subset N$ is an essential submodule then also $N \in \mathsf{M}$.

Example 1.4. Suppose A is left noetherian. If the ideal $\mathfrak{a}$ has the left Artin-Rees property (e.g. when $\mathfrak{a}$ is generated by normalizing elements) then the localizing subcategory $\mathsf{M}_{\mathfrak{a}}$ is stable. See [MR, Theorem 2.2 and Proposition 2.6]. More generally, if the set Z of ideals has the Artin-Rees property then M_Z is stable, see [BM, Proposition 2.9].

In this paper the most important examples of localizing subcategories arise from dimension functions.

Definition 1.5. Let M be an abelian category. An *exact dimension function* on M is a function $\dim : \mathsf{M} \to \{-\infty\} \cup \mathbb{R} \cup \{\text{infinite ordinals}\}$, satisfying the following axioms:

(i) $\dim 0 = -\infty$.
(ii) For every short exact sequence $0 \to M' \to M \to M'' \to 0$ one has $\dim M = \max\{\dim M', \dim M''\}$.
(iii) If $M = \lim_{\alpha \to} M_\alpha$ and each $M_\alpha \to M$ is an injection then $\dim M = \sup\{\dim M_\alpha\}$.

When $\mathsf{M} = \mathsf{Mod}\, A$ for a left noetherian ring A, often a dimension function will satisfy a further axiom. Recall that if $\mathfrak{p}$ is a prime ideal then an $A/\mathfrak{p}$-module M is called *torsion* if for any $m \in M$ there is a regular element $a \in A/\mathfrak{p}$ such that $am = 0$.

Definition 1.6. Let A be a left noetherian ring. A *spectral exact dimension function* on $\mathsf{Mod}\, A$ is an exact dimension function dim satisfying the extra axiom

(iv) If $\mathfrak{p}M = 0$ for some prime ideal $\mathfrak{p}$, and M is a torsion $A/\mathfrak{p}$-module, then $\dim M \leq \dim A/\mathfrak{p} - 1$.

In this paper the dimension functions will all take values in $\{-\infty\} \cup \mathbb{Z}$.

Remark 1.7. The definition of spectral exact dimension function is standard in ring theory, although usually one restricts to the subcategory $\mathsf{Mod}_{\mathrm{f}}\, A$ of finite (i.e. finitely generated) modules, where condition (iii) becomes trivial. Cf. [MR, Section 6.8.4].

Example 1.8. Let dim be an exact dimension function on $\mathsf{Mod}\, A$. For an integer q let

$$\mathsf{M}_q(\dim) := \{M \in \mathsf{Mod}\, A \mid \dim M \leq q\}.$$

Then $\mathsf{M}_q(\dim)$ is a localizing subcategory.

Here is a different kind of localizing subcategory.

Example 1.9. Given a left denominator set $S \subset A$ (cf. [MR, Paragraph 2.1.13]) we define

$$\mathfrak{F}_S := \{\mathfrak{a} \subset A \text{ left ideal } \mid \mathfrak{a} \cap S \neq \emptyset\}.$$

According to [Ste, Section II.3 and Example in Section VI.9] this is a left Gabriel topology. We denote the localizing subcategory by M_S. Letting $A_S = S^{-1}A$ be the left ring of fractions, for every module M one has an exact sequence

$$0 \to \Gamma_{\mathsf{M}_S} M \to M \to A_S \otimes_A M.$$

Now we want to pass to derived categories. Let $\mathsf{D}(\mathsf{Mod}\, A)$ be the derived category of A-modules, and let $\mathsf{D}^+(\mathsf{Mod}\, A)$ be the full subcategory of bounded below complexes. As usual $\mathsf{C}(\mathsf{Mod}\, A)$ denotes the abelian category of complexes. Our references are [RD, Section I] and [KS, Section I].

Suppose $M \in \mathsf{C}^+(\mathsf{Mod}\, A)$. By an *injective resolution of M in $\mathsf{C}^+(\mathsf{Mod}\, A)$* we mean a quasi-isomorphism $M \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, A)$ with each I^q an injective module.

Lemma 1.10. *Let M be a localizing subcategory of $\mathsf{Mod}\, A$. Then there is a right derived functor*

$$\mathrm{R}\Gamma_{\mathsf{M}} : \mathsf{D}^+(\mathsf{Mod}\, A) \to \mathsf{D}^+(\mathsf{Mod}\, A).$$

Proof. Given $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$ take any injective resolution $M \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, A)$, and let $\mathrm{R}\Gamma_{\mathsf{M}} M := \Gamma_{\mathsf{M}} I$ (cf. [RD, Theorem I.5.1]). □

Note that $\mathrm{R}\Gamma_{\mathsf{M}} M \in \mathsf{D}^+_{\mathsf{M}}(\mathsf{Mod}\, A)$, the full triangulated subcategory whose objects are complexes with cohomology in M.

Remark 1.11. One can define $\mathrm{R}\Gamma_{\mathsf{M}} M$ for an unbounded complex as $\mathrm{R}\Gamma_{\mathsf{M}} M := \Gamma_{\mathsf{M}} I$ where $M \to I$ is a quasi-isomorphism to a K-injective complex I, cf. [Sp].

Definition 1.12. The qth cohomology of M with support in M is defined to be $\mathrm{H}^q_{\mathsf{M}} M := \mathrm{H}^q \mathrm{R}\Gamma_{\mathsf{M}} M$.

For the purposes of this paper it will be useful to introduce a notion of flasque modules. Recall that a sheaf $\mathcal{M}$ on a topological space X is called flasque (or flabby) if for any two open subsets $V \subset U$ the restriction map $\Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$ is surjective. It follows that for any closed subset Z the cohomology sheaves $\mathcal{H}^q_Z \mathcal{M}$, $q > 0$, are zero. The following definition is somewhat ad hoc, but we try to justify it in the subsequent examples.

Definition 1.13. Suppose $\mathsf{M} \subset \mathsf{Mod}\, A$ is a localizing subcategory. A module M is called M-*flasque* if $\mathrm{H}^q_{\mathsf{M}} M = 0$ for all $q > 0$.

Example 1.14. Suppose M is stable. For any $M \in \mathsf{M}$ the minimal injective resolution $M \to I^0 \to I^1 \to \cdots$ is in $\mathsf{C}^+(\mathsf{M})$, and hence M is M-flasque.

Example 1.15. Suppose $S \subset A$ is a left denominator set (Example 1.9), and assume that the localizing subcategory M_S is stable. Then a module M is M_S-flasque iff the canonical homomorphism $M \to A_S \otimes_A M$ is surjective. To see why this is true, first observe that for an injective module I the module $\Gamma_{\mathsf{M}_S} I$ is also injective (because of stability). So $I/\Gamma_{\mathsf{M}_S} I$ is an injective S-torsion-free module. Since A_S is a flat right A-module [Ste, Proposition II.3.5], we get from [Ste, Proposition V.2.11] that $A_S \otimes_A I \cong A_S \otimes_A (I/\Gamma_{\mathsf{M}_S} I) \cong I/\Gamma_{\mathsf{M}_S} I$. Hence there is an exact sequence $0 \to \Gamma_{\mathsf{M}_S} I \to I \to A_S \otimes_A I \to 0$. Using an injective resolution of M we deduce that $\mathrm{H}^q_{\mathsf{M}_S} M = 0$ for $q \geq 2$ and the sequence $M \to A_S \otimes_A M \to \mathrm{H}^1_{\mathsf{M}_S} M \to 0$ is exact.

A module M is called *finitely resolved* if it has a free resolution

$$\cdots \to A^{r_2} \to A^{r_1} \to A^{r_0} \to M \to 0$$

where all the $r_i < \infty$.

Definition 1.16. A localizing subcategory M is called *locally finitely resolved* if there is a cofinal inverse system $\{\mathfrak{a}_i\}$ in the filter $\mathfrak{F}_{\mathsf{M}}$ consisting of finitely resolved left ideals.

If A is left noetherian then any localizing subcategory M is automatically locally finitely resolved. But the next examples shows this is a more general phenomenon.

Example 1.17. Let A be any ring and a a central regular element. Define $\mathfrak{m} := (a)$. Then the localizing subcategory $\mathsf{M}_{\mathfrak{m}}$ (cf. Example 1.2) is locally finitely resolved. This generalizes to a regular sequence $a_1, \ldots, a_n$ of normalizing elements.

Example 1.18. Let $\mathbb{K}$ be a commutative ring and $A := \mathbb{K}\langle x_1, \ldots, x_n \rangle$, a free associative algebra. Let $\mathfrak{m} := (x_1, \ldots, x_n)$ be the augmentation ideal. Then $\mathsf{M}_{\mathfrak{m}}$ is locally finitely resolved.

Example 1.19. Suppose A is a connected graded algebra over some field $\mathbb{K}$, and $\mathfrak{m}$ is the augmentation ideal. If A is Ext-finite in the sense of [VdB1], i.e. every $\mathrm{Ext}^q_A(\mathbb{K}, \mathbb{K})$ is finite as $\mathbb{K}$-module, then the localizing subcategory $\mathsf{M}_{\mathfrak{m}}$ is locally finitely resolved. This is Van den Bergh's original setup in [VdB1].

Proposition 1.20. *Suppose* M *is locally finitely resolved. Then* $\mathrm{H}^q_{\mathsf{M}}$ *commutes with direct limits. Therefore the direct limit of* M*-flasque modules is* M*-flasque.*

Proof. Let $\{\mathfrak{a}_i\}$ be a cofinal inverse system in the filter $\mathfrak{F}_{\mathsf{M}}$ with all the left ideals $\mathfrak{a}_i$ finitely resolved. Say $M = \lim_{j \to} M_j$ for some direct system $\{M_j\}$ of A-modules. Since the left module $A/\mathfrak{a}_i$ is finitely resolved we get

$$\mathrm{Ext}^q_A(A/\mathfrak{a}_i, M) \cong \lim_{j \to} \mathrm{Ext}^q_A(A/\mathfrak{a}_i, M_j).$$

Hence for any q we have

$$\begin{aligned}\mathrm{H}^q_{\mathsf{M}} M &\cong \lim_{i\to} \mathrm{Ext}^q_A(A/\mathfrak{a}_i, M)\\ &\cong \lim_{i\to}\lim_{j\to} \mathrm{Ext}^q_A(A/\mathfrak{a}_i, M_j)\\ &\cong \lim_{j\to}\lim_{i\to} \mathrm{Ext}^q_A(A/\mathfrak{a}_i, M_j)\\ &\cong \lim_{j\to} \mathrm{H}^q_{\mathsf{M}} M_j.\end{aligned}$$

□

Since any injective module is M-flasque it follows that there are enough M-flasque modules: any module embeds into an M-flasque one. Hence for any $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$ there is a quasi-isomorphism $M \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, A)$ with each I^q an M-flasque module. We call such a quasi-isomorphism an *M-flasque resolution of M in $\mathsf{C}^+(\mathsf{Mod}\, A)$*.

Proposition 1.21. *Let $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$ and $M \to I$ an M-flasque resolution in $\mathsf{C}^+(\mathsf{Mod}\, A)$. Then the canonical morphism $\Gamma_{\mathsf{M}} I \to \mathrm{R}\Gamma_{\mathsf{M}} I$ is an isomorphism, and hence $\mathrm{R}\Gamma_{\mathsf{M}} M \cong \Gamma_{\mathsf{M}} I$.*

Proof. If $J \in \mathsf{C}^+(\mathsf{Mod}\, A)$ is an acyclic complex of M-flasque modules then $\Gamma_{\mathsf{M}} J$ is also acyclic. Now use [RD, Theorem I.5.1]. □

Thus we can compute $\mathrm{R}\Gamma_{\mathsf{M}}$ using M-flasque resolutions.

Let $\mathbb{K}$ be a commutative base ring and let A and B be associative unital $\mathbb{K}$-algebras. We denote by B^{op} the opposite algebra, and $A \otimes B^{\mathrm{op}} := A \otimes_{\mathbb{K}} B^{\mathrm{op}}$. Thus an $(A \otimes B^{\mathrm{op}})$-module M is, in conventional notation, a $\mathbb{K}$-central A-B-bimodule ${}_A M_B$. When $A = B$ then $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ is the enveloping algebra.

Proposition 1.22. *Let A and B be $\mathbb{K}$-algebras with B flat over $\mathbb{K}$. Then there is a derived functor*

$$\mathrm{R}\Gamma_{\mathsf{M}} : \mathsf{D}^+(\mathsf{Mod}\, A \otimes B^{\mathrm{op}}) \to \mathsf{D}^+(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$$

commuting with the forgetful functor $\mathsf{D}^+(\mathsf{Mod}\, A \otimes B^{\mathrm{op}}) \to \mathsf{D}^+(\mathsf{Mod}\, A)$. In particular an $A \otimes B^{\mathrm{op}}$-module M is M-flasque iff it is M-flasque as A-module.

Proof. Since B^{op} is flat over $\mathbb{K}$, any injective $A \otimes B^{\mathrm{op}}$-module is also an injective A-module. □

The next theorem is inspired by [VdB1, Theorem 4.8]. We shall use it in our discussion of Cousin complexes in the next section.

Theorem 1.23. *Let A and B be flat $\mathbb{K}$-algebras and let $\mathsf{M} \subset \mathsf{Mod}\, A$ and $\mathsf{N} \subset \mathsf{Mod}\, B^{\mathrm{op}}$ be stable, locally finitely resolved, localizing subcategories. Suppose $M \in \mathsf{D}^+(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$ satisfies $\mathrm{H}^q_{\mathsf{M}} M \in \mathsf{N}$ and $\mathrm{H}^q_{\mathsf{N}} M \in \mathsf{M}$ for all integers q. Then there is a functorial isomorphism*

$$\mathrm{R}\Gamma_{\mathsf{M}} M \cong \mathrm{R}\Gamma_{\mathsf{N}} M \ \text{ in } \ \mathsf{D}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}}).$$

We precede the proof by three lemmas.

Lemma 1.24. *In the situation of the theorem, but without the stability assumption, if I is an injective $A \otimes B^{\mathrm{op}}$-module, then $\Gamma_{\mathsf{M}} I$ is an N-flasque B^{op}-module.*

Proof. Let $\mathfrak{F}_{\mathsf{M}}$ be the filter of left ideals associated with M. Then

$$\Gamma_{\mathsf{M}} I = \varinjlim_{\mathfrak{a} \in \mathfrak{F}_{\mathsf{M}}} \mathrm{Hom}_A(A/\mathfrak{a}, I).$$

Because B^{op} is flat over $\mathbb{K}$ each $\mathrm{Hom}_A(A/\mathfrak{a}, I)$ is an injective B^{op}-module, and hence it is N-flasque. By Proposition 1.20 the direct limit of N-flasque modules is N-flasque. □

Lemma 1.25. *In the situation of the theorem, but without the stability assumption, there is a functorial isomorphism*

$$\mathrm{R}\Gamma_{\mathsf{N}} \mathrm{R}\Gamma_{\mathsf{M}} M \cong \mathrm{R}(\Gamma_{\mathsf{N}} \Gamma_{\mathsf{M}}) M \ \textit{in} \ \mathsf{D}^{+}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}}).$$

Proof. Take an injective resolution $M \to I$ in $\mathsf{C}^{+}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$. Then $\mathrm{R}(\Gamma_{\mathsf{N}} \Gamma_{\mathsf{M}}) M = \Gamma_{\mathsf{N}} \Gamma_{\mathsf{M}} I$, and $\mathrm{R}\Gamma_{\mathsf{M}} M = \Gamma_{\mathsf{M}} I$. According to Lemma 1.24, $\Gamma_{\mathsf{M}} I$ is a complex of N-flasque B^{op} modules, so by Proposition 1.21, $\mathrm{R}\Gamma_{\mathsf{N}} \Gamma_{\mathsf{M}} I = \Gamma_{\mathsf{N}} \Gamma_{\mathsf{M}} I$. □

Lemma 1.26. *In the situation of the theorem, let $N \in \mathsf{D}^{+}_{\mathsf{N}}(\mathsf{Mod}\, B^{\mathrm{op}})$. Then the natural morphism $\mathrm{R}\Gamma_{\mathsf{N}} N \to N$ in $\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}})$ is an isomorphism.*

Proof. It suffices (by "way-out" reasons, see [RD, Proposition 7.1(iv)]) to consider a single module $N \in \mathsf{N}$. But by the stability assumption, the minimal injective resolution $N \to I^0 \to I^1 \to \cdots$ is in N, so $I = \Gamma_{\mathsf{N}} I$. □

Proof of Theorem 1.23. For any bimodule N write

$$\Gamma_{\mathsf{M} \cap \mathsf{N}} N := \Gamma_{\mathsf{N}} \Gamma_{\mathsf{M}} N = \Gamma_{\mathsf{N}} N \cap \Gamma_{\mathsf{M}} N = \Gamma_{\mathsf{M}} \Gamma_{\mathsf{N}} N \subset N,$$

It suffices by symmetry to prove that $\mathrm{R}\Gamma_{\mathsf{M}} M \cong \mathrm{R}\Gamma_{\mathsf{M} \cap \mathsf{N}} M$. Since $\mathrm{R}\Gamma_{\mathsf{M}} M \in \mathsf{D}^{+}_{\mathsf{N}}(\mathsf{Mod}\, B^{\mathrm{op}})$, Lemma 1.26 says that there is a functorial isomorphism $\mathrm{R}\Gamma_{\mathsf{M}} M \cong \mathrm{R}\Gamma_{\mathsf{N}} \mathrm{R}\Gamma_{\mathsf{M}} M$. Finally by Lemma 1.25 there is a functorial isomorphism $\mathrm{R}\Gamma_{\mathsf{M} \cap \mathsf{N}} M \cong \mathrm{R}\Gamma_{\mathsf{N}} \mathrm{R}\Gamma_{\mathsf{M}} M$. □

Example 1.27. Theorem 1.23 does not hold in general without the stability assumption. Here is a counterexample. Take $\mathbb{K} = \mathbb{C}$ and $A = B = \mathrm{U}(\mathfrak{sl}_2)$, the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. Let $\mathsf{M}_0 := \mathsf{M}_0(\mathrm{GKdim})$ be the full subcategory of $\mathsf{Mod}\, A$ consisting of modules of Gelfand-Kirillov dimension 0 (unions of A-modules that are finite over $\mathbb{K}$), and let $\mathsf{N} := \mathsf{Mod}\, B^{\mathrm{op}}$. Then all hypotheses of Theorem 1.23 hold, except that M_0 is not stable. If we take $M \in \mathsf{M}_0$ to be the simple module with $\mathrm{rank}_{\mathbb{K}} M = 1$, then it follows from [AjSZ, Proposition 7.5] that $\mathrm{H}^3_{\mathsf{M}_0} M \cong M$, but of course $\mathrm{H}^3_{\mathsf{N}} M = 0$.

