Perverse Sheaves and Dualizing Complexes on Schemes

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This talk is about the "commutative" aspects of the theory developed in the paper [YZ2], which is joint work with James Zhang.

1. Notation, and Review of Derived Categories

Throughout the lecture $\Bbbk$ will denote a base field, and A will be a finitely generated commutative $\Bbbk$-algebra. We denote by $\mathsf{Mod}\, A$ the category of A-modules.

The objects of the derived category $\mathsf{D}(\mathsf{Mod}\, A)$ are complexes of objects

$$M = \big(\cdots \to M^{-1} \xrightarrow{\partial^{-1}} M^0 \xrightarrow{\partial^0} M^1 \xrightarrow{\partial} \cdots \big).$$

The morphisms $\psi : M \to N$ are of the form $\psi = \phi_2^{-1} \circ \phi_1$ where $\phi_1 : M \to L$ is a homomorphism of complexes and $\phi_2 : N \to L$ is a quasi-isomorphism.

In $\mathsf{D}(\mathsf{Mod}\, A)$ we have the full subcategories $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ of bounded complexes, $\mathsf{D}^{+}(\mathsf{Mod}\, A)$ of bounded below complexes, and also $\mathsf{Mod}\, A$, which is identified with complexes concentrated in degree 0.

By $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)$ we denote the subcategory of complexes whose cohomology modules are finitely generated modules. Combining superscripts and subscripts we get the intersections of the respective subcategories.

Of utmost importance for us is the derived functor RHom. Given complexes $M, N \in \mathsf{D}(\mathsf{Mod}\, A)$ there is a well defined object

$$\mathrm{RHom}_A(M, N) \in \mathsf{D}(\mathsf{Mod}\, A)$$

depending functorially on M and N.

If $M, N \in \mathsf{Mod}\, A$ then we recover the familiar Exts:

$$\mathrm{H}^i \, \mathrm{RHom}_A(M, N) = \mathrm{Ext}^i_A(M, N).$$

Also

$$\mathrm{H}^i \, \mathrm{RHom}_A(M, N) \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(M, N[i])$$

where $N[i]$ is the shifted complex.

There are similar definitions for schemes. Let X be a finite type $\Bbbk$-scheme. Denote by $\mathsf{Mod}\, \mathcal{O}_X$ the category of $\mathcal{O}_X$-modules. We have a derived category $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$, with full subcategories $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, \mathcal{O}_X)$, $\mathsf{D}^{+}(\mathsf{Mod}\, \mathcal{O}_X)$, $\mathsf{D}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ etc. Here the subscript "c" stands for coherent cohomology sheaves. The internal Hom is

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X).$$

2. Dualizing Complexes

Dualizing complexes in were introduced by Grothendieck in the 1960's, in the book [RD].

A dualizing complex over the ring A is (omitting details) a complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ such that the contravariant functor

$$\mathrm{RHom}_A(-, R) : \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$$

is a duality (i.e. a contravariant equivalence).

Likewise on the scheme X a dualizing complex is a complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ such that

$$\mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{R})$$

is a duality of the category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

In case X is affine, say $X = \operatorname{Spec} A$, then the two definitions coincide.

Grothendieck's duality theory is relative. If Y is another such scheme and $f : X \to Y$ is a morphism, then there is a functor

$$f^! : \mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$$

called the twisted inverse image. It sends dualizing complexes to dualizing complexes.

If we denote by $\pi_X : X \to \operatorname{Spec} \Bbbk$ the structural morphism then the complex $\mathcal{R}_X := \pi_X^! \Bbbk$ is a dualizing complex on X with special properties.

In particular if X is smooth of dimension n then there is a canonical isomorphism

$$\mathcal{R}_X \cong \Omega^n_{X/\Bbbk}[n].$$

Here $\Omega^n_{X/\Bbbk}$ is the sheaf of top degree differential forms and $\Omega^n_{X/\Bbbk}[n]$ is the complex concentrated in degree $-n$.

Given a proper morphism $f : X \to Y$ there is a nondegenerate trace morphism

$$\mathrm{Tr}_f : \mathrm{R}f_* \mathcal{R}_X \to \mathcal{R}_Y,$$

which induces an isomorphism

$$\mathrm{R}f_* \mathcal{R}_X \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R}f_* \mathcal{O}_X, \mathcal{R}_Y).$$

Grothendieck duality is an important tool in many areas of algebraic geometry, including moduli spaces and arithmetic geometry.

Some interesting recent work on commutative duality has been done by Neeman [Ne], Lipman et al. [AJL] and Conrad [Co]. For explicit construction of residue complexes see [Ye] and its references.

3. Gluing Dualizing Complexes

One of the subtle aspects of Grothendieck duality is the passage between local and global features. A particular problem that occupies much of the book [RD] is the following: suppose $X = \bigcup U_i$ is an affine open covering, and there are given dualizing complexes $\mathcal{R}_i \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_{U_i})$ with isomorphisms

$$\phi_{i,j} : \mathcal{R}_i|_{U_i \cap U_j} \xrightarrow{\simeq} \mathcal{R}_j|_{U_i \cap U_j}$$

in $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_{U_i \cap U_j})$ satisfying the cocycle condition. Then one is looking for a dualizing complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ such that $\mathcal{R}|_{U_i} \cong \mathcal{R}_i$.

