THE DERIVED PICARD GROUP AND REPRESENTATIONS OF QUIVERS

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0. Introduction

Let A be a ring, and let $\mathsf{D}(\mathsf{Mod}\, A)$ be the derived category of A-modules. The derived Picard group $\mathrm{DPic}(A)$ is the group of auto-equivalences of $\mathsf{D}(\mathsf{Mod}\, A)$ generated by tilting complexes. This is a generalization of the (noncommutative) Picard group $\mathrm{Pic}(A)$, namely the group generated by invertible bimodules, which acts on $\mathsf{Mod}\, A$ by auto-equivalences (Morita Theory).

The group $\mathsf{D}(\mathsf{Mod}\, A)$ becomes interesting when A is nonlocal and noncommutative. This takes us into the realm of noncommutative algebraic geometry.

Today I will talk about the case when A is a finite dimensional k-algebra (k is a field). Here the geometry is completely described by a quiver (directed graph). For a hereditary algebra A we can also completely describe the structure of the group $\mathsf{D}(\mathsf{Mod}\, A)$ – using another quiver ... .

This is joint work with J. Miyachi (Tokyo).

1. Some background

Suppose A is a (unital, associative) k-algebra. Let $\mathsf{Mod}\, A$ be the category of left A-modules. Recall that Morita Theory tells us what are all k-linear auto-equivalences $F : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$. They are (up to isomorphism) $FM = P \otimes_A M$, where P is a k-central invertible bimodule (i.e. $A \cong P \otimes_A Q \cong Q \otimes_A P$ for some bimodule Q). We call the group generated by the invertible bimodules P (with operation $P \otimes_A Q$) the (noncommutative) Picard group of A, denoted $\mathrm{Pic}_k(A)$.

Remark 1.1. If A is commutative then $\mathrm{Pic}_k(A)$ contains the automorphism group $\mathrm{Aut}_k(A)$ and the usual (commutative) Picard group $\mathrm{Pic}_A(A)$ as subgroups.

Now let us look at the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. We will not attempt to say too much about it now. All we need to know are a few facts. The objects of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ are the bounded complexes of A-modules

$$M^{\cdot} = \left(\cdots \to 0 \to M^p \xrightarrow{\delta} M^{p+1} \xrightarrow{\delta} \cdots \to M^q \to 0 \to \cdots \right)$$

where $\delta \circ \delta = 0$. The category $\mathsf{Mod}\, A$ embeds in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ as the complexes concentrated in degree 0. And $M[n]$ is the complex with M in degree $-n$. If $T^{\cdot}$ is a complex of bimodules, then one can define the derived tensor product $T^{\cdot} \otimes^{\mathrm{L}}_A M^{\cdot}$. We call $T^{\cdot}$ a tilting complex if the functor $FM^{\cdot} = T^{\cdot} \otimes^{\mathrm{L}}_A M^{\cdot}$ is an auto-equivalence of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$.

Example 1.2. If P is an invertible bimodule then $P[n]$ is a tilting complex.

Definition 1.3. The derived Picard group $\mathrm{DPic}_k(A)$ is the group generated by tilting complexes, with operation $T^{\cdot} \otimes^{\mathrm{L}}_A S^{\cdot}$.

We note that $\mathrm{DPic}_k(A)$ contains two natural subgroups: a copy of $\mathbb{Z}$ (represented by the complexes $A[n]$), and $\mathrm{Pic}_k(A)$.

Theorem 1.4 (Y., Zimmermann-Rouquier-...)**.** *If A is either local or commutative, then* $\mathrm{DPic}_k(A) = \mathrm{Pic}_k(A) \times \mathbb{Z}$.

However:

Example 1.5 (Y)**.** Consider the smallest nonlocal, noncommutative k-algebra $A = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. The bimodule $A^* := \mathrm{Hom}_k(A, k)$ turns out to be a tilting complex. Let τ be the class of $A^*[-1]$ in $\mathrm{DPic}(A)$. Then one can show that $\tau \notin \mathrm{Pic}_k(A) \times \mathbb{Z}$.

This raises:

Problem 1.6. Study the structure of the group $\mathrm{DPic}_k(A)$ and its relation to the geometry of the noncommutative ring A.

2. Quivers and Path Algebras

We will concentrate on a finite dimensional k-algebra A. Let us assume that k is algebraically closed, and that A is basic (i.e. $A/\mathfrak{p} = k$ for every prime ideal). Since every k-algebra is Morita equivalent to a basic one the latter is really no restriction.