2. Cousin Functors

Cousin complexes in commutative algebraic geometry were introduced by Grothendieck in [RD]. Several people (including Lipman, private communication) had suggested extending the construction to more general settings. The noncommutative version below already appeared in [Ye2], but the powerful Theorem 2.11 is new.

Suppose A is a ring, and we are given an increasing filtration

$$\cdots \subset \mathsf{M}_{q-1} \subset \mathsf{M}_q \subset \mathsf{M}_{q+1} \subset \cdots \subset \mathsf{Mod}\, A$$

by localizing subcategories, indexed by $\mathbb{Z}$. We shall sometimes write $\mathsf{M}^q := \mathsf{M}_{-q}$, so that $\{\mathsf{M}^p\}_{p \in \mathbb{Z}}$ is a decreasing filtration. This is to conform to the convention that decreasing filtrations go with cochain complexes. We say the filtration $\mathsf{M} = \{\mathsf{M}_q\} = \{\mathsf{M}^p\}$ is *bounded* if there are $q_0 \leq q_1$ such that $\mathsf{M}_{q_0 - 1} = 0$ and $\mathsf{M}_{q_1} = \mathsf{Mod}\, A$.

Example 2.1. Suppose dim is an exact dimension function that is bounded, in the sense that there are integers $q_0 \leq q_1$ such that for any nonzero module M, $q_0 \leq \dim M \leq q_1$. Define $\mathsf{M}_q(\dim)$ as in Example 1.8. Then $\mathsf{M} = \{\mathsf{M}_q(\dim)\}$ is a bounded filtration of $\mathsf{Mod}\, A$ by localizing subcategories. Conversely, given a bounded filtration $\mathsf{M} = \{\mathsf{M}_q\}$ by localizing subcategories, we can define $\dim M := \inf\{q \mid M \in \mathsf{M}_q\}$, and this will be a bounded exact dimension function.

Example 2.2. Specializing the previous example, let A be a finitely generated commutative algebra over a field $\mathbb{K}$ (or more generally A is a catenary commutative noetherian ring of Krull dimension $\operatorname{Kdim} A < \infty$) and $X := \operatorname{Spec} A$. Taking $\dim = \operatorname{Kdim}$, $\mathcal{Z}_q = \{Z \subset X \text{ closed}, \dim Z \leq q\}$ and $\mathcal{Z}^q := \mathcal{Z}_{-q}$ we get $\Gamma_{\mathsf{M}^p} = \Gamma_{\mathcal{Z}^p}$. This is the kind of filtration by codimension (coniveau) used in [RD, Chapter IV]. The bounds are $q_1 = \dim X$ and $q_0 = 0$.

Suppose $\mathsf{M} = \{\mathsf{M}_q\}$ is a collection of localizing subcategories of $\mathsf{Mod}\, A$. We call a module M M-*flasque* if it is M_q-flasque for all q (Definition 1.13).

For a module M and $d \geq 0$ we write $\Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}} M := \Gamma_{\mathsf{M}^p} M / \Gamma_{\mathsf{M}^{p+d}} M$.

Lemma 2.3. *There is a right derived functor* $\mathrm{R}\Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}}$ *that fits into a functorial triangle for* $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$*:*

$$\mathrm{R}\Gamma_{\mathsf{M}^{p+d}} M \to \mathrm{R}\Gamma_{\mathsf{M}^p} M \to \mathrm{R}\Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}} M \to \mathrm{R}\Gamma_{\mathsf{M}^{p+d}} M[1].$$

If $M \to I$ *is a flasque resolution then* $\mathrm{R}\Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}} M = \Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}} I$.

Proof. If $I \in \mathsf{D}^+(\mathsf{Mod}\, A)$ is an acyclic complex of M-flasque modules then from the exact sequence of complexes

$$0 \to \Gamma_{\mathsf{M}^{p+d}} I \to \Gamma_{\mathsf{M}^p} I \to \Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}} I \to 0$$

we see that $\Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}} I$ is also acyclic. Thus we can define $\mathrm{R}\Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}} M := \Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}} I$ when $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$ and $M \to I$ is a flasque resolution (cf. proof of Proposition 1.21). $\square$

We set $\mathrm{H}^q_{\mathsf{M}^p / \mathsf{M}^{p+d}} M := \mathrm{H}^q \mathrm{R}\Gamma_{\mathsf{M}^p / \mathsf{M}^{p+d}} M$.

Proposition 2.4. *Let* $\mathsf{M} = \{\mathsf{M}^p\}$ *be a bounded filtration of* $\mathsf{Mod}\, A$ *by localizing subcategories. Then for every* $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$ *there is a convergent spectral sequence*

$$E_1^{p,q} = \mathrm{H}^{p+q}_{\mathsf{M}^p / \mathsf{M}^{p+1}} M \Rightarrow \mathrm{H}^{p+q} M,$$

functorial in M.

Proof. Pick an M-flasque resolution $M \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, A)$. The decreasing filtration $\{\Gamma_{\mathsf{M}^p} I\}$ is bounded in the sense of [Mac, Sections XI.3 and XI.8], i.e. $\Gamma_{\mathsf{M}^{p_0}} I = I$ and $\Gamma_{\mathsf{M}^{p_1}} I = 0$ for some $p_0 \leq p_1$. Hence by [Mac, Theorem XI.3.1] we get a convergent spectral sequence

$$E_1^{p,q} = \mathrm{H}^{p+q} \Gamma_{\mathsf{M}^p / \mathsf{M}^{p+1}} I \Rightarrow \mathrm{H}^{p+q} I.$$

Now $\mathrm{H}^{p+q} \Gamma_{\mathsf{M}^p / \mathsf{M}^{p+1}} I = \mathrm{H}^{p+q}_{\mathsf{M}^p / \mathsf{M}^{p+1}} M$ and $\mathrm{H}^{p+q} I = \mathrm{H}^{p+q} M$. If $I_1 \to I_2$ is a homomorphism between M-flasque complexes then there is a map between the two spectral sequences; and if $I_1 \to I_2$ is a quasi-isomorphism then the two spectral sequences are isomorphic. $\square$

Definition 2.5. (Grothendieck, [RD]) Given a bounded filtration $\mathsf{M} = \{\mathsf{M}^p\}$ of $\mathsf{Mod}\, A$ and a complex $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$ define the complex $\mathrm{E}_{\mathsf{M}} M$ as follows. For any p

$$(\mathrm{E}_{\mathsf{M}} M)^p := E_1^{p,0} = \mathrm{H}^p_{\mathsf{M}^p \,/\, \mathsf{M}^{p+1}} M$$

in the spectral sequence above, and the coboundary operator is

$$\mathrm{d}_1^{p,0} : (\mathrm{E}_{\mathsf{M}} M)^p = E_1^{p,0} \to (\mathrm{E}_{\mathsf{M}} M)^{p+1} = E_1^{p+1,0}.$$

Thus $\mathrm{E}_{\mathsf{M}} M$ is the row $q = 0$ in the E_1 page of the spectral sequence. We obtain an additive functor

$$\mathrm{E}_{\mathsf{M}} : \mathsf{D}^+(\mathsf{Mod}\, A) \to \mathsf{C}^+(\mathsf{Mod}\, A)$$

called the *Cousin functor*.

Unlike the commutative situation, here the complex $\mathrm{E}_{\mathsf{M}} M$ can behave quite oddly – see below.

Definition 2.6. Given an exact dimension function dim on $\mathsf{Mod}\, A$ we say an A-module M is dim-*pure of dimension* q if $\dim M' = \dim M = q$ for all nonzero submodules $M' \subset M$.

Remark 2.7. In the commutative case (see Example 2.2) let $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$ and let $\mathcal{M} := \mathcal{O}_X \otimes_A M$ denote the corresponding complex of quasi-coherent sheaves on X. Then for any p, q one has

$$\mathrm{H}^{p+q}_{\mathsf{M}^p \,/\, \mathsf{M}^{p+1}} M \cong \Gamma(X, \mathcal{H}^{p+q}_{\mathcal{Z}^p / \mathcal{Z}^{p+1}} \mathcal{M}) \cong \bigoplus_x \mathrm{H}^{p+q}_x \mathcal{M}$$

where x runs over the points in X of $\dim \overline{\{x\}} = -p$ and H^{p+q}_x is local cohomology. In the language of [RD], the sheaf $\mathcal{H}^{p+q}_{\mathcal{Z}^p / \mathcal{Z}^{p+1}} \mathcal{M}$ lies on the $\mathcal{Z}^p / \mathcal{Z}^{p+1}$-skeleton of X. In particular this means the A-module $(\mathrm{E}_{\mathsf{M}} M)^{-q} = \mathrm{H}^{-q}_{\mathsf{M}_q \,/\, \mathsf{M}_{q-1}} M$ is M-flasque and Kdim-pure of dimension q. Note that this implies $\mathrm{E}_{\mathsf{M}} \mathrm{E}_{\mathsf{M}} M = \mathrm{E}_{\mathsf{M}} M$. A complex N such that each N^{-q} is M-flasque and Kdim-pure of dimension q is called a "Cousin complex" in [RD, Section IV.3]. However for a noncommutative ring A the complex $\mathrm{E}_{\mathsf{M}} M$ will seldom be a Cousin complex in this sense; cf. the next example.

Example 2.8. Consider $\mathbb{K} = \mathbb{C}$ and $A = \mathrm{U}(\mathfrak{sl}_2)$ as in Example 1.27. Let $\mathsf{M} = \{\mathsf{M}_q(\mathrm{GKdim})\}$ be the filtration by Gelfand-Kirillov dimension and $M \in \mathsf{M}_0$ the simple A-module. Then $(\mathrm{E}_{\mathsf{M}} M)^0 = \mathrm{H}^0_{\mathsf{M}_0} M \cong M$. Since $\mathrm{H}^3_{\mathsf{M}_0} M \neq 0$ we see that $(\mathrm{E}_{\mathsf{M}} M)^0$ is not M-flasque.

Proposition 2.9. *Suppose $A \to B$ is a homomorphism of rings, $\mathsf{M}(A) = \{\mathsf{M}_q(A)\}$ and $\mathsf{M}(B) = \{\mathsf{M}_q(B)\}$ are bounded filtrations of $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, B$ respectively by localizing subcategories, with Cousin functors $\mathrm{E}_{\mathsf{M}(A)}$ and $\mathrm{E}_{\mathsf{M}(B)}$, satisfying:*

(i) *For any $M \in \mathsf{Mod}\, B$ and any q, $\Gamma_{\mathsf{M}_q(B)} M = \Gamma_{\mathsf{M}_q(A)} M$.*
(ii) *If $I \in \mathsf{Mod}\, B$ is injective then it is $\mathsf{M}(A)$-flasque.*

Then there is an isomorphism $\mathrm{E}_{\mathsf{M}(B)} M \cong \mathrm{E}_{\mathsf{M}(A)} M$, functorial in $M \in \mathsf{D}^+(\mathsf{Mod}\, B)$.

Proof. Choose an injective resolution $M \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, B)$. Then $M \to I$ is an $\mathsf{M}(A)$-flasque resolution in $\mathsf{C}^+(\mathsf{Mod}\, A)$, the filtered complexes $\Gamma_{\mathsf{M}_q(B)} I$ and $\Gamma_{\mathsf{M}_q(A)} I$ coincide, and the spectral sequence defines both $\mathrm{E}_{\mathsf{M}(B)} M$ and $\mathrm{E}_{\mathsf{M}(A)} M$. □

Now let $\mathbb{K}$ be a commutative base ring and as before $\otimes = \otimes_{\mathbb{K}}$.

Corollary 2.10. *Let A be a $\mathbb{K}$-algebra, $\mathsf{M} = \{\mathsf{M}^p\}$ a bounded filtration of $\mathsf{Mod}\, A$ by localizing subcategories, and B a flat $\mathbb{K}$-algebra. Suppose $M \in \mathsf{D}^+(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$. Then the Cousin functor $\mathrm{E}_{\mathsf{M}} M$ commutes with the forgetful functor $\mathsf{Mod}\, A \otimes B^{\mathrm{op}} \to \mathsf{Mod}\, A$.*

Proof. Write $\mathsf{M}(A)$ and $\mathsf{M}(A \otimes B^{\mathrm{op}})$ for the filtrations of $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, A \otimes B^{\mathrm{op}}$ respectively, and apply Proposition 2.9. $\square$

For the next theorem it will be important to distinguish between morphisms in $\mathsf{D}(\mathsf{Mod}\, A)$ and $\mathsf{C}(\mathsf{Mod}\, A)$, so let $\mathrm{Q} : \mathsf{C}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$ be the localization functor (identity on objects).

Theorem 2.11. *Let A be a ring, $\mathsf{M} = \{\mathsf{M}^p\}$ a bounded filtration of $\mathsf{Mod}\, A$ by localizing subcategories and E_{M} the associated Cousin functor. Let $M \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ a complex satisfying*

(∗) $\mathrm{H}^{p+q}_{\mathsf{M}^p / \mathsf{M}^{p+1}} M = 0$ *for all $q \neq 0$ and all p.*

Then there is an isomorphism $M \cong \mathrm{Q}\mathrm{E}_{\mathsf{M}} M$ in $\mathsf{D}(\mathsf{Mod}\, A)$.

Proof. This is really the implication (ii) ⇒ (iii) in [RD, Proposition IV.3.1]. We shall explain the minor modification needed in the proof to make it apply to our situation. Also we shall sketch the main ideas of the proof using our notation, so the interested reader can find it easier to consult the rather lengthy proof in [RD].

The result in [RD] refers to the abelian category $\mathsf{Ab}\, X$ of sheaves of abelian groups on a topological space X. The space X has a filtration $\{Z^p\}$ by closed subsets, inducing a filtration $\mathsf{M} = \{\mathsf{M}^p\}$ of $\mathsf{Ab}\, X$, with $\Gamma_{\mathsf{M}^p} = \Gamma_{Z^p}$. With this notation the proof involves homological algebra only, hence it applies to $\mathsf{Mod}\, A$ as well.

Here is the sketch. Let us abbreviate $\mathrm{E} := \mathrm{E}_{\mathsf{M}}$. Define

$$\tau_{\geq p}\mathrm{E}M := (\cdots \to 0 \to (\mathrm{E}M)^p \to (\mathrm{E}M)^{p+1} \to \cdots)$$

to be the truncation in $\mathsf{C}(\mathsf{Mod}\, A)$. One shows by descending induction on p that there are (noncanonical) isomorphisms

$$\phi_p : \mathrm{R}\Gamma_{\mathsf{M}^p} M \xrightarrow{\simeq} \mathrm{Q}\tau_{\geq p}\mathrm{E}M \text{ in } \mathsf{D}(\mathsf{Mod}\, A) \tag{2.12}$$

such that the diagrams

$$\begin{tikzcd} \mathrm{H}^p_{\mathsf{M}^p / \mathsf{M}^{p+1}} M \arrow[d, "="] & \mathrm{H}^p_{\mathsf{M}^p} M \arrow[l] \arrow[d, "\mathrm{H}^p(\phi_p)"] \\ (\mathrm{E}M)^p & \mathrm{H}^p \tau_{\geq p}\mathrm{E}M \arrow[l] \end{tikzcd} \tag{2.13}$$

commute. The horizontal arrows are the canonical ones.

The starting point is that for large enough $p = p_{\mathrm{big}}$, $\mathrm{H}^p M = \mathrm{H}^p \mathrm{E}M = 0$. For such p one shows that $\mathrm{H}^q_{\mathsf{M}^p} M = 0$ if $q \neq p$. Hence there is an isomorphism $\mathrm{R}\Gamma_{\mathsf{M}^p} M \cong \mathrm{Q}(\mathrm{H}^p_{\mathsf{M}^p} M)[-p]$. Also one shows that $\mathrm{H}^q \tau_{\geq p}\mathrm{E}M = 0$ if $q \neq p$, so that $\mathrm{Q}\tau_{\geq p}\mathrm{E}M \cong \mathrm{Q}(\mathrm{H}^p \tau_{\geq p}\mathrm{E}M)[-p]$. Since the modules $\mathrm{H}^p_{\mathsf{M}^p} M$ and $\mathrm{H}^p \tau_{\geq p}\mathrm{E}M$ are canonically isomorphic in this case, we get an isomorphism (2.12) for $p = p_{\mathrm{big}}$.

In the inductive step, depicted in Figure 1, we have two canonical triangles (in which the morphisms α_{p-1} and d^{p-1} have degree $+1$), canonical isomorphisms ψ_p and ψ_{p-1} (arising from the assumption (∗)) and an isomorphism ϕ_p that's already

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathsf{M}^p/\mathsf{M}^{p+1}} M \arrow[dd, "\psi_p"] & & & \mathrm{R}\Gamma_{\mathsf{M}^{p-1}/\mathsf{M}^p} M \arrow[dll, "\alpha_{p-1}"'] \arrow[dd, "\psi_{p-1}"] & \\
& \mathrm{R}\Gamma_{\mathsf{M}^p} M \arrow[ul, "\beta_p"'] \arrow[rrr] \arrow[dd, "\phi_p"] & & & \mathrm{R}\Gamma_{\mathsf{M}^{p-1}} M \arrow[ul, "\beta_{p-1}"'] \arrow[dd, dashed, "\phi_{p-1}"] \\
\mathrm{Q}(\mathrm{E}M)^p[-p] & & & \mathrm{Q}(\mathrm{E}M)^{p-1}[-p+1] \arrow[dll, "\mathrm{d}^{p-1}"'] & \\
& \mathrm{Q}\tau_{\geq p}\mathrm{E}M \arrow[ul] \arrow[rrr] & & & \mathrm{Q}\tau_{\geq p-1}\mathrm{E}M \arrow[ul]
\end{tikzcd}$$

FIGURE 1

been constructed. The square on the left commutes because diagram (2.13) commutes.

By definition of the Cousin complex it follows that $\mathrm{d}^{p-1} = \mathrm{H}^{p-1}(\beta_p \alpha_{p-1})$. Since $\mathrm{H}^p \tau_{\geq p} \mathrm{E}M \subset (\mathrm{E}M)^p$, diagram (2.13) implies that

$$\mathrm{H}^p(\phi_p \alpha_{p-1}) = \mathrm{H}^{p-1}(\mathrm{d}^{p-1}\psi_{p-1}) : \mathrm{H}^{p-1}_{\mathsf{M}^{p-1}/\mathsf{M}^p} M \to \mathrm{H}^p \tau_{\geq p} \mathrm{E}M.$$

Therefore $\phi_p \alpha_{p-1} = d^{p-1}\psi_{p-1}$, and hence, by the axioms of triangulated categories, there is an isomorphism ϕ_{p-1} making diagram in Figure 1 commutative. Note that diagram (2.13) for $p-1$ commutes too so the induction continues. □

Remark 2.14. In [RD] a complex satisfying condition $(*)$ of the theorem is called a Cohen-Macaulay complex w.r.t. the filtration. And indeed in the commutative case (Example 2.2), for an A-module M of $\operatorname{Kdim} M = d$, M is a Cohen-Macaulay module iff the complex $M[d]$ satisfies $(*)$; cf. [RD, page 239]. For a noncommutative ring A these notions diverge.

Corollary 2.15. *Let A be a $\mathbb{K}$-algebra, $\mathsf{M} = \{\mathsf{M}^p\}$ a bounded filtration of $\mathsf{Mod}\, A$ by localizing subcategories, and B a flat $\mathbb{K}$-algebra. Suppose $M \in \mathsf{D}^+(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$ satisfies condition $(*)$ of the theorem. Then there is an isomorphism $M \cong \mathrm{Q}\mathrm{E}_{\mathsf{M}} M$ in $\mathsf{D}^+(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$ commuting with the forgetful functor $\mathsf{Mod}\, A \otimes B^{\mathrm{op}} \to \mathsf{Mod}\, A$.*

Proof. Invoke the theorem with $A \otimes B^{\mathrm{op}}$ instead of A, and use Corollary 2.10. □

We shall also need the next propositions.

Proposition 2.16. *Suppose A and B are flat $\mathbb{K}$-algebras, and $\mathsf{M} = \{\mathsf{M}^p\}$ and $\mathsf{N} = \{\mathsf{N}^p\}$ are bounded filtrations of $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, B^{\mathrm{op}}$ respectively by stable, locally finitely resolved, localizing subcategories. Let $M \in \mathsf{D}^+(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$ be a complex satisfying $\mathrm{H}^q_{\mathsf{M}^p} M \in \mathsf{N}^p$ and $\mathrm{H}^q_{\mathsf{N}^p} M \in \mathsf{M}^p$ for all p, q. Then there is a functorial isomorphism*

$$\mathrm{E}_{\mathsf{M}} M \cong \mathrm{E}_{\mathsf{N}} M \ \textit{ in } \ \mathsf{C}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}}).$$

Proof. Choose an injective resolution $M \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$. Denote by $\Gamma_{\mathsf{M}} I$ the filtered complex with filtration $\{\Gamma_{\mathsf{M}^p} I\}_{p \in \mathbb{Z}}$, and by $\Gamma_{\mathsf{M} \cap \mathsf{N}} I$ the filtered complex with filtration $\{\Gamma_{\mathsf{M}^p} \Gamma_{\mathsf{N}^p} I\}_{p \in \mathbb{Z}}$. By the proof of Theorem 1.23, the homomorphism of filtered complexes $\Gamma_{\mathsf{M} \cap \mathsf{N}} I \to \Gamma_{\mathsf{M}} I$ induces an isomorphism on the E_1 pages of the spectral of the sequences from Proposition 2.4. Similarly for $\Gamma_{\mathsf{M} \cap \mathsf{N}} I \to \Gamma_{\mathsf{N}} I$. □

Proposition 2.17. *Let* dim *be a bounded exact dimension function on* $\mathsf{Mod}\, A$*, and let* $\mathsf{M} = \{\mathsf{M}_q(\dim)\}$ *be a the induced filtration of* $\mathsf{Mod}\, A$*. Suppose the complexes* $M, I \in \mathsf{C}^+(\mathsf{Mod}\, A)$ *satisfy:*

(i) *Each module* M^{-q} *and* I^{-q} *is* M*-flasque and* dim*-pure of dimension* q*.*
(ii) *Each module* I^{-q} *is injective.*

Then

(1) $\mathrm{E}_{\mathsf{M}} \mathrm{Q} M = M$ *and* $\mathrm{E}_{\mathsf{M}} \mathrm{Q} I = I$*.*
(2) *The functor* E_{M} *induces an isomorphism*

$$\mathrm{Hom}_{\mathsf{D}^+(\mathsf{Mod}\, A)}(\mathrm{Q}M, \mathrm{Q}I) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{C}^+(\mathsf{Mod}\, A)}(M, I)$$

with inverse induced by Q.

Proof. 1. Clear, since $\Gamma_{\mathsf{M}^p / \mathsf{M}^{p+1}} M^{-p} \cong M^{-p}$ and the same for I.

2. Since I is a bounded below complex of injectives we have an isomorphism

$$\mathrm{H}^0 \, \mathrm{Hom}_A(M, I) \cong \mathrm{Hom}_{\mathsf{D}^+(\mathsf{Mod}\, A)}(\mathrm{Q}M, \mathrm{Q}I).$$

The purity implies that $\mathrm{Hom}_A(M, I)^{-1} = 0$ and hence we get an isomorphism

$$\mathrm{Hom}_{\mathsf{C}^+(\mathsf{Mod}\, A)}(M, I) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}^+(\mathsf{Mod}\, A)}(\mathrm{Q}M, \mathrm{Q}I)$$

induced by Q. Finally given a morphism $\phi : M \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, A)$ we have $\mathrm{E}_{\mathsf{M}} \mathrm{Q}(\phi) = \phi$. □

3. Rigid Dualizing Complexes

Dualizing complexes were introduced by Grothendieck [RD]. The noncommutative variant was studied in [Ye1], and the notion of rigid dualizing complex is due to Van den Bergh [VdB1]. Let us recall the definitions. From here to the end of the paper $\mathbb{K}$ denotes a base field, and as before $\otimes = \otimes_{\mathbb{K}}$.

An A-module M is said to be *finite* if it is finitely generated. A homomorphism of rings $A \to B$ is called *finite* if B is a finite A-module on both sides. A $\mathbb{K}$-algebra A is called *affine* if it finitely generated.

Definition 3.1 ([Ye1], [YZ2])**.** Let A be a left noetherian $\mathbb{K}$-algebra and B a right noetherian $\mathbb{K}$-algebra. A complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$ is called a *dualizing complex over* (A, B) if:

(i) R has finite injective dimension over A and B^{op}.
(ii) R has finite cohomology modules over A and B^{op}.
(iii) The canonical morphisms $B \to \mathrm{RHom}_A(R, R)$ in $\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{e}})$ and $A \to \mathrm{RHom}_{B^{\mathrm{op}}}(R, R)$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ are both isomorphisms.

In case $A = B$, we shall say that R is a dualizing complex over A.

Condition (i) is equivalent to the existence of a quasi-isomorphism $R \to I$ in $\mathsf{C}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$ with each bimodule I^q injective over A and B^{op}.

In this paper, whenever we mention a dualizing complex over (A, B) we implicitly assume that A and B^{op} are left noetherian $\mathbb{K}$-algebras.

Example 3.2. When A is commutative and R is a dualizing complex over A consisting of central bimodules, then R is a dualizing complex in the sense of [RD, Section V.2].

Definition 3.3 ([VdB1])**.** Suppose R is a dualizing complex over a noetherian $\mathbb{K}$-algebra A. If there is an isomorphism

$$\phi : R \xrightarrow{\simeq} \mathrm{RHom}_{A^{\mathrm{e}}}(A, R \otimes R)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ we call the pair (R, ϕ) a *rigid dualizing complex.*

In the definition above $\mathrm{Hom}_{A^{\mathrm{e}}}$ is with respect to the outside A^{e}-module structure of $R \otimes R$, and the isomorphism ρ is with respect to the remaining inside A^{e}-module structure.

A rigid dualizing complex over A is unique, up to an isomorphism in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, see [VdB1, Proposition 8.2].

Remark 3.4. "Rigid dualizing complex" is a relative notion, in the sense that it depends on the homomorphism $\mathbb{K} \to A$. Cf. [YZ2, Example 3.13].

Definition 3.5 ([YZ2])**.** Suppose $A \to B$ is a finite homomorphism of $\mathbb{K}$-algebras and (R_A, ϕ_A) and (R_B, ϕ_B) are rigid dualizing complexes over A and B respectively. A *rigid trace* is a morphism $\mathrm{Tr}_{B/A} : R_B \to R_A$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ satisfying the two conditions below.

(i) $\mathrm{Tr}_{B/A}$ induces isomorphisms

$$R_B \cong \mathrm{RHom}_A(B, R_A) \cong \mathrm{RHom}_{A^{\mathrm{op}}}(B, R_A)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

(ii) The diagram

$$\begin{array}{ccc} R_B & \xrightarrow{\phi_B} & \mathrm{RHom}_{B^{\mathrm{e}}}(B, R_B \otimes R_B) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr} \otimes \mathrm{Tr}} \\ R_A & \xrightarrow{\phi_A} & \mathrm{RHom}_{A^{\mathrm{e}}}(A, R_A \otimes R_A) \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ is commutative.

According to [YZ2, Theorem 3.2] a rigid trace, if it exists, is unique. Taking $A = B$ this implies that any two rigid dualizing complexes (R, ϕ) and (R', ϕ') are *uniquely* isomorphic in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, see [YZ2, Corollary 3.4]. Often we shall omit explicit mention of the isomorphism ϕ.

Lemma 3.6. *Let R be a dualizing complex over (A, B), and let $C := \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})}(R)$.*

(1) *The left action of the center $\mathrm{Z}(A)$ on R, and the right action of $\mathrm{Z}(B)$ on R, induce isomorphisms of $\mathbb{K}$-algebras $\mathrm{Z}(A) \cong C \cong \mathrm{Z}(B)$. These make R into a complex of C-bimodules (not necessarily central).*

(2) *Let $M \in \mathsf{D}(\mathsf{Mod}\, A)$. Then the two C-module structures on $\mathrm{Ext}^q_A(M, R)$ coincide.*

(3) *If $M \in \mathsf{D}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$ is C-central then the three C-module structures on $\mathrm{Ext}^q_{A \otimes B^{\mathrm{op}}}(M, R \otimes R)$ coincide.*

(4) *If $A = B$ and R is rigid then the automorphism of $\mathrm{Z}(A)$ in item (1) is the identity.*

Proof. The first item is a slight variation of [YZ2, Lemma 3.3] and [Ye4, Lemma 5.4]. In item (2) the two actions of C on $\operatorname{Ext}^q_A(M,R)$ correspond to the left action of A on R (and on M), and the right action of B on R. Since $\operatorname{Ext}^q_A(M,R) = \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(M,R[q])$ these actions commute. Likewise in item (3). Item (4) is [YZ2, Proposition 3.5]. □

Lemma 3.7. *Let A be a left noetherian $\mathbb{K}$-algebra and $L \in \mathsf{D}^-_{\mathrm{f}}(\mathsf{Mod}\,A)$.*

(1) *Let B be some $\mathbb{K}$-algebra, let N be a flat B-module and let $M \in \mathsf{D}(\mathsf{Mod}(A \otimes B^{\mathrm{op}}))$. Then the canonical morphism*
$$\operatorname{RHom}_A(L,M) \otimes_B N \to \operatorname{RHom}_A(L, M \otimes_B N)$$
is an isomorphism.

(2) *Suppose $A \to A'$ a ring homomorphism such that A' is a flat A^{op}-module. Let $M \in \mathsf{D}(\mathsf{Mod}\,A')$. Then the canonical morphism*
$$\operatorname{RHom}_A(L,M) \to \operatorname{RHom}_{A'}(A' \otimes_A L, M)$$
is an isomorphism.

Proof. (1) Choose a quasi-isomorphism $P \to L$ where P is a bounded above complex of finite free A-modules. Then the homomorphism of complexes
$$\operatorname{Hom}_A(P,M) \otimes_B N \to \operatorname{Hom}_A(P, M \otimes_B N)$$
is bijective.

(2) With $P \to L$ as above we get a free resolution $A' \otimes_A P \to A' \otimes_A L$ as A'-modules, and
$$\operatorname{Hom}_A(P,M) \to \operatorname{Hom}_{A'}(A' \otimes_A P, M)$$
is bijective. □

The next theorem relates rigid dualizing complexes and central localization.

Theorem 3.8. *Let R be a dualizing complex over (A,B), and identify $C \cong \mathrm{Z}(A) \cong \mathrm{Z}(B)$ as in Lemma 3.6. Suppose $S \subset C$ is a multiplicatively closed set, and let $C_S := S^{-1}C$, $A_S := C_S \otimes_C A$ and $B_S := C_S \otimes_C B$ be the localizations. Then:*

(1) *The complex*
$$R_S := A_S \otimes_A R \otimes_B B_S$$
is a dualizing complex over (A_S, B_S).