The geometry of A is described by a quiver $\vec{\Delta}$. Here is a nonorthodox definition of $\vec{\Delta}$. The set of vertices of $\vec{\Delta}$ is $\mathrm{Spec}\, A$, namely the set of prime ideals. Let $\mathfrak{r}$ be the Jacobson radical of A, and for each $\mathfrak{p}_i \in \mathrm{Spec}\, A$ let $e_i \in A/\mathfrak{r}$ be the corresponding central idempotent. So

$$\mathfrak{r}/\mathfrak{r}^2 = \bigoplus_{i,j} e_i(\mathfrak{r}/\mathfrak{r}^2)e_j \cong \bigoplus_{i,j} \frac{\mathfrak{p}_i \cap \mathfrak{p}_j}{\mathfrak{p}_i\mathfrak{p}_j}$$

as A-A-bimodules. Define $d_{i,j} := \dim_k e_i(\mathfrak{r}/\mathfrak{r}^2)e_j$. Then there are $d_{i,j}$ arrows $\mathfrak{p}_i \to \mathfrak{p}_j$.

Example 2.1. Take $A = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. The prime ideals are $\mathfrak{p}_1 = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix}$ and $\mathfrak{p}_2 = \begin{bmatrix} k & k \\ 0 & 0 \end{bmatrix}$. Clearly $\mathfrak{r}/\mathfrak{r}^2 = \mathfrak{r} = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$. So the quiver $\vec{\Delta}$ is $\underset{\mathfrak{p}_1}{\bullet} \longrightarrow \underset{\mathfrak{p}_2}{\bullet}$, a Dynkin quiver of type A_2.

On the other hand, given a finite quiver $\vec{\Delta}$ we can construct an algebra, called the path algebra $k\vec{\Delta}$. A path in $\vec{\Delta}$ is either a trivial path e_x for each vertex x, or a sequence of arrows $\alpha_1\alpha_2\cdots\alpha_m$, such that α_{i+1} starts where α_i ends:

$$\underset{x_1}{\bullet} \xrightarrow{\alpha_1} \underset{x_2}{\bullet} \xrightarrow{\alpha_2} \underset{x_3}{\bullet} \xrightarrow{\alpha_3} \underset{x_4}{\bullet}$$

The paths form a basis of $k\vec{\Delta}$ as vector space, and multiplication of paths is by concatenation if possible, or 0.

Example 2.2. Take the quiver $\vec{\Delta} = \underset{x_1}{\bullet} \xrightarrow{\alpha} \underset{x_2}{\bullet}$. Then e_{x_1}, e_{x_2} are orthogonal idempotents; $e_{x_1}\alpha = \alpha = \alpha e_{x_2}$; and $\alpha e_{x_1} = \alpha^2 = e_{x_2}\alpha = 0$. Thus $k\vec{\Delta} \cong \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, with

$$e_{x_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ e_{x_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Here is a classical structure theorem for finite dimensional algebras. Recall that a ring A is hereditary if it has global dimension 1, or equivalently if every left or right ideal is a projective module.

Theorem 2.3 (Gabriel). *Let A be a finite dimensional k-algebra with quiver $\vec{\Delta}$. Then*

1. *$A \cong k\vec{\Delta}/I$ for an ideal $I \subset \mathfrak{a}^2$, where $\mathfrak{a} \subset k\vec{\Delta}$ is the ideal generated by the arrows.*
2. *$I = 0$ iff A is hereditary.*

In the rest of the talk $A = k\vec{\Delta}$ will be a hereditary algebra, and the quiver $\vec{\Delta}$ connected. An A-module is also called a representation of the quiver $\vec{\Delta}$. As customary we write $\mathsf{mod}\, A$ for the category of finitely generated A-modules (=finite dimensional representations).

Some remarks:

Remark 2.4. According to the definitions of Cunts-Quillen, a finite dimensional algebra A is smooth iff it is hereditary.

Remark 2.5. The isomorphism $A \cong k\vec{\Delta}$ should be viewed as a noncommutative analog of the Cohen theorem for commutative rings: $A \cong k[[t_1, \dots, t_n]]$. Indeed the arrows in $k\vec{\Delta}$ are lifts of a basis of $\mathfrak{r}/\mathfrak{r}^2$.

Remark 2.6. Beilinson showed that the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{P}^1)$ of coherent sheaves on the projective line $\mathbf{P}^1$ is equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, k\vec{\Delta}_2)$, for a certain quiver $\vec{\Delta}_2$. According to Kontsevich-Rosenberg one has $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{NP}^n) \approx \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, k\vec{\Delta}_{n+1})$, where $\mathbf{NP}^n$ is their noncommutative projective n-space.