(2) *If $A = B$, R is a rigid dualizing complex over A, and A^{e} is noetherian, then R_S is a rigid dualizing complex over A_S.*

Proof. (1) This is proved in a special case (when A is commutative and $A_S = A_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$) in the course of the proof of [YZ2, Theorem 1.11(1)]; but the same proof works here too. Among other things one gets that $R_S \cong A_S \otimes_A R$ in $\mathsf{D}(\mathsf{Mod}\,A_S \otimes A^{\mathrm{op}})$ and $R_S \cong R \otimes_A A_S$ in $\mathsf{D}(\mathsf{Mod}\,A \otimes A_S^{\mathrm{op}})$

(2) We consider $R \otimes R$ as a left (resp. right) A^{e}-module via the outside (resp. inside) action. Since A^{e} is noetherian and $A^{\mathrm{e}} \to (A_S)^{\mathrm{e}}$ is flat, by Lemma 3.7(1) we obtain an isomorphism
$$\begin{aligned} R_S &\cong R \otimes_{A^{\mathrm{e}}} (A_S)^{\mathrm{e}} \\ &\cong \operatorname{RHom}_{A^{\mathrm{e}}}(A, R \otimes R) \otimes_{A^{\mathrm{e}}} (A_S)^{\mathrm{e}} \\ &\cong \operatorname{RHom}_{A^{\mathrm{e}}}(A, (R \otimes R) \otimes_{A^{\mathrm{e}}} (A_S)^{\mathrm{e}}) \end{aligned}$$

in $\mathsf{D}(\mathsf{Mod}(A_S)^{\mathrm{e}})$. Now

$$(R \otimes R) \otimes_{A^{\mathrm{e}}} (A_S)^{\mathrm{e}} \cong R_S \otimes R_S$$

in $\mathsf{D}(\mathsf{Mod}(A^{\mathrm{e}} \otimes (A_S)^{\mathrm{e}}))$. Finally using Lemma 3.7(2) we get

$$\begin{aligned} \mathrm{RHom}_{A^{\mathrm{e}}}(A, R_S \otimes R_S) &\cong \mathrm{RHom}_{(A_S)^{\mathrm{e}}}((A_S)^{\mathrm{e}} \otimes_A A, R_S \otimes R_S) \\ &\cong \mathrm{RHom}_{(A_S)^{\mathrm{e}}}(A_S, R_S \otimes R_S). \end{aligned}$$

□

If M is a bimodule over a ring A then the centralizer of M is

$$\mathrm{Z}_A(M) := \{a \in A \mid am = ma \text{ for all } m \in M\}.$$

Thus $\mathrm{Z}_A(A) = \mathrm{Z}(A)$. A ring homomorphism $A \to B$ is called *centralizing* if $B = A \cdot \mathrm{Z}_B(A)$. An *invertible bimodule* over A is a bimodule L such there exists another bimodule $L^\vee$ with $L \otimes_A L^\vee \cong L^\vee \otimes_A A \cong A$. If C is a commutative ring then a central invertible C-bimodule is the same as a projective C-module of rank 1.

Proposition 3.9. *Suppose C is a commutative affine $\mathbb{K}$-algebra. Then C has a rigid dualizing complex R_C consisting of central bimodules. If C is Cohen-Macaulay and equidimensional of dimension n then we can choose $R_C = \boldsymbol{\omega}_C[n]$ where $\boldsymbol{\omega}_C$ is a central bimodule, and if C is Gorenstein then $\boldsymbol{\omega}_C$ is invertible.*

Proof. First assume $C = \mathbb{K}[\boldsymbol{t}] = \mathbb{K}[t_1, \ldots, t_n]$, a polynomial algebra. Then the bimodule C is a dualizing complex. Because $\mathrm{Ext}^n_{C^{\mathrm{e}}}(C, C^{\mathrm{e}}) \cong C$ and $\mathrm{Ext}^q_{C^{\mathrm{e}}}(C, C^{\mathrm{e}}) = 0$ for $q \neq n$ it follows that the dualizing complex $C[n]$ is rigid.

Next take any affine algebra C. Choose a finite homomorphism $\mathbb{K}[\boldsymbol{t}] \to C$. Let $\mathbb{K}[\boldsymbol{t}] \to I$ be an injective resolution of the module $\mathbb{K}[\boldsymbol{t}]$ in $\mathsf{C}^{\mathrm{b}}(\mathsf{Mod}\, C)$ and define $R_C := \mathrm{Hom}_{\mathbb{K}[\boldsymbol{t}]}(C, I[n]) \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, C^{\mathrm{e}})$. So R_C consists of central C-bimodules, and $R_C = \mathrm{RHom}_{\mathbb{K}[\boldsymbol{t}]}(C, \mathbb{K}[\boldsymbol{t}][n])$. According to the calculations in the proof of [Ye4, Proposition 5.7], R_C is a rigid dualizing complex.

Finally suppose C is Cohen-Macaulay and equidimensional of dimension n. Choose a noether normalization, that is a finite (and necessarily injective) homomorphism $\mathbb{K}[\boldsymbol{t}] = \mathbb{K}[t_1, \ldots, t_n] \to C$, cf. [Ei, Theorem 13.3]. According to [Ei, Corollary 18.17], C is a projective $\mathbb{K}[\boldsymbol{t}]$-module. Hence $\boldsymbol{\omega}_C := \mathrm{Hom}_{\mathbb{K}[\boldsymbol{t}]}(C, \mathbb{K}[\boldsymbol{t}])$ is a rigid dualizing complex. If moreover C is Gorenstein then the bimodule C is also a dualizing complex, and by the uniqueness of dualizing complexes over commutative algebras (cf. [RD, Theorem V.3.1]) we find that $\boldsymbol{\omega}_C$ must be an invertible bimodule. □

Corollary 3.10. *Suppose C is a commutative affine $\mathbb{K}$-algebra, R_C is a rigid dualizing complex over C and $C \to A$ is a finite centralizing homomorphism of $\mathbb{K}$-algebras. Then $R_A := \mathrm{RHom}_C(A, R_C)$ is a rigid dualizing complex over A.*

Proof. Because of Proposition 3.9 and the uniqueness of rigid dualizing complexes we may assume R_C is a complex of central C-bimodules. Now proceed as in the proof of [Ye4, Proposition 5.7]. □

Proposition 3.11. *Suppose A is a noetherian affine $\mathbb{K}$-algebra finite over its center and $A \to B$ is a finite centralizing homomorphism. Let R_A and R_B be rigid dualizing complexes over A and B respectively. The the rigid trace $\mathrm{Tr}_{B/A} : R_B \to R_A$ exists.*

Proof. See [Ye4, Proposition 5.8], noting that the morphism $\mathrm{Tr}_{B/A}$ constructed there satisfies axioms of rigid trace, as can be seen using the calculations done in the proof of [Ye4, Proposition 5.7]. □

Remark 3.12. An alternative approach to proving the last three results is via noetherian connected filtrations; see [YZ2, Theorem 7.16] and text prior to it.

Example 3.13. A rigid dualizing complex of a commutative $\mathbb{K}$-algebra C need not be central. Let R_C be as in Proposition 3.9 above, and let N^0 be any non-central C-bimodule (e.g. $N^0 = A^{\sigma}$, the twisted bimodule with σ a nontrivial automorphism of A). Define the complex $N := (N^0 \xrightarrow{=} N^1)$. Then $R_C \cong R_C \oplus N$ in $\mathsf{D}(\mathsf{Mod}\, C^{\mathrm{e}})$, so the latter is a non-central rigid dualizing complex.

Example 3.14. Assume C is a smooth commutative $\mathbb{K}$-algebra of relative dimension n. Let $\Omega^n_{C/\mathbb{K}}$ be the module of degree n Kähler differentials. The canonical isomorphism (fundamental class of the diagonal)

$$\Omega^n_{C/\mathbb{K}} \cong \mathrm{Ext}^n_{C^{\mathrm{e}}}(C, \Omega^{2n}_{C^{\mathrm{e}}/\mathbb{K}})$$

makes $\Omega^n_{C/\mathbb{K}}[n]$ into a rigid dualizing complex. More generally for any C, if $\pi : \operatorname{Spec} C \to \operatorname{Spec} \mathbb{K}$ is the structural morphism, then the twisted inverse image $\pi^!\mathbb{K}$ of [RD] is the rigid dualizing complex of C.

4. Residual Complexes

In this section we examine a refined notion of dualizing complex, again generalizing from commutative algebraic geometry. Some graded examples have been studied by Ajitabh [Aj] and the first author [Ye2]. The main result here is Theorem 4.8, which guarantees the existence of a residual complex.

Suppose R is a dualizing complex over (A, B) – where A and B^{op} are left noetherian $\mathbb{K}$-algebras – and let M be a finite A-module. The grade of M with respect to R is

$$j_{R;A}(M) := \inf\{q \mid \mathrm{Ext}^q_A(M, R) \neq 0\} \in \mathbb{Z} \cup \{\infty\}.$$

Similarly define $j_{R;B^{\mathrm{op}}}$ for a B^{op}-module.

Definition 4.1 ([Ye2], [YZ2]). Let R be a dualizing complex over (A, B). We say that R is an *Auslander dualizing complex* if it satisfies the following two conditions:

(i) For every finite A-module M, integer q and B^{op}-submodule $N \subset \mathrm{Ext}^q_A(M, R)$, one has $j_{R;B^{\mathrm{op}}}(N) \geq q$.
(ii) The same holds after exchanging A and B^{op}.

Definition 4.2. For a finite A-module M the *canonical dimension* is

$$\mathrm{Cdim}_{R;A}\, M := -j_{R;A}(M).$$

It is known that if R is an Auslander dualizing complex then the canonical dimension $\mathrm{Cdim}_{R;A}$ is a spectral exact dimension function on $\mathsf{Mod}\, A$ (cf. Definition 1.6 and [YZ2, Theorem 2.10]). By symmetry there is a spectral exact dimension function $\mathrm{Cdim}_{R;B^{\mathrm{op}}}$ on $\mathsf{Mod}\, B^{\mathrm{op}}$.

Definition 4.3. A complex $R \in \mathsf{C}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$ is called a *residual complex* over (A, B) if the following conditions are satisfied:

(i) R is a dualizing complex.
(ii) Each bimodule R^{-q} is an injective module over A and over B^{op}.

(iii) R is Auslander, and each bimodule R^{-q} is $\mathrm{Cdim}_{R;A}$-pure and $\mathrm{Cdim}_{R;B^{\mathrm{op}}}$-pure of dimension q (Definition 2.6).

Let us denote by $\mathrm{Q} : \mathsf{C}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$ the localization functor. If $\mathsf{M} = \{\mathsf{M}_q(\mathrm{Cdim}_{R;A})\}$ then from Proposition 2.17(1) we see that $\mathrm{E}_{\mathsf{M}}\mathrm{Q}R = R$ for a residual complex R.

A complex $I \in \mathsf{D}^+(\mathsf{Mod}\, A)$ is called a minimal injective complex if the module I^q is injective and $\mathrm{Ker}(I^q \to I^{q+1}) \subset I^q$ is essential, for all q. Given $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$, there is a quasi-isomorphism $M \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, A)$, where I is a minimal injective complex. Such I is unique (up to a non-unique isomorphism), and it is called the *minimal injective resolution* of M (cf. [Ye1] Lemma 4.2). If M has finite injective dimension then I is bounded.

Definition 4.4. Let R be an Auslander dualizing complex over (A, B), and let I be the minimal injective resolution of R in $\mathsf{C}^+(\mathsf{Mod}\, A)$. Suppose each A-module I^{-q} is $\mathrm{Cdim}_{R;A}$-pure of dimension q. Then we say R has a *pure minimal injective resolution* over A. Likewise for B^{op}.

According to [Ye2, Lemma 2.15], a residual complex R is a minimal injective resolution of itself, on both sides. Thus R has pure minimal injective resolutions.

Proposition 4.5. *Suppose R is an Auslander dualizing complex over (A, B) that has a pure minimal injective resolution over A. Then the subcategories $\mathsf{M}_q(\mathrm{Cdim}_{R;A}) \subset \mathsf{Mod}\, A$ are stable for all q. Likewise with B^{op} and A exchanged.*

Proof. We may use the proof of [Ye2, Proposition 2.7]. □

Remark 4.6. If the subcategories $\mathsf{M}_q(\mathrm{Cdim}_{R;A}) \subset \mathsf{Mod}\, A$ are stable for all q then A is called a left pure algebra. As shown in [AjSZ], many familiar algebras with Auslander dualizing complexes do not admit residual complexes – indeed, are not even pure algebras (cf. Example 1.27).

Lemma 4.7. *Let R be a residual complex over (A, B). Then the ring homomorphisms (left and right multiplication)*

$$\mathrm{Z}(A), \mathrm{Z}(B^{\mathrm{op}}) \to \mathrm{Hom}_{\mathsf{C}(\mathsf{Mod}\, A\otimes B^{\mathrm{op}})}(R, R)$$

are bijective.

Proof. Since R consists of injective A-modules, $\mathrm{Hom}_A(R, R) = \mathrm{RHom}_A(R, R)$. So $\mathrm{H}^0 \mathrm{Hom}_A(R, R) = B^{\mathrm{op}} \cdot 1_R \cong B^{\mathrm{op}}$. By the purity assumption $\mathrm{Hom}_A(R, R)^{-1} = 0$, so

$$\mathrm{Hom}_{\mathsf{C}(\mathsf{Mod}\, A)}(R, R) = \mathrm{H}^0 \mathrm{Hom}_A(R, R) = B^{\mathrm{op}} \cdot 1_R \subset \mathrm{Hom}_{\mathsf{C}(\mathsf{Mod}\, \mathbb{K})}(R, R).$$

We see that

$$\mathrm{Hom}_{\mathsf{C}(\mathsf{Mod}\, A\otimes B^{\mathrm{op}})}(R, R) = \mathrm{Z}_{B^{\mathrm{op}}}(B^{\mathrm{op}} \cdot 1_R) = \mathrm{Z}(B^{\mathrm{op}}) \cdot 1_R.$$

The equality with $\mathrm{Z}(A) \cdot 1_R$ is proved symmetrically. □

Theorem 4.8. *Suppose R is an Auslander dualizing complex over (A, B) that has pure minimal injective resolutions over A and over B^{op}. Let $\mathsf{M} = \{\mathsf{M}_q(\mathrm{Cdim}_{R;A})\}$ be the filtration of $\mathsf{Mod}\, A$ determined by R and let $\mathrm{E} := \mathrm{E}_{\mathsf{M}}$ be the associated Cousin functor. Then $\mathrm{E}R$ is a residual complex, and there is a unique isomorphism*

$$\phi : R \xrightarrow{\simeq} \mathrm{QE}R \text{ in } \mathsf{D}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$$

such that

$$\mathrm{E}(\phi) : \mathrm{E}R \to \mathrm{EQE}R = \mathrm{E}R \textit{ in } \mathsf{C}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$$

is the identity.

Proof. If we decide to forget the B^{op}-module structure of R, we may use the minimal injective resolution I of R as A-module to compute $\mathrm{R}\Gamma_{\mathsf{M}^p\,/\,\mathsf{M}^{p+1}} R$. By the purity assumption

$$\mathrm{H}^{p+q}\mathrm{R}\Gamma_{\mathsf{M}^p\,/\,\mathsf{M}^{p+1}} R = \begin{cases} I^p & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

We conclude that $(\mathrm{E}_{\mathsf{M}} R)^{-q} \cong I^{-q}$ as A-modules, and so $(\mathrm{E}R)^{-q}$ is an injective A-module, $\mathrm{Cdim}_{R;A}$-pure of dimension q.

According to Proposition 4.5 we see that the hypotheses of Proposition 2.16 hold with $M = R$, $\mathsf{M} = \{\mathsf{M}_q(\mathrm{Cdim}_{R;A})\}$ and $\mathsf{N} = \{\mathsf{M}_q(\mathrm{Cdim}_{R;B^{\mathrm{op}}})\}$. This tells us that $\mathrm{E}_{\mathsf{M}} R \cong \mathrm{E}_{\mathsf{N}} R$ as complexes of bimodules. By the previous paragraph applied to B^{op} instead of A, $(\mathrm{E}_{\mathsf{N}} R)^{-q}$ is an injective B^{op}-module, $\mathrm{Cdim}_{R;B^{\mathrm{op}}}$-pure of dimension q.

Formula (4.9) says that Corollary 2.15 holds here. We deduce the existence of an isomorphism $\phi' : R \xrightarrow{\simeq} \mathrm{QE}R$ in $\mathsf{D}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$. According to Lemma 4.7 there is some invertible element $a \in \mathrm{Z}(A)$ such that $\mathrm{E}(\phi') : \mathrm{E}R \to \mathrm{EQE}R = \mathrm{E}R$ is multiplication by a. The isomorphism $\phi := a^{-1}\phi' : R \to \mathrm{QE}R$ has the desired property. □

Here is a class of algebras to which the results of this section apply. Recall that a ring A is right bounded if every essential right ideal of A contains an ideal which is essential as a right ideal. A ring A is a right FBN (fully bounded noetherian) if A is right noetherian and every prime factor ring of A is right bounded. An *FBN ring* is a ring A that is both right and left FBN [GW, Chapter 8].

A dualizing complex R over two algebras A and B is called *weakly bifinite* if for every bimodule M which is a subquotient of A, the bimodules $\mathrm{Ext}^q_A(M, R)$ are all finite on both sides; and the same is true with A and B^{op} interchanged.

An exact dimension function dim, defined on $\mathsf{Mod}\, A$ and on $\mathsf{Mod}\, B^{\mathrm{op}}$, is called *symmetric* if $\dim_A M = \dim_{B^\circ} M$ for every bimodule M finite on both sides.

Theorem 4.10. *Let A and B be FBN $\mathbb{K}$-algebras and let R be an Auslander dualizing complex over (A, B) which is weakly bifinite and such that* Cdim_R *is symmetric. Then R has pure minimal injective resolutions on both sides, and therefore the Cousin complex* $\mathrm{E}R$ (*notation as in Theorem* 4.8) *is a residual complex.*

For the proof we will need two lemmas and some notation. Let $\mathfrak{p}$ be a prime ideal of a left noetherian ring A. Write $S_{A/\mathfrak{p}}(0)$ for the set of regular elements of $A/\mathfrak{p}$. This is a denominator set in $A/\mathfrak{p}$, and the ring of fractions $Q(\mathfrak{p}) = \mathrm{Frac}\, A/\mathfrak{p} = S_{A/\mathfrak{p}}(0)^{-1}A/\mathfrak{p}$ is simple artinian.

Given a finite A-module M, it's *reduced* (*Goldie*) *rank* at $\mathfrak{p}$ is

$$\mathrm{rank}_{\mathfrak{p}}(M) := \mathrm{length}_{Q(\mathfrak{p})}\, Q(\mathfrak{p}) \otimes_A M.$$

For $M = A/\mathfrak{p}$ we write $\mathrm{r}(\mathfrak{p}) := \mathrm{rank}_{\mathfrak{p}}(A/\mathfrak{p})$. Let $J(\mathfrak{p}) = J_A(\mathfrak{p})$ be the indecomposable injective A-module with associated prime $\mathfrak{p}$. The injective hull of $A/\mathfrak{p}$ as A-module is then $J(\mathfrak{p})^{\mathrm{r}(\mathfrak{p})}$.

Suppose A is a prime ring. Recall that an element $m \in M$ is *torsion* if $am = 0$ for some regular element $a \in A$. M is a *torsion module* if all its elements are torsion;

otherwise it is a *non-torsion module.* M is *torsion-free* if the only torsion element in it is 0.

Lemma 4.11. *Suppose A is a prime left noetherian ring, and M, L are non-torsion A-modules, with M finite.*

(1) *There is an injective homomorphism $f : A \hookrightarrow L^r$, where $r = \mathrm{r}(0)$ is the Goldie rank.*
(2) *Let* $\dim$ *be an exact dimension function on* $\mathsf{Mod}\, A$. *Then* $\dim L = \dim A$.
(3) *There is a nonzero homomorphism $g : M \to L$.*

The proof of this lemma is standard, cf. [GW, Corollary 6.26(b)].

The following is essentially proved in [Br, Lemma 2.3]. We state it for any minimal injective complex instead of a minimal resolution of a module.

Lemma 4.12. *Suppose A is a left noetherian ring. Let I be a minimal injective complex of A-modules. Let $\mathfrak{p}$ be a prime ideal of A, and let $\mu_i(\mathfrak{p})$ be the multiplicity of $J(\mathfrak{p})$ in I^i. Then*

(1) *The image of the map $\partial^{i-1} : \mathrm{Hom}_A(A/\mathfrak{p}, I^{i-1}) \to \mathrm{Hom}_A(A/\mathfrak{p}, I^i)$ is a torsion $A/\mathfrak{p}$-module.*
(2) $\mu_i(\mathfrak{p}) = \mathrm{rank}_{\mathfrak{p}}(\mathrm{Hom}_A(A/\mathfrak{p}, I^i)) = \mathrm{rank}_{\mathfrak{p}}(\mathrm{Ext}^i_A(A/\mathfrak{p}, I))$.
(3) *Let $\mathfrak{p}$ and $\mathfrak{q}$ be two primes of A and M an $A/\mathfrak{p}$-$A/\mathfrak{q}$-bimodule. Assume M is nonzero, torsion-free as $(A/\mathfrak{q})^{\mathrm{op}}$-module, and finite non-torsion as $A/\mathfrak{p}$-module. If I^i contains a copy of $J(\mathfrak{p})$, then $\mathrm{Ext}^i_A(M, I)$ is a non-torsion $A/\mathfrak{q}$-module.*

Proof. (1) This is true because the kernel of this map is an essential submodule.

(2) The first equality is clear. The second follows from (1) because factoring out $A/\mathfrak{p}$-torsion the complex $\mathrm{Hom}_A(A/\mathfrak{p}, I)$ has zero coboundary maps.

(3) By assumption there is a nonzero left ideal L of $A/\mathfrak{p}$ contained in I^i. Let $Z^i := \mathrm{Ker}(\partial^i : I^i \to I^{i+1})$. Replacing L by $L \cap Z^i$, we may assume $L \subset Z^i$. Since L and M are non-torsion $A/\mathfrak{p}$-modules and M is finite, Lemma 4.11 says there is a nonzero map $f : M \to L$. We claim that f is nonzero in $\mathrm{Ext}^i_A(M, I)$. Otherwise there is a map $g : M \to I^{i-1}$ such that $f = \partial^{i-1} g$. Let $M' := g(M) \subset I^{i-1}$. Then $M' \cap Z^{i-1}$ is essential in M' and hence $M'/(M' \cap Z^{i-1})$ is a torsion $A/\mathfrak{p}$-module. Now $f(M)$ is a quotient of $M'/(M' \cap Z^{i-1})$, so it is a nonzero torsion $A/\mathfrak{p}$-module. This contradicts the fact that any nonzero submodule of L must be a torsion-free $A/\mathfrak{p}$-module. Therefore we proved that f is nonzero in $\mathrm{Ext}^i_A(M, I)$.

Now let a be a regular element of $A/\mathfrak{q}$. Since $M \cong Ma$, M/Ma is a torsion $A/\mathfrak{p}$-module. Hence Ma is not contained in the kernel of $f : M \to L$. This implies that $af : M \to L$ is nonzero. By the claim we proved in the last paragraph, we see that af is nonzero in $\mathrm{Ext}^i_A(M, I)$. So f is non-torsion in $\mathrm{Ext}^i_A(M, I)$. □

Proof of Theorem 4.8. Let I be the minimal injective resolution of R as complex of A-modules. First we will show that each I^{-i} is essentially pure of $\mathrm{Cdim}_{R;A} = i$, meaning that I^{-i} contains an essential submodule that's pure of $\mathrm{Cdim}_{R;A} = i$. It suffices to show that if M is a $\mathrm{Cdim}_{R;A}$-critical submodule of I^{-i}, then $\mathrm{Cdim}_{R;A} M = i$.

The critical module M is uniform. Since A is FBN the injective hull of M is $J(\mathfrak{q})$ for some prime ideal $\mathfrak{q}$. Replacing M by a nonzero submodule we can assume M is a left ideal of $A/\mathfrak{q}$, so it is a torsion-free $A/\mathfrak{q}$-module. By Lemma 4.12(3),

$E := \mathrm{Ext}_A^{-i}(A/\mathfrak{q}, R)$ is a non-torsion $A/\mathfrak{q}$-module. In particular $E \neq 0$ and hence $\mathrm{Cdim}_{R;A} A/\mathfrak{q} \geq i$. By Lemma 4.11 we get $\mathrm{Cdim}_{R;A} E = \mathrm{Cdim}_{R;A} A/\mathfrak{q}$. From the weakly bifinite hypothesis, E is noetherian on both sides. Hence, by the symmetry of Cdim_R, we have $\mathrm{Cdim}_{R;A} E = \mathrm{Cdim}_{R;B^{\mathrm{op}}} E$. According to [YZ2, Theorem 2.14] we have $\mathrm{Cdim}_{R;B^{\mathrm{op}}} E \leq i$. We conclude that $\mathrm{Cdim}_{R;A} A/\mathfrak{q} = i$. Again by Lemma 4.11 we get $\mathrm{Cdim}_{R;A} M = \mathrm{Cdim}_{R;A} A/\mathfrak{q} = i$.

Next we show that the $\mathrm{Cdim}_{R;A}$ is a constant on the cliques of A. If there is a link $\mathfrak{q} \rightsquigarrow \mathfrak{p}$, then there is a nonzero $A/\mathfrak{q}$-$A/\mathfrak{p}$-bimodule M that is a subquotient of A and is torsion free on both sides. By Lemma 4.11 and the symmetry of Cdim_R we have

$$\begin{aligned}\mathrm{Cdim}_{R;A} A/\mathfrak{q} = \mathrm{Cdim}_{R;A} M &= \mathrm{Cdim}_{R;B^{\mathrm{op}}} M \\ &= \mathrm{Cdim}_{R;B^{\mathrm{op}}} A/\mathfrak{p} = \mathrm{Cdim}_{R;A} A/\mathfrak{p}.\end{aligned}$$

The FBN ring A satisfies the second layer condition [MR, 4.3.14]. We know that $\mathrm{Cdim}_{R;A}$ is constant on cliques. It follows from [MR, Proposition 4.3.13] that an indecomposable injective A-module has pure $\mathrm{Cdim}_{R;A}$ (cf. [AjSZ, Theorem 4.2]). So the minimal injective resolution I is pure.

All the above works also for the minimal injective resolution of R as a complex of B^{op}-modules. □

Remark 4.13. Let A be a noetherian affine PI Hopf algebra over $\mathbb{K}$ of finite injective dimension n. Brown and Goodearl [BG] show that A is Auslander-Gorenstein. Using this one can show that the Auslander dualizing complex $A[n]$ is pre-balanced and has pure minimal injective resolutions (see [YZ2]). According to Theorem 4.8, $\mathrm{E}(A[n])$ is a residual complex.

5. The Residue Complex of an Algebra

In this section we define the residue complex of an algebra, combining the results if Sections 3 and 4. The main result here is Theorem 5.4 which explains the functoriality of residue complexes. Here as before $\mathbb{K}$ is the base field.

If a $\mathbb{K}$-algebra A has a rigid dualizing complex R (Definition 3.1) that is Auslander (Definition 4.1) then we shall usually write Cdim_A instead of $\mathrm{Cdim}_{R;A}$. This dimension function depends only on the $\mathbb{K}$-algebra A.

Definition 5.1. A *residue complex* over A is a rigid dualizing complex (R, ϕ) such that R is also a residual complex (Definition 4.3).

The uniqueness of the residue complex will be made clear later in this section (Corollary 5.5). In [Ye2] the name "strong residue complex" was used for the same notion (in the graded case).

Let $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)$ denote the subcategory of complexes with finite cohomology modules. The next result will be used in the proof of Theorem 5.4.

Proposition 5.2 (Local Duality). *Let R be a dualizing complex over (A, B) and $\mathsf{M} \subset \mathsf{Mod}\, A$ a localizing subcategory. Then there is a functorial isomorphism*

$$\mathrm{R}\Gamma_{\mathsf{M}} M \cong \mathrm{RHom}_{B^{\mathrm{op}}}(\mathrm{RHom}_A(M, R), \mathrm{R}\Gamma_{\mathsf{M}} R)$$

for $M \in \mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

Proof. Take a quasi-isomorphism $M \to I$ in $\mathsf{C}^+(\mathsf{Mod}\, A)$ with each I^q injective over A, and a quasi-isomorphism $R \to J$ in $\mathsf{C}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$, where each J^q is injective over A and over B^{op}. Write $DM := \mathrm{RHom}_A(M, R)$ and $D^{\mathrm{op}}N := \mathrm{RHom}_{B^{\mathrm{op}}}(N, R)$. Using Lemma 4.7 we get a commutative diagram in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathsf{M}} M \arrow[r, "\alpha"] \arrow[d, "="] & \mathrm{R}\Gamma_{\mathsf{M}} D^{\mathrm{op}} D M \arrow[r, "\beta"] \arrow[d, "="] & \mathrm{RHom}_{B^{\mathrm{op}}}(DM, \mathrm{R}\Gamma_{\mathsf{M}} R) \arrow[d, "="] \\
\Gamma_{\mathsf{M}} I \arrow[r] & \Gamma_{\mathsf{M}} \mathrm{Hom}_{B^{\mathrm{op}}}(\mathrm{Hom}_A(I, J), J) \arrow[r, "\gamma"] & \mathrm{Hom}_{B^{\mathrm{op}}}(\mathrm{Hom}_A(I, J), \Gamma_{\mathsf{M}} J)
\end{tikzcd}$$

with the bottom row consisting of morphisms in $\mathsf{C}^+(\mathsf{Mod}\, A)$. The homomorphism γ is actually bijective. And since $M \in \mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)$, $M \to D^{\mathrm{op}} D M$ is an isomorphism, and hence so is α. The isomorphism we want is $\beta\alpha$. □

Remark 5.3. The proposition above generalizes [RD, Theorem IV.6.2] (resp. [Ye1, Theorem 4.18]), when A is commutative local with maximal ideal (resp. connected graded with augmentation ideal) $\mathfrak{m}$, and M is the category of $\mathfrak{m}$-torsion A-modules.

Denote by $\mathrm{Q} : \mathsf{C}(\mathsf{Mod}\, A^{\mathrm{e}}) \to \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ the localization functor.

Theorem 5.4. *Let $A \to B$ be a finite centralizing homomorphism between noetherian $\mathbb{K}$-algebras. Suppose the two conditions below hold.*

(i) *There are rigid dualizing complexes R_A and R_B and the rigid trace morphism $\mathrm{Tr}_{B/A} : R_B \to R_A$ exists.*
(ii) *R_A is an Auslander dualizing complex and it has pure minimal injective resolutions on both sides.*

Then:

(1) *R_B is an Auslander dualizing complex and it has pure minimal injective resolutions on both sides.*
(2) *Denote by $\mathrm{E}_A : \mathsf{D}^+(\mathsf{Mod}\, A^{\mathrm{e}}) \to \mathsf{C}^+(\mathsf{Mod}\, A^{\mathrm{e}})$ and $\mathrm{E}_B : \mathsf{D}^+(\mathsf{Mod}\, B^{\mathrm{e}}) \to \mathsf{C}^+(\mathsf{Mod}\, B^{\mathrm{e}})$ the Cousin functors associated to the dimension functions $\mathrm{Cdim}_{R_A;A}$ and $\mathrm{Cdim}_{R_B;B}$ respectively. Then $\mathrm{E}_A M \cong \mathrm{E}_B M$ functorially for $M \in \mathsf{D}^+(\mathsf{Mod}\, B^{\mathrm{e}})$.*
(3) *Let $\mathcal{K}_A := \mathrm{E}_A R_A$ and $\mathcal{K}_B := \mathrm{E}_B R_B \cong \mathrm{E}_A R_B$ be the two residual complexes, so we have a morphism $\mathrm{E}_A(\mathrm{Tr}_{B/A}) : \mathcal{K}_B \to \mathcal{K}_A$ in $\mathsf{C}(\mathsf{Mod}\, A^{\mathrm{e}})$. Let $\phi_A : R_A \xrightarrow{\simeq} \mathrm{Q}\mathcal{K}_A$ and $\phi_B : R_B \xrightarrow{\simeq} \mathrm{Q}\mathcal{K}_B$ be the isomorphisms from Theorem 4.8. Then the diagram*

$$\begin{tikzcd}
R_B \arrow[r, "\phi_B"] \arrow[d, "\mathrm{Tr}"] & \mathrm{Q}\mathcal{K}_B \arrow[d, "\mathrm{Q}\mathrm{E}_A(\mathrm{Tr})"] \\
R_A \arrow[r, "\phi_A"] & \mathrm{Q}\mathcal{K}_A
\end{tikzcd}$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ is commutative.
(4) *$\mathrm{E}_A(\mathrm{Tr}_{B/A})$ induces isomorphisms*

$$\mathcal{K}_B \cong \mathrm{Hom}_A(B, \mathcal{K}_A) \cong \mathrm{Hom}_{A^{\mathrm{op}}}(B, \mathcal{K}_A)$$

in $\mathsf{C}(\mathsf{Mod}\, A^{\mathrm{e}})$.

Proof. 1. According to [YZ2, Proposition 3.9]

$$R_B \cong \mathrm{RHom}_A(B, R_A) \cong \mathrm{Hom}_A(B, \mathcal{K}_A)$$

in $\mathsf{D}(\mathsf{Mod}(B \otimes A^{\mathrm{op}}))$. Therefore the complex $\mathrm{Hom}_A(B, \mathcal{K}_A)$ is an injective resolution of R_B in $\mathsf{K}^+(\mathsf{Mod}\, B)$. Choose elements $b_1, \ldots, b_r \in \mathrm{Z}_B(A)$ which generate B as an A-module. This gives rise to a surjection $A^r \twoheadrightarrow B$ of A-bimodules, and hence to an inclusion $\mathrm{Hom}_A(B, \mathcal{K}_A^{-q}) \subset (\mathcal{K}_A^{-q})^r$. So $\mathrm{Hom}_A(B, \mathcal{K}_A^{-q})$ is $\mathrm{Cdim}_{R_A;A}$-pure of dimension q as an A-module. By [YZ2, Proposition 3.9], the dualizing complex R_B is Auslander, and $\mathrm{Cdim}_{R_A;A} M = \mathrm{Cdim}_{R_B;B} M$ for any B-module M. Thus $\mathrm{Hom}_A(B, \mathcal{K}_A^{-q})$ is $\mathrm{Cdim}_{R_B;B}$-pure as B-module. We conclude that the injective resolution $\mathrm{Hom}_A(B, \mathcal{K}_A)$ is $\mathrm{Cdim}_{R_B;B}$-pure. But one easily sees that a pure injective resolution must be minimal. Symmetrically all the above applies to the right resolution $\mathrm{Hom}_{A^{\mathrm{op}}}(B, \mathcal{K}_A)$ of R_B.