3. The Auslander-Reiten Quiver

One of the important invariants of the algebra A is the set of isomorphism classes of indecomposable modules. If this set is finite A is said to have finite representation type.

Theorem 3.1 (Gabriel). *A has finite representation type iff $\vec{\Delta}$ is a Dynkin quiver of type A_n, D_n, E_6, E_7 or E_8.*

The Auslander-Reiten quiver $\vec{\Gamma}(\mathsf{mod}\, A)$ is defined as follows. Its vertices are the isomorphism classes of indecomposable A-modules, and there are d arrows $x \to y$, where d is the dimension of the space of irreducible homomorphisms $\mathrm{Irr}(M_x, M_y)$ (this is like $\mathfrak{r}/\mathfrak{r}^2$).

Example 3.2. Again we look at $A = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. The indecomposable modules are the column spaces

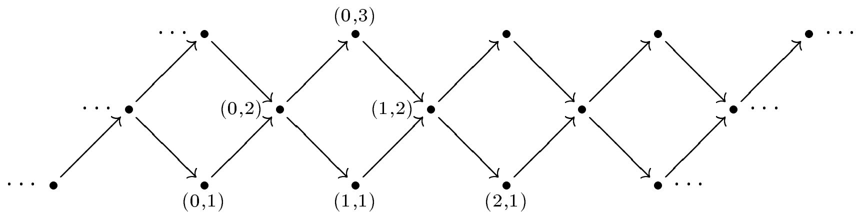
$$P_1 = \begin{bmatrix} k \\ 0 \end{bmatrix},\ P_2 = \begin{bmatrix} k \\ k \end{bmatrix},\ I_2 = \begin{bmatrix} 0 \\ k \end{bmatrix}.$$

The irreducible homomorphisms are the obvious ones $P_1 \hookrightarrow P_2$, $P_2 \twoheadrightarrow I_2$. The quiver $\vec{\Gamma}(\mathsf{mod}\, A)$ is

$$\begin{tikzcd} & \overset{P_2}{\bullet} \arrow[dr] & \\ \underset{P_1}{\bullet} \arrow[ur] & & \underset{I_2}{\bullet} \end{tikzcd}$$

Happel showed how to define the Auslander-Reiten quiver $\vec{\Gamma} = \vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$. This is an infinite quiver. Usually it is very complicated, but it contains a nice subquiver denoted $\vec{\mathbb{Z}}\vec{\Delta}$ that is easy to describe ($\vec{\mathbb{Z}}\vec{\Delta}$ was introduced by Riedtmann). When $\vec{\Delta}$ is a Dynkin quiver, then in fact $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)) = \vec{\mathbb{Z}}\vec{\Delta}$.

Example 3.3. Take $\vec{\Delta} = \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$. Then $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)) = \vec{\mathbb{Z}}\vec{\Delta}$ is



The vertices in $\vec{\Gamma}(\mathsf{mod}\, A)$ are labeled.

The group $\mathrm{DPic}_k(A)$ acts naturally on the quiver $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$ (by quiver automorphisms). The element $\tau \in \mathrm{DPic}_k(A)$, represented by the tilting complex $A^*[-1]$, is called the translation. In the example above it is a shift by 1 to the left. Using this action we proved the following theorems. Let $\mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})$ denote the group of quiver automorphisms.

Theorem 3.4 (Miyachi and Y.)**.** *Suppose A has finite representation type. Then*

$$\mathrm{DPic}_k(A) \cong \mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})^{\langle\tau\rangle}.$$

Example 3.5. If $\vec{\Delta}$ is of Dynkin type A_n then $\mathrm{DPic}_k(A) \cong \mathbb{Z}$ if n is even, and $\mathrm{DPic}_k(A) \cong \mathbb{Z} \times \mathbb{Z}/(2)$ if n is odd.

We denote by $\mathrm{Aut}((\vec{\mathbb{Z}}\vec{\Delta})_0; d)$ the group of permutations of the vertex set $(\vec{\mathbb{Z}}\vec{\Delta})_0$ that preserve the arrow-multiplicity d.

Theorem 3.6 (Miyachi and Y.)**.** *If A has infinite representation type then there is an isomorphism of groups*

$$\mathrm{DPic}_k(A) \cong \big(\mathrm{Aut}((\vec{\mathbb{Z}}\vec{\Delta})_0; d)^{\langle\tau\rangle} \ltimes \mathrm{Pic}^0_k(A)\big) \times \mathbb{Z}.$$

$\mathrm{Pic}^0_k(A)$ is a connected linear algebraic group, and is trivial when $\vec{\Delta}$ is a tree.

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