2. Applying the functor E_B to the isomorphism $R_B \cong \mathrm{Hom}_A(B, \mathcal{K}_A)$ in $\mathsf{D}(\mathsf{Mod}\, B \otimes A^{\mathrm{op}})$, and using the fact that $\mathrm{Hom}_A(B, \mathcal{K}_A)$ is a pure injective complex of B-modules, we obtain $\mathcal{K}_B \cong \mathrm{Hom}_A(B, \mathcal{K}_A)$ in $\mathsf{C}(\mathsf{Mod}\, B \otimes A^{\mathrm{op}})$. Thus in particular $\mathrm{Hom}_A(B, \mathcal{K}_A)$ is a complex of B-B-bimodules. By symmetry also $\mathcal{K}_B \cong \mathrm{Hom}_{A^{\mathrm{op}}}(B, \mathcal{K}_A)$ as complexes of bimodules.

Denote by $\mathsf{M}_q(A) := \{M \in \mathsf{Mod}\, A \mid \mathrm{Cdim}_{R_A;A} M \leq q\}$ and likewise $\mathsf{M}_q(B)$. We get filtrations $\mathsf{M}(A) = \{\mathsf{M}_q(A)\}$ and $\mathsf{M}(B) = \{\mathsf{M}_q(B)\}$. Since $\Gamma_{\mathsf{M}_q(A)} \mathcal{K}_A = \mathcal{K}_A^{\geq -q}$, Proposition 5.2 tells us that

$$\begin{aligned} \mathrm{R}\Gamma_{\mathsf{M}_q(A)} M &\cong \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, \mathcal{K}_A), \mathcal{K}_A^{\geq -q}) \\ &\cong \mathrm{Hom}_{B^{\mathrm{op}}}(\mathrm{Hom}_B(M, \mathcal{K}_B), \mathcal{K}_B^{\geq -q}) \cong \mathrm{R}\Gamma_{\mathsf{M}_q(B)} M \end{aligned}$$

functorially for $M \in \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, B)$. In particular $\mathrm{H}^p_{\mathsf{M}_q(B)} M \cong \mathrm{H}^p_{\mathsf{M}_q(A)} M$ functorially for finite B-modules M. Passing to direct limits (using Proposition 1.20) this becomes true for all B-modules. Hence if M is an $\mathsf{M}(B)$-flasque B-module, it is also $\mathsf{M}(A)$-flasque. By Proposition 2.9 it follows that $\mathrm{E}_A M \cong \mathrm{E}_B M$ functorially for $M \in \mathsf{D}^+(\mathsf{Mod}\, B^{\mathrm{e}})$.

3. Next we analyze the morphism $\mathrm{QE}_A(\mathrm{Tr}_{B/A}) \in \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})}(R_B, R_A)$. By [YZ2, Lemma 3.3],

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})}(R_B, R_A) = \mathrm{Z}_B(A) \cdot \mathrm{Tr}_{B/A},$$

so $\mathrm{QE}_A(\mathrm{Tr}_{B/A}) = b \cdot \mathrm{Tr}_{B/A}$ for some (unique) $b \in \mathrm{Z}_B(A)$. We shall prove that $b = 1$. If we forget the A^{op}-module structure, then

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(\mathrm{Q}\mathcal{K}_B, \mathrm{Q}\mathcal{K}_A) = B \cdot \mathrm{Tr}_{B/A}\,.$$

From Proposition 2.17 we get that there is a bijection

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(\mathrm{Q}\mathcal{K}_B, \mathrm{Q}\mathcal{K}_A) \cong \mathrm{Hom}_{\mathsf{C}(\mathsf{Mod}\, A)}(\mathcal{K}_B, \mathcal{K}_A)$$

induced by E, and the inverse is induced by Q. Hence we obtain

$$\mathrm{Tr}_{B/A} = \mathrm{QE}_A(\mathrm{Tr}_{B/A}) \in \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(\mathrm{Q}\mathcal{K}_B, \mathrm{Q}\mathcal{K}_A).$$

This implies that $b = 1$.

4. By part 3, if we apply the functor $\mathrm{Hom}_A(B, -)$ to the homomorphism of complexes $\mathrm{E}_A(\mathrm{Tr}_{B/A}) : \mathcal{K}_B \to \mathcal{K}_A$ we get a quasi-isomorphism $\mathcal{K}_B \to \mathrm{Hom}_A(B, \mathcal{K}_A)$. But these are minimal injective complexes of B-modules, so it must actually be

an isomorphism of complexes. By symmetry also $\mathcal{K}_B \to \mathrm{Hom}_{A^{\mathrm{op}}}(B, \mathcal{K}_A)$ is an isomorphism of complexes. □

Corollary 5.5 (Uniqueness of Residue Complex)**.** *Suppose $(\mathcal{K}_A, \phi)$ and $(\mathcal{K}'_A, \phi')$ are two residue complexes over A. Then there is a unique isomorphism $\tau : \mathcal{K}'_A \xrightarrow{\simeq} \mathcal{K}_A$ in $\mathsf{C}(\mathsf{Mod}\, A^{\mathrm{e}})$ that's compatible with ϕ' and ϕ, i.e. a rigid trace.*

Proof. Write $B := A$ and $(\mathcal{K}_B, \phi_B) := (\mathcal{K}'_A, \phi')$. By [YZ2, Theorem 3.2] we get a unique isomorphism $\mathrm{Tr}_{B/A} : \mathcal{K}_B \xrightarrow{\simeq} \mathcal{K}_A$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ that's a rigid trace. According to part 3 of the theorem above, $\tau := \mathrm{E}_A(\mathrm{Tr}_{B/A})$ satisfies $\mathrm{Q}(\tau) = \mathrm{Tr}_{B/A}$, so it too is a rigid trace. □

Corollary 5.6. *If in the previous theorem $B = A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ then there is equality*

$$\mathrm{Hom}_A(A/\mathfrak{a}, \mathcal{K}_A) = \mathrm{Hom}_{A^{\mathrm{op}}}(A/\mathfrak{a}, \mathcal{K}_A) \subset \mathcal{K}_A.$$

Proof. By part 4 of the theorem we get an isomorphism of $B \otimes A^{\mathrm{op}}$-modules $\mathcal{K}^q_B \cong \mathrm{Hom}_A(B, \mathcal{K}^q_A) \subset \mathcal{K}^q_A$ for every q. This implies that $\mathrm{Hom}_A(B, \mathcal{K}^q_A)$ is annihilated by $\mathfrak{a}$ on the right too, and hence $\mathrm{Hom}_A(B, \mathcal{K}_A) \subset \mathrm{Hom}_{A^{\mathrm{op}}}(B, \mathcal{K}_A)$. By symmetry there is equality. □

Remark 5.7. Corollary 5.6 is pretty surprising. The ideal $\mathfrak{a}$ will in general not be generated by central elements. On the other hand the centralizer $\mathrm{Z}_A(\mathcal{K}_A) = \mathrm{Z}(A)$. So there is no obvious reason for the left annihilator of $\mathfrak{a}$ in $\mathcal{K}_A$ to coincide with the right annihilator.

Corollary 5.8. *Let $A \to B$ and $B \to C$ be finite centralizing homomorphisms. Assume the hypotheses of Theorem 5.4, and also that the rigid dualizing complex R_C and the rigid trace $\mathrm{Tr}_{C/B}$ exist. Then*

$$\mathrm{E}_A(\mathrm{Tr}_{C/A}) = \mathrm{E}_A(\mathrm{Tr}_{B/A})\, \mathrm{E}_B(\mathrm{Tr}_{C/B}) : \mathcal{K}_C \to \mathcal{K}_A.$$

Proof. By [YZ2, Corollary 3.8] the morphism $\mathrm{Tr}_{C/A} := \mathrm{Tr}_{B/A} \mathrm{Tr}_{C/B}$ is a rigid trace. According to Theorem 5.4 the residue complex $\mathcal{K}_C = \mathrm{E}_B R_C$ exists, and $\mathrm{E}_B(\mathrm{Tr}_{C/B}) = \mathrm{E}_A(\mathrm{Tr}_{C/B})$. □

Here are a few examples of algebras with residue complexes.

Example 5.9. If A is a commutative affine (i.e. finitely generated) $\mathbb{K}$-algebra and R_A is its rigid dualizing complex then the complex $\mathcal{K}_A := \mathrm{E} R_A$ is a residue complex. It consists of central bimodules, and is the residue complex of A also in the sense of [RD]; cf. Example 3.14. For a finite homomorphism $A \to B$ of commutative algebras the trace morphism $\mathrm{Tr}_{B/A}$ coincides with that of [RD].

Example 5.10. Consider a commutative artinian local $\mathbb{K}$-algebra A whose residue field $A/\mathfrak{m}$ is finitely generated over $\mathbb{K}$ (i.e. A is residually finitely generated). Then $A \cong \mathrm{Frac}\, A_0$, the ring of fractions of some commutative affine $\mathbb{K}$-algebra $A_0 \subset A$, and by Theorem 3.8 A has a rigid dualizing complex $R_A \cong A \otimes_{A_0} R_{A_0}$. Because of the uniqueness of dualizing complexes for commutative algebras, $R_A \cong \mathcal{K}(A)[n]$ where $\mathcal{K}(A) := \mathrm{H}^{-n}_{\mathfrak{m}} R_A$ is an injective hull of $A/\mathfrak{m}$ and $n = \dim A_0 = \mathrm{tr.deg}_{\mathbb{K}}(A/\mathfrak{m})$.

If $A \to B$ is a finite homomorphism of such artinian algebras then the rigid trace $\mathrm{Tr}_{B/A} : \mathcal{K}(B)[n] \to \mathcal{K}(A)[n]$ exists.

Now if A is a residually finitely generated commutative noetherian complete local $\mathbb{K}$-algebra we can define $\mathcal{K}(A) := \varinjlim \mathcal{K}(A/\mathfrak{m}^i)$. The functorial A-module

$\mathcal{K}(A)$ is called the *dual module* of A. Cf. [Ye3] and [Ye5] for alternative approaches, applications and references to other related work.

Example 5.11. If A is a noetherian affine $\mathbb{K}$-algebra finite over its center C and $\mathcal{K}_C$ is the residue complex of C then $\mathcal{K}_A := \operatorname{Hom}_C(A, \mathcal{K}_C)$ is the residue complex of A. If $A \to B$ is a finite centralizing homomorphism then the rigid trace $\operatorname{Tr}_{B/A} : \mathcal{K}_B \to \mathcal{K}_A$ is gotten by applying $\operatorname{Hom}_C(-, \mathcal{K}_C)$ to $A \to B$. We see that the theory of residual complexes for algebras finite over the center is very close to the commutative theory.

Proposition 5.12. *The first Weyl algebra over the field* $\mathbb{C}$ *has a residue complex.*

Proof. Recall that the first Weyl algebra is $A := \mathbb{C}\langle x, y\rangle/(yx - xy - 1)$. According to [YZ2, Example 6.20], $R_A := A[2]$ is a rigid Auslander dualizing complex over A, and $\operatorname{Cdim} = \operatorname{GKdim}$. The ring of fractions $Q = \operatorname{Frac} A$ is a division ring, and the global dimension of A is 1. Therefore the minimal injective resolution of A in $\mathsf{Mod}\, A$ is $0 \to A \to I^0 \to I^1 \to 0$ and $I^0 \cong Q$. We see that I^0 is pure of $\operatorname{GKdim} I^0 = \operatorname{GKdim} A = 2$. Since I^1 is a torsion A-module we get $\operatorname{GKdim} I^1 \leq 1$; but since there are no A-modules of $\operatorname{GKdim} = 0$ it follows that I^1 is pure of $\operatorname{GKdim} = 1$. So A has a pure injective resolution on the left. The same is true on the right too. We see that R_A has pure minimal injective resolutions on both sides, so $\mathcal{K}_A := \mathrm{E} R_A$ is a residue complex. Moreover $\mathcal{K}_A^{-2} = Q$ and $\mathcal{K}_A^{-1} = Q/A$. □

Proposition 5.13. *Let* $\mathfrak{g}$ *be a nilpotent* 3*-dimensional Lie algebra over* $\mathbb{C}$ *and* $A := \mathrm{U}(\mathfrak{g})$ *the universal enveloping algebra. Then* A *has a residue complex.*

Proof. We may assume A is not commutative, so A is generated by x, y, z with z central and $[x, y] = z$. By [YZ2, Proposition 6.18] and [Ye5, Theorem A] the complex $R_A := A[3]$ is a rigid Auslander dualizing complex, and $\operatorname{Cdim} = \operatorname{GKdim}$. Consider a minimal injective resolution $0 \to A \to I^0 \to I^1 \to I^2 \to I^3 \to 0$ of A in $\mathsf{Mod}\, A$. For any $\lambda \in \mathbb{C}$ consider the ideal $\mathfrak{a} = A \cdot (z - \lambda)$. The localizing subcategory $\mathsf{M}_{(z-\lambda)} = \mathsf{M}_{\mathfrak{a}} \subset \mathsf{Mod}\, A$ is stable (cf. Example 1.4). We get a direct sum decomposition of A-modules indexed by $\operatorname{Spec} \mathbb{C}[z]$:

$$I^q \cong \Big(\bigoplus_{\lambda \in \mathbb{C}} \Gamma_{(z-\lambda)} I^q \Big) \oplus \big(\mathbb{C}(z) \otimes_{\mathbb{C}[z]} I^q \big).$$

For any $\lambda \in \mathbb{C}$, $\operatorname{RHom}_A(A/(z-\lambda), R_A)$ is the rigid dualizing complex of $A/(z-\lambda)$, and $\operatorname{Hom}_A(A/(z-\lambda), I[3])$ is its minimal injective resolution as complex of left modules. Since the algebra $A/(z-\lambda)$ is isomorphic to either the commutative polynomial algebra ($\lambda = 0$) or to the first Weyl algebra, we see that $\operatorname{Hom}_A(A/(z-\lambda), I^q)$ is pure of $\operatorname{GKdim} = 3 - q$, for $1 \leq q \leq 3$.

Fix q and λ. Introduce a filtration F on $N := \Gamma_{(z-\lambda)} I^q$ by $F^{-j} N := \operatorname{Hom}_A(A/(z-\lambda)^j, I^q)$. Then for $j \geq 1$ multiplication by $z - \lambda$ is a bijection $\operatorname{gr}_F^{-j-1} N \xrightarrow{\simeq} \operatorname{gr}_F^{-j} N$. It follows that $\operatorname{gr}_F^{-j} N$ is pure of $\operatorname{GKdim} = 3 - q$. Therefore also $N = \bigcup F^{-j} N$ is pure of $\operatorname{GKdim} = 3 - q$.

The direct sum complement $\mathbb{C}(z) \otimes_{\mathbb{C}[z]} I^q$ is a B-module, where $B := \mathbb{C}(z) \otimes_{\mathbb{C}[z]} A$. In fact $\mathbb{C}(z) \otimes_{\mathbb{C}[z]} I$ is a minimal injective resolution of B in $\mathsf{Mod}\, B$. But B is isomorphic the first Weyl algebra over the field $\mathbb{C}(z)$. Therefore $I^0 \cong \operatorname{Frac} B$ is pure of $\operatorname{GKdim} = 3$, and $\mathbb{C}(z) \otimes_{\mathbb{C}[z]} I^1$ is pure of $\operatorname{GKdim} = 2$.

We conclude that each I^q is pure of $\operatorname{GKdim} = 3 - q$. By symmetry the same is true on the right too. So $\mathcal{K}_A := \mathrm{E} R_A$ is a residue complex over A. □

Remark 5.14. There are nilpotent Lie algebras $\mathfrak{g}$ such that $\mathrm{U}(\mathfrak{g})$ does not have a residue complex. Indeed one can find such a Lie algebra with a surjection from $A = \mathrm{U}(\mathfrak{g})$ to the second Weyl algebra B. B is not pure, so it does not have a residue complex. Hence by Theorem 5.4, A does not have a residue complex.

Example 5.15. Let A be a 3-dimensional Sklyanin algebra over the algebraically closed field $\mathbb{K}$. The whole apparatus of Cousin functors can be implemented also in the $\mathbb{Z}$-graded module category $\mathsf{GrMod}\, A$– actually this was already done in [Ye2] – and in particular Theorem 4.8 is true in the graded sense. According to [Aj] the minimal injective resolutions of A in $\mathsf{GrMod}\, A$ and in $\mathsf{GrMod}\, A^{\mathrm{op}}$ are pure. On the hand the balanced dualizing complex, which is also rigid in the graded sense, is $R_A = \boldsymbol{\omega}_A[3]$ where $\boldsymbol{\omega}_A = A^{\sigma}$ for some automorphism σ. We conclude that $\mathcal{K}_A := \mathrm{E} R_A$ is a graded residue complex over A. Note that this result was proved in [Ye2] by a direct (and rather involved) calculation of Öre localizations with respect to σ-orbits in the elliptic curve associated to A.

Question 5.16. In case the rigid Auslander dualizing complex R exists but there is no residue complex (e.g. $A = \mathrm{U}(\mathfrak{sl}_2)$), is it still true that $\mathrm{E}_A R \cong \mathrm{E}_{A^{\mathrm{op}}} R$? What can be said about this complex?

In the following section we will discuss residue complexes over PI algebras in detail.

6. The Residue Complex of a PI Algebra

In this section we look at an affine noetherian PI algebra A over the base field $\mathbb{K}$. We show that – under a certain technical assumption – such an algebra A has a residue complex $\mathcal{K}_A$. Furthermore in Theorem 6.14 we give a detailed description of the structure of $\mathcal{K}_A$. The material on PI rings needed here can be found in [MR, Section 13].

Proposition 6.1. *Suppose A is an affine prime PI $\mathbb{K}$-algebra with center C. Then there is a nonzero element $s \in C$ such that the localization A_s is an Azumaya algebra over its center $\mathrm{Z}(A_s) = C_s$, and C_s is a regular commutative affine $\mathbb{K}$-algebra.*

Proof. By the Artin-Procesi Theorem [MR, Theorem 13.7.14] and [MR, Proposition 13.7.4] we may find $s_1 \in C$, $s_1 \neq 0$ such that A_{s_1} is an Azumaya algebra over its center $C_1 := \mathrm{Z}(A_{s_1})$. The commutative prime $\mathbb{K}$-algebra C_1 is affine, and hence by [Mat, page 246, Theorem 73] there is a nonzero element $s_2 \in C_1$ such that the localization $C_2 := (C_1)_{s_2}$ is regular. By Posner's Theorem [MR, Theorem 13.6.5] the fraction fields coincide: $\operatorname{Frac} C = \operatorname{Frac} C_2$. Because C_2 is affine we may find $s \in C$ (the product of the denominators of a finite set of $\mathbb{K}$-algebra generators of C_2) such that $C_s = (C_2)_s$. Hence C_s is also regular, affine over $\mathbb{K}$, and A_s is Azumaya with center C_s. $\square$

Theorem 6.2. *Let A be an affine prime noetherian PI $\mathbb{K}$-algebra with center C and Gelfand-Kirillov dimension $\operatorname{GKdim} A = n$. Assume A has a rigid dualizing complex R_A.*

(1) *Let $s \in C$ be a nonzero element such that the localization A_s is an Azumaya algebra with center C_s, and C_s is a regular affine $\mathbb{K}$-algebra. Then there is an isomorphism*

$$C_s \otimes_C R_A \cong \boldsymbol{\omega}_{C_s}[n] \otimes_C A$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$*, where* $\boldsymbol{\omega}_{C_s}$ *is a projective* C_s*-module of rank* 1*.*

(2) *Let* $K := \operatorname{Frac} C$ *and* $Q := \operatorname{Frac} A$*. Then*

$$K \otimes_C R_A \cong Q[n]$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$*.*

Proof. (1) By [ASZ, Proposition 4.4] the algebra A^{e} is noetherian. Therefore according to Theorem 3.8 the complex $R_{A_s} := A_s \otimes_A R_A \otimes_A A_s$ is a rigid dualizing complex over A_s. Moreover,

$$R_{A_s} \cong R_A \otimes_C C_s \cong C_s \otimes_C R_A$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

By [MR, Proposition 8.2.13] we have $\operatorname{GKdim} A = \operatorname{GKdim} A_s$, and hence $n = \operatorname{GKdim} C_s = \operatorname{Kdim} C_s$ (Krull dimension). According to Proposition 3.9 the rigid dualizing complex of C_s is $\boldsymbol{\omega}_{C_s}[n]$ with $\boldsymbol{\omega}_{C_s}$ a projective C_s-module of rank 1. From Corollary 3.10 we see that $\operatorname{Hom}_{C_s}(A_s, \boldsymbol{\omega}_{C_s})$ is a rigid dualizing complex over A_s. Finally the reduced trace $A_s \to C_s$ induces a bimodule isomorphism $A_s \cong \operatorname{Hom}_{C_s}(A_s, C_s)$. Therefore

$$R_{A_s} \cong \boldsymbol{\omega}_{C_s} \otimes_{C_s} A_s \cong \boldsymbol{\omega}_{C_s} \otimes_C A$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

(2) Follows from (1). □

Recall that a *connected graded* $\mathbb{K}$-algebra is an $\mathbb{N}$-graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ such that $A_0 = \mathbb{K}$ and $\operatorname{rank}_{\mathbb{K}} A_i < \infty$ for all i. By a filtration of A we mean an ascending filtration $F = \{F_i A\}_{i \in \mathbb{Z}}$ by $\mathbb{K}$-modules such that $F_i A \cdot F_j \subset F_{i+j} A$. The associated graded algebra is denoted by $\operatorname{gr}^F A$.

Definition 6.3. A *noetherian connected filtration* of a $\mathbb{K}$-algebra A is a filtration F such that $\operatorname{gr}^F A$ is a noetherian connected graded $\mathbb{K}$-algebra.

In [YZ2, Definition 6.1] the condition was that the Rees algebra $\operatorname{Rees}^F A$ should be a noetherian connected graded $\mathbb{K}$-algebra; but as mentioned there the two conditions are in fact equivalent.

It is not hard to see that if A admits a noetherian connected filtration then A itself is noetherian and affine over $\mathbb{K}$.

Remark 6.4. If A is affine and finite over its center then it admits a noetherian connected filtration (see [YZ2, Example 6.14]); but this case is in a sense too easy. There are known examples of PI algebras not finite over their centers that admit noetherian connected filtrations (e.g. [YZ2, Example 6.15]), and for a long time it was an open problem whether they all do. The first counterexample was recently discovered by Stafford [SZ].

The notions of symmetric dimension function and weakly bifinite dualizing complex were defined just before Theorem 4.10.

Proposition 6.5. *Suppose* A *is a PI algebra admitting a noetherian connected filtration. Then* A *has an Auslander rigid dualizing complex* R*, the canonical dimension* $\operatorname{Cdim} = \operatorname{Cdim}_R$ *is symmetric, and* R *is weakly bifinite.*

Proof. Let F be a noetherian connected filtration of A. Then $\operatorname{gr}^F A$ is a noetherian connected graded PI $\mathbb{K}$-algebra. By [YZ2, Corollary 6.9] A has an Auslander dualizing complex R, and $\operatorname{Cdim}_R = \operatorname{GKdim}$ (Gelfand-Kirillov dimension) on the categories $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, A^{\mathrm{op}}$. Since GKdim is symmetric (see [MR, Proposition 8.3.14(ii)]), so is Cdim_R.

Now take a bimodule M that's a subquotient of A. Then M admits a two-sided good filtration F (i.e. $\operatorname{gr}^F M$ is a finite module over $\operatorname{gr}^F A$ on both sides), and by [YZ2, Proposition 6.21] we get $\operatorname{Ext}^i_A(M,R) \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(M,R)$ as bimodules. Hence this bimodule is finite on both sides. We conclude that R is weakly bifinite. $\square$

Theorem 6.6. *Let A be an affine noetherian PI algebra admitting a noetherian connected filtration.*

1. *A has a residue complex $\mathcal{K}_A$.*
2. *Let $B = A/\mathfrak{a}$ be a quotient algebra. Then B has a residue complex $\mathcal{K}_B$, there is a rigid trace $\operatorname{Tr}_{B/A} : \mathcal{K}_B \to \mathcal{K}_A$ that is an actual homomorphism of complexes of bimodules, and $\operatorname{Tr}_{B/A}$ induces an isomorphism*

$$\mathcal{K}_B \cong \operatorname{Hom}_A(B, \mathcal{K}_A) = \operatorname{Hom}_{A^{\mathrm{op}}}(B, \mathcal{K}_A) \subset \mathcal{K}_A.$$

Proof. (1) Is immediate from Proposition 6.5 and Theorem 4.10. (2) Follows from (1), Theorem 5.4 and Corollary 5.6. $\square$

Given a set Z of ideals of A we defined a localizing subcategory $\mathsf{M}_Z \subset \mathsf{Mod}\, A$ in Example 1.2. Now let us write Z^{op} for the same set, but considered as a set of ideals in the ring A^{op}, and let $\mathsf{M}_{Z^{\mathrm{op}}} \subset \mathsf{Mod}\, A^{\mathrm{op}}$ be the localizing subcategory. Denote by Γ_Z and $\Gamma_{Z^{\mathrm{op}}}$ the two torsion functors respectively.

Corollary 6.7. *Assume A is like in the theorem, and let Z be a set of ideals of A. Then*

$$\Gamma_Z \mathcal{K}_A = \Gamma_{Z^{\mathrm{op}}} \mathcal{K}_A \subset \mathcal{K}_A.$$

Proof. Apply Theorem 6.6(2) to the ideals $\mathfrak{a} = \mathfrak{b}_1 \cdots \mathfrak{b}_n$ where $\mathfrak{b}_1, \ldots, \mathfrak{b}_n \in Z$, using formulas (1.1) and (1.3). $\square$

Corollary 6.8. *Assume A is like in the theorem. Let S be a denominator set in A, with localization A_S. Define $\mathcal{K}_{A_S} := A_S \otimes_A \mathcal{K}_A \otimes_A A_S$.*

1. *$\mathcal{K}_{A_S} \cong A_S \otimes_A \mathcal{K}_A \cong \mathcal{K}_A \otimes_A A_S$ as complexes of A-bimodules.*
2. *$\mathcal{K}_{A_S}$ is a dualizing complex over A_S.*

Proof. (1) Let $Z := \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S \neq \emptyset\}$, and define Gabriel filters $\mathcal{F}_Z$ and $\mathcal{F}_S$ like in Examples 1.2 and 1.9. According to [Ste, Theorem VII.3.4] these two filters are equal. Hence (cf. Example 1.15) for each q there is an exact sequence

$$0 \to \Gamma_Z \mathcal{K}_A^q \to \mathcal{K}_A^q \to A_S \otimes_A \mathcal{K}_A^q \to 0.$$

By symmetry there is an exact sequence

$$0 \to \Gamma_{Z^{\mathrm{op}}} \mathcal{K}_A^q \to \mathcal{K}_A^q \to \mathcal{K}_A^q \otimes_A A_S \to 0.$$

But by Corollary 6.7, $\Gamma_Z \mathcal{K}_A^q = \Gamma_{Z^{\mathrm{op}}} \mathcal{K}_A^q$, which implies $A_S \otimes_A \mathcal{K}_A^q \cong \mathcal{K}_A^q \otimes_A A_S$. Finally use the fact that $A_S \otimes_A A_S = A_S$.

(2) This is proved just like [YZ2, Theorem 1.11(1)] (cf. proof of Theorem 3.8(1)). $\square$

Let us remind the reader the definition of a *clique* in the prime spectrum $\operatorname{Spec} A$. Suppose $\mathfrak{p}$ and $\mathfrak{q}$ are two prime ideals. If there is a bimodule M that's a subquotient of $(\mathfrak{p} \cap \mathfrak{q})/(\mathfrak{p}\mathfrak{q})$ and is nonzero torsion-free as $A/\mathfrak{p}$-module and as $(A/\mathfrak{q})^{\mathrm{op}}$-module, then we say there is a (second layer) link $\mathfrak{p} \rightsquigarrow \mathfrak{q}$. The links make $\operatorname{Spec} A$ into a quiver, and the cliques are its connected components.

Example 6.9. Suppose $[A : \mathbb{K}] < \infty$. The lemma below implies that (up to multiplicity of arrows) the link quiver of A coincides with the quiver defined by Gabriel in the context of representation theory (see [MY]). Cliques in this case stand in bijection to blocks of A (indecomposable factors), and also to $\operatorname{Spec} \mathrm{Z}(A)$.

Lemma 6.10. *Let A be an artinian ring with Jacobson radical $\mathfrak{r}$ and maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Then the inclusions $\mathfrak{r} \subset \mathfrak{p}_i$ induce an isomorphism of A-bimodules*
$$\frac{\mathfrak{r}}{\mathfrak{r}^2} \cong \bigoplus\nolimits_{i,j} \frac{\mathfrak{p}_i \cap \mathfrak{p}_j}{\mathfrak{p}_i \mathfrak{p}_j}.$$

Proof. This proof was communicated to us by K. Goodearl. Choose orthogonal idempotents $e_i \in A$ lifting the central idempotents in $A/\mathfrak{r}$, so that
$$\mathfrak{p}_i = A(1 - e_i) + \mathfrak{r} = (1 - e_i)A + \mathfrak{r}.$$
We have
$$(1 - e_i)(\mathfrak{p}_i \cap \mathfrak{p}_j) = (1 - e_i)\mathfrak{p}_j \subset \mathfrak{p}_i \mathfrak{p}_j,$$
and likewise on the right, so each element of $(\mathfrak{p}_i \cap \mathfrak{p}_j)/(\mathfrak{p}_i \mathfrak{p}_j)$ comes from some element in $e_i(\mathfrak{p}_i \cap \mathfrak{p}_j)e_j$. But $e_i(\mathfrak{p}_i \cap \mathfrak{p}_j)e_j = e_i \mathfrak{r} e_j$. We see that the canonical homomorphism $f : e_i \mathfrak{r} e_j \to (\mathfrak{p}_i \cap \mathfrak{p}_j)/(\mathfrak{p}_i \mathfrak{p}_j)$ is surjective. Obviously $e_i \mathfrak{r}^2 e_j \subset \operatorname{Ker}(f)$. On the other hand
$$\mathfrak{p}_i \mathfrak{p}_j = (1 - e_i)A(1 - e_j) + (1 - e_i)\mathfrak{r} + \mathfrak{r}(1 - e_j) + \mathfrak{r}^2,$$
so $e_i \mathfrak{p}_i \mathfrak{p}_j e_j = e_i \mathfrak{r}^2 e_j$. Thus $\operatorname{Ker}(f) = e_i \mathfrak{r}^2 e_j$. Finally the isomorphism is obtained by the decomposition
$$\frac{\mathfrak{r}}{\mathfrak{r}^2} = \bigoplus_{i,j} e_i \frac{\mathfrak{r}}{\mathfrak{r}^2} e_j = \bigoplus_{i,j} \frac{e_i \mathfrak{r} e_j}{e_i \mathfrak{r}^2 e_j}.$$
□

Definition 6.11. Let A be a noetherian $\mathbb{K}$-algebra with an Auslander rigid dualizing complex, such that the canonical dimension Cdim is weakly symmetric. The *q-skeleton* of $\operatorname{Spec} A$ is the set
$$\{\mathfrak{p} \in \operatorname{Spec} A \mid \operatorname{Cdim} A/\mathfrak{p} = q\}.$$

Proposition 6.12. *Suppose A is a PI $\mathbb{K}$-algebra admitting some noetherian connected filtration. Then the q-skeleton of* $\operatorname{Spec} A$ *is a union of cliques.*

Proof. Within the proof of Theorem 4.10 it is shown that Cdim is constant on cliques in $\operatorname{Spec} A$. □

Proposition 6.13. *Suppose A is a prime PI $\mathbb{K}$-algebra admitting a noetherian connected filtration. Let $n := \operatorname{Cdim} A$ and $Q := \operatorname{Frac} A$ the ring of fractions. Then $\mathcal{K}_A^{-n} \cong Q$ as A-bimodules.*

Proof. Since $\mathcal{K}_A^{-n}$ is pure of $\operatorname{Cdim} = n$ it is a torsion-free A-module, and it follows that $Q \otimes_A \mathcal{K}_A^{-n} \cong \mathcal{K}_A^{-n}$. On the other hand for $q < n$, $\mathcal{K}_A^{-q}$ is pure of $\operatorname{Cdim} = q < n$, so it is a torsion A-module and $Q \otimes_A \mathcal{K}_A^{-q} = 0$. We see that $Q \otimes_A \mathcal{K}_A \cong \mathcal{K}_A^{-n}[n]$. But by Theorem 6.2 we get $Q \otimes_A \mathcal{K}_A \cong Q[n]$. □

Given a prime ideal $\mathfrak{p}$ in a ring A let

$$S(\mathfrak{p}) = S_A(\mathfrak{p}) := \{a \in A \mid a + \mathfrak{p} \text{ is regular in } A/\mathfrak{p}\}.$$

For a set Z of prime ideals we write

$$S(Z) := \bigcap_{\mathfrak{p} \in Z} S(\mathfrak{p}).$$

According to [Mu, Theorem 10 and remarks in Section 3], if A is a noetherian PI affine $\mathbb{K}$-algebra and Z is a clique of prime ideals in $\operatorname{Spec} A$ then $S(Z)$ is a denominator set. We get a ring of fractions

$$A_{S(Z)} := S(Z)^{-1} \cdot A = A \cdot S(Z)^{-1}.$$

Furthermore for each $\mathfrak{p} \in Z$ one has

$$A/\mathfrak{p} \otimes_A A_{S(Z)} = Q(\mathfrak{p}) = \operatorname{Frac} A/\mathfrak{p} = (A/\mathfrak{p})_{S_{A/\mathfrak{p}}(0)}.$$

For a prime ideal $\mathfrak{p}$ denote by $J_A(\mathfrak{p})$ the indecomposable injective A-module with associated prime $\mathfrak{p}$ (it is unique up to isomorphism). Let $\mathrm{r}(\mathfrak{p})$ denote the Goldie rank of $A/\mathfrak{p}$.

We say a clique Z_1 is a specialization (resp. immediate specialization) of a clique Z_0 if there exist prime ideals $\mathfrak{p}_i \in Z_i$ with $\mathfrak{p}_0 \subset \mathfrak{p}_1$ (resp. and Z_i is in the $(q-i)$-skeleton of $\operatorname{Spec} A$ for some q).

Here is the main result of this section.

Theorem 6.14. *Let A be a PI $\mathbb{K}$-algebra admitting a noetherian connected filtration, and let $\mathcal{K}_A$ be its residue complex.*

(1) *For every q there is a canonical A-bimodule decomposition*

$$\mathcal{K}_A^{-q} = \bigoplus_Z \Gamma_Z \mathcal{K}_A^{-q}$$

where Z runs over the cliques in the q-skeleton of $\operatorname{Spec} A$.

(2) *Fix one clique Z in the q-skeleton of* $\operatorname{Spec} A$. *Then $\Gamma_Z \mathcal{K}_A^{-q} = \Gamma_{Z^{\mathrm{op}}} \mathcal{K}_A^{-q}$ is an injective left (resp. right) $A_{S(Z)}$-module, and its socle as left (resp. right) $A_{S(Z)}$-module is the essential submodule*

$$\bigoplus_{\mathfrak{p} \in Z} \mathcal{K}_{A/\mathfrak{p}}^{-q} \cong \bigoplus_{\mathfrak{p} \in Z} \operatorname{Hom}_A(A/\mathfrak{p}, \mathcal{K}_A^{-q}) \subset \Gamma_Z \mathcal{K}_A^{-q}.$$

(3) *There is a (noncanonical) decomposition of left (resp. right) $A_{S(Z)}$-modules*

$$\Gamma_Z \mathcal{K}_A^{-q} \cong \bigoplus_{\mathfrak{p} \in Z} J_A(\mathfrak{p})^{\mathrm{r}(\mathfrak{p})} \cong \bigoplus_{\mathfrak{p} \in Z} J_{A^{\mathrm{op}}}(\mathfrak{p})^{\mathrm{r}(\mathfrak{p})}.$$

(4) *$\Gamma_Z \mathcal{K}_A^{-q}$ is an indecomposable A-bimodule.*

(5) *Suppose Z_i is a clique in the $(q-i)$-skeleton of* $\operatorname{Spec} A$, *for $i = 0, 1$. Then Z_1 is an immediate specialization of Z_0 iff the composed homomorphism*

$$\delta_{(Z_0, Z_1)} : \Gamma_{Z_0} \mathcal{K}_A^{-q} \hookrightarrow \mathcal{K}_A^{-q} \to \mathcal{K}_A^{-q+1} \twoheadrightarrow \Gamma_{Z_1} \mathcal{K}_A^{-q+1}$$

is nonzero.

Proof. 1. Let I be an indecomposable injective module with associated prime $\mathfrak{p} \in Z$, where Z is a clique in the q-skeleton of $\operatorname{Spec} A$. Since A satisfies the second layer condition, we get $\Gamma_Z I = I$ and also $\Gamma_{S(Z)} I = 0$. It follows that $I \to A_{S(Z)} \otimes_A I$ is bijective. Therefore for any other clique Z' in the q-skeleton of $\operatorname{Spec} A$ we must have $\Gamma_{Z'} I = 0$. Because $\mathcal{K}_A^{-q}$ is an injective module, pure of dimension q, we get the left module decomposition $\mathcal{K}_A^{-q} = \bigoplus_Z \Gamma_Z \mathcal{K}_A^{-q}$. By Corollary 6.7 and symmetry this is a bimodule decomposition.

2. Clearly $\Gamma_Z \mathcal{K}_A^{-q}$ is an injective $A_{S(Z)}$-bimodule, and $\bigoplus_{\mathfrak{p} \in Z} \mathcal{K}_{A/\mathfrak{p}}^{-q} \subset \Gamma_Z \mathcal{K}_A^{-q}$ is essential. Write $Q(\mathfrak{p}) := \operatorname{Fract} A/\mathfrak{p}$; then $\bigoplus_{\mathfrak{p}} \mathcal{K}_{A/\mathfrak{p}}^{-q} \cong \bigoplus_{\mathfrak{p}} Q(\mathfrak{p})$ is a semi-simple (left and right) $A_{S(Z)}$-module.

3. This is because the injective hull of $Q(\mathfrak{p})$ as A-module is $J_A(\mathfrak{p})^{\mathrm{r}(\mathfrak{p})}$.

4. Assume by contraposition that $\Gamma_Z \mathcal{K}_A^{-q} = M_1 \oplus M_2$ as bimodules, with $M_i \neq 0$. Then the socle $V = \bigoplus_{\mathfrak{p} \in Z} \mathcal{K}_{A/\mathfrak{p}}^{-q}$ of $\Gamma_Z \mathcal{K}_A^{-q}$ (as left or right $A_{S(Z)}$-module) also decomposes into $V = V_1 \oplus V_2$ with $V_i = M_i \cap V = \bigoplus_{\mathfrak{p} \in Z_i} \mathcal{K}_{A/\mathfrak{p}}^{-q}$ and $Z = Z_1 \amalg Z_2$, $Z_i \neq \emptyset$. Take $\mathfrak{p}_i \in Z_i$ such that there is a second layer link $\mathfrak{p}_1 \rightsquigarrow \mathfrak{p}_2$. Recall that this means there is a bimodule surjection

$$\mathfrak{r} = \frac{\mathfrak{p}_1 \cap \mathfrak{p}_2}{\mathfrak{p}_1 \mathfrak{p}_2} \twoheadrightarrow N$$

with N a nonzero torsion-free module over $A/\mathfrak{p}_1$ and $(A/\mathfrak{p}_2)^{\mathrm{op}}$. Then replacing A with $A/\mathfrak{p}_1\mathfrak{p}_2$ we retain the link, only now $Z_i = \{\mathfrak{p}_i\}$ and $V_i \cong Q(\mathfrak{p}_i)$ as bimodules.

Let $B := A_{S(Z)}$. According to Corollary 6.8, $\mathcal{K}_B := B \otimes_A \mathcal{K}_A \otimes_A B$ is a dualizing complex over B. As in the proof of Proposition 6.13 we get $\mathcal{K}_B^{-p} = 0$ for all $p < q$, and $\mathcal{K}_B^{-q} = \mathcal{K}_A^{-q}$. Hence $\mathcal{K}_B = \mathcal{K}_B^{-q}[q] = \mathcal{K}_A^{-q}[q]$.

By Lemma 3.6

$$\mathrm{Z}(B) \cong \operatorname{End}_{\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{e}})}(\mathcal{K}_B) \cong \operatorname{End}_{B^{\mathrm{e}}}(\mathcal{K}_B^{-q})$$

as rings. Take $\pi \in \operatorname{End}_{B^{\mathrm{e}}}(\mathcal{K}_B^{-q})$ to be the projection $\mathcal{K}_B^{-q} \twoheadrightarrow M_1$. So π is left multiplication by a central idempotent $e \in B$. Since $Q(\mathfrak{p}_i) \cong V_i \subset M_i$ we see that $e \cdot Q(\mathfrak{p}_1) = Q(\mathfrak{p}_1)$ and $e \cdot Q(\mathfrak{p}_2) = 0$.

Now the bimodule

$$N_B := B \otimes_A N \otimes_A B \cong Q(\mathfrak{p}_1) \otimes_A N \otimes_A Q(\mathfrak{p}_2) \neq 0.$$

Being a subquotient of B, e centralizes N_B. We get a contradiction $e \cdot N_B = N_B$, $N_B \cdot e = 0$.

5. First assume there is specialization, and choose prime ideals $\mathfrak{p}_0, \mathfrak{p}_1$ as evidence. Then the algebra $B := A/\mathfrak{p}_0 \otimes_A A_{S(Z_1)}$ is nonzero, having $Q(\mathfrak{p}_1)$ as a quotient. Thus B is prime. The complex $\mathcal{K}_B$, with $\mathcal{K}_B^{-q} = \mathcal{K}_{A/\mathfrak{p}_0}^{-q}$, $\mathcal{K}_B^{-q+1} = \Gamma_{Z_1} \mathcal{K}_{A/\mathfrak{p}_0}^{-q+1}$ and $\mathcal{K}_B^{i} = 0$ otherwise, is dualizing by Corollary 6.8. If $\delta : \mathcal{K}_B^{-q} \to \mathcal{K}_B^{-q+1}$ were zero this would imply that $B \cong \mathrm{H}^0 \operatorname{Hom}_B(\mathcal{K}_B, \mathcal{K}_B)$ is decomposable as bimodule, contradicting it being a prime ring.

Conversely assume $\delta_{(Z_0,Z_1)} \neq 0$ and pick some $\phi \in \Gamma_{Z_0} \mathcal{K}_A^{-q}$ s.t. $\delta_{(Z_0,Z_1)}(\phi) \in \Gamma_{Z_1} \mathcal{K}_A^{-q+1}$ is nonzero. By part (3) we can find $a_i \in \mathfrak{p}_{1,i} \in Z_1$ such that

$$0 \neq \psi = a_1 \cdots a_m \delta_{(Z_0,Z_1)}(\phi) \in \mathcal{K}_{A/\mathfrak{p}_1}^{-q+1} \cong Q(\mathfrak{p}_1)$$

for some $\mathfrak{p}_1 \in Z_1$. On the other hand there are primes $\mathfrak{p}_{0,1}, \ldots, \mathfrak{p}_{0,n} \in Z_0$ s.t. $\phi \mathfrak{p}_{0,1} \cdots \mathfrak{p}_{0,n} = 0$, which implies that $\psi \mathfrak{p}_{0,1} \cdots \mathfrak{p}_{0,n} = 0$. We conclude that $Q(\mathfrak{p}_1)\mathfrak{p}_{0,1} \cdots \mathfrak{p}_{0,m} = 0$, and therefore $\mathfrak{p}_0 := \mathfrak{p}_{0,i} \subset \mathfrak{p}_1$ for some i. $\square$

Example 6.15. Assume A is finite over $\mathbb{K}$, and let $A = \prod A_i$ be the block decomposition, i.e. each A_i is an indecomposable bimodule. Then $\operatorname{Spec} A_i$ is a clique in $\operatorname{Spec} A$ and

$$\mathcal{K}_A = \mathcal{K}_A^0 = \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K}) = \bigoplus_i \operatorname{Hom}_{\mathbb{K}}(A_i, \mathbb{K})$$

is a decomposition into indecomposable bimodules (cf. Example 6.9).

Example 6.16. Generalizing the previous example, consider a noetherian affine $\mathbb{K}$-algebra A finite over its center C. It is well known that $\mathfrak{q} \mapsto \mathfrak{p} = C \cap \mathfrak{q}$ is a bijection from the cliques $Z \subset \operatorname{Spec} A$ to $\operatorname{Spec} C$ (see [GW, Theorem 11.20]). For $\mathfrak{p} \in \operatorname{Spec} C$ denote by $\widehat{C}_{\mathfrak{p}}$ the $\mathfrak{p}$-adic completion. The complete semilocal ring $\widehat{C}_{\mathfrak{p}} \otimes_C A$ has center $\widehat{C}_{\mathfrak{p}}$ and is indecomposable. On the other hand, say $\dim C/\mathfrak{p} = n$. Then

$$\Gamma_Z \mathcal{K}_A^{-n} = \Gamma_{\mathfrak{p}} \mathcal{K}_A^{-n} \cong \Gamma_{\mathfrak{p}} \operatorname{Hom}_C(A, \mathcal{K}_C^{-n}) \cong \operatorname{Hom}_{\widehat{C}_{\mathfrak{p}}}(\widehat{C}_{\mathfrak{p}} \otimes_C A, \mathcal{K}(\widehat{C}_{\mathfrak{p}}))$$

where $\mathcal{K}(\widehat{C}_{\mathfrak{p}})$ is the dual module from Example 5.10. Since $\operatorname{Hom}_{\widehat{C}_{\mathfrak{p}}}(-, \mathcal{K}(\widehat{C}_{\mathfrak{p}}))$ is a duality for finite $\widehat{C}_{\mathfrak{p}}$-modules we see that the indecomposability of the bimodule $\Gamma_Z \mathcal{K}_A^{-n}$ is equivalent to the indecomposability of $\widehat{C}_{\mathfrak{p}} \otimes_C A$.

Example 6.17. Consider the PI algebra $A = \mathbb{K}\langle x, y\rangle/(yx - qxy, y^2)$, $q \in \mathbb{K}$. Assume $\mathbb{K}$ is algebraically closed, so the spectrum of A consists of the prime ideals $\mathfrak{p} := (y)$ and $\mathfrak{m}_\lambda := (y, x - \lambda)$ where $\lambda \in \mathbb{K}$. We note that

$$\frac{\mathfrak{m}_\lambda \cap \mathfrak{m}_\mu}{\mathfrak{m}_\lambda \mathfrak{m}_\mu} \cong \begin{cases} \mathbb{K} \cdot y \neq 0 & \text{if } \mu = q\lambda \\ 0 & \text{otherwise.} \end{cases}$$

Thus the cliques are $\{\mathfrak{m}_{q^i \lambda} \mid i \in \mathbb{Z}\}$ and of course $\{\mathfrak{p}\}$. We see that if q is not a root of unity then we get infinite cliques.

Example 6.18. Take the quantum plane $A := \mathbb{K}\langle x, y\rangle/(yx - qxy)$ with q a primitive lth root of unity in $\mathbb{K}$. The center is $C := \mathbb{K}[x^l, y^l]$. Assume $\mathbb{K}$ is algebraically closed. Let us describe the indecomposable bimodules $\Gamma_Z \mathcal{K}_A^{-i}$ and their decomposition into indecomposable left modules $\Gamma_Z \mathcal{K}_A^{-i} \cong \bigoplus_{\mathfrak{p} \in Z} J_A(\mathfrak{p})^{\mathrm{r}(\mathfrak{p})}$.

a) Generically $Q(0)$ is a division ring, and thus $\mathcal{K}_A^{-2} \cong J_A(0)$.

b) If $\mathfrak{q} \in \operatorname{Spec} C$ is a curve or a point ($i = 1, 0$) in the Azumaya locus of A (i.e. $x^l, y^l \notin \mathfrak{q}$) then $\mathfrak{p} := A\mathfrak{q}$ is prime. By Tsen's Theorem the Brauer group $\operatorname{Br}(\boldsymbol{k}(\mathfrak{q}))$ of the residue field $\boldsymbol{k}(\mathfrak{q})$ is trivial. Therefore $Q(\mathfrak{p}) \cong \mathrm{M}_l(\boldsymbol{k}(\mathfrak{q}))$. We conclude that $Z := \{\mathfrak{p}\}$ is a clique, $\mathrm{r}(\mathfrak{p}) = l$ and $\Gamma_Z \mathcal{K}_A^{-i} \cong J_A(\mathfrak{p})^l$.

c) If $\mathfrak{q} = y^l C$ then $\mathfrak{p} := yA$ is prime and $Q(\mathfrak{p})$ is commutative. We conclude that $Z := \{\mathfrak{p}\}$ is a clique, $\mathrm{r}(\mathfrak{p}) = 1$ and $\Gamma_Z \mathcal{K}_A^{-1} \cong J_A(\mathfrak{p})$. Likewise if $\mathfrak{q} = x^l C$.

d) If $\mathfrak{n} = y^l C + (x^l - \lambda^l)C \in \operatorname{Spec} C$ with $\lambda \neq 0$ then the clique lying above $\mathfrak{n}$ is $Z := \{\mathfrak{m}_{q^j \lambda} \mid j = 0, \ldots, l-1\}$; notation as in the previous example. The Goldie rank is $\mathrm{r}(\mathfrak{m}_{q^j \lambda}) = 1$ and $\Gamma_Z \mathcal{K}_A^0 \cong \bigoplus_{j=0}^{l-1} J_A(\mathfrak{m}_{q^j \lambda})$. Likewise with y and x interchanged.

e) Finally the clique lying above $\mathfrak{n} := y^l C + x^l C$ is $Z := \{\mathfrak{m}\}$ where $\mathfrak{m} := xA + yA$, $\mathrm{r}(\mathfrak{m}) = 1$ and $\Gamma_Z \mathcal{K}_A^0 \cong J_A(\mathfrak{m})$.

Question 6.19. Are Theorems 6.6 and 6.14 valid without assuming the existence of a noetherian connected filtration?

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Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, ISRAEL
E-mail address: `amyekut@math.bgu.ac.il`
URL: `http://www.math.bgu.ac.il/~amyekut`

Department of Mathematics, University of Washington, Seattle, WA 98195, USA
E-mail address: `zhang@math.washington.edu`