In [RD] the solution to the gluing problem was to use the Cousin functor

$$\mathrm{E} : \mathsf{D}^{+}(\mathsf{Mod}\, \mathcal{O}_X) \to \mathsf{C}^{+}(\mathsf{Mod}\, \mathcal{O}_X)$$

where the latter is the abelian category of complexes. The Cousin complex $\mathrm{E}\mathcal{M}$ is a direct sum of local cohomologies of $\mathcal{M}$. If $\mathcal{R}$ is a dualizing complex then there is an isomorphism $\mathcal{R} \cong \mathrm{E}\mathcal{M}$.

The gluing data $\{\phi_{i,j}\}$ above becomes actual isomorphisms of complexes of sheaves

$$\mathrm{E}(\phi_{i,j}) : \mathrm{E}\mathcal{R}_i|_{U_i \cap U_j} \xrightarrow{\simeq} \mathrm{E}\mathcal{R}_j|_{U_i \cap U_j},$$

and thus gluing can be achieved.

The point of this talk is to suggest a totally different solution to the gluing problem.

4. Rigid Dualizing Complexes

Van den Bergh [VdB] made the following definition. A dualizing complex R over a $\Bbbk$-algebra A is called *rigid* is there is an isomorphism

$$\rho : R \xrightarrow{\simeq} \mathrm{RHom}_{A \otimes A}(A, R \otimes R)$$

in $\mathsf{D}(\mathsf{Mod}(A \otimes A))$.

In [VdB] and [YZ1] it is proved that the pair (R, ρ) is unique up to a unique isomorphism. Hence we can talk about *the rigid dualizing complex* of A.

Every finitely generated commutative $\Bbbk$-algebra A has a rigid dualizing complex (R_A, ρ_A).

If $f : A \to B$ is a finite homomorphism then there is a nondegenerate rigid trace morphism

$$\mathrm{Tr}_f : R_B \to R_A$$

in $\mathsf{D}(\mathsf{Mod}\, A)$ that induces an isomorphism

$$R_B \cong \mathrm{RHom}_A(B, R_A).$$

If A_S is a localization of A at some multiplicatively closed set S then $R_A \otimes_A A_S$ is a rigid dualizing complex over A_S.

Example 4.1. If A is finite over $\Bbbk$ then $A^* := \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$ is the rigid dualizing complex of A. Duality here is nothing but the duality of linear algebra since

$$\mathrm{RHom}_A(M, A^*) = \mathrm{Hom}_{\Bbbk}(M, \Bbbk).$$

Example 4.2. If A is a smooth $\Bbbk$-algebra of dimension n then $\Omega^n_{A/\Bbbk}[n]$ is the rigid dualizing complex of A.

Now suppose $X = \bigcup U_i$ is an affine open cover as before. For each $A_i := \Gamma(U_i, \mathcal{O}_X)$ we have a rigid dualizing complex R_i, and by localization it will sheafify to a dualizing complex $\mathcal{R}_i$ on U_i. Moreover, by the uniqueness of rigid dualizing complexes, the collection of dualizing complexes $\{\mathcal{R}_i\}$ will automatically have gluing data $\{\phi_{i,j}\}$.

It remains to find a way to glue them. We could resort to Cousin complexes, but there is an even easier way, as we now show.

5. Perverse Coherent Sheaves

For a complex $\mathcal{M} \in \mathsf{D}^+(\mathsf{Mod}\, \mathcal{O}_X)$ we define

$$\mathrm{H}^j_{\mathsf{M}_i} \mathcal{M} := \lim_{Z \to} \mathrm{H}^j_Z \mathcal{M}$$

as Z runs over the closed sets in X with $\dim Z \leq i$ and $\mathrm{H}^j_Z \mathcal{M}$ is cohomology with supports in Z.

Let $p : \mathbb{Z} \to \mathbb{Z}$ be some function, which is called the perversity. Following [BBD] and [KS] we define full subcategories

$$ {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0} := \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \dim \operatorname{Supp} \mathrm{H}^j \mathcal{M} < i \text{ for all } i, j \text{ with } j > p(i)\} $$

and

$$ {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0} := \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \mathrm{H}^j_{\mathsf{M}_i} \mathcal{M} = 0 \text{ for all } i, j \text{ with } j < p(i)\}. $$

If the pair

$$ \big({}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0}, {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0}\big) $$

is a t-structure (some technical condition) then the heart

$$ {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{0} := {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0} \cap {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0} \subset \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) $$

is called the category of *perverse coherent $\mathcal{O}_X$-modules.* It is an abelian category, where a sequence of morphisms

$$ 0 \to \mathcal{M}' \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{M}'' \to 0 $$

in ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{0}$ is exact precisely when

$$ \mathcal{M}' \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{M}'' \to \mathcal{M}'[1] $$

is a triangle in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$.

Example 5.1. Take the perversity $p(i) = 0$. Then the heart is

$$ {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{0} = \mathsf{Coh}\,\mathcal{O}_X, $$

the category of coherent sheaves.

Theorem 5.2. ([YZ2]) *With the perversity $p(i) = -i$, the pair*

$$ \big({}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0}, {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0}\big) $$

is a t-structure.

Theorem 5.3. ([YZ2]) *The assignment $U \mapsto {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_U)^{0}$, for $U \subset X$ open, is a stack on X, namely it is "a sheaf of abelian categories".*

The theorem says perverse sheaves can be glued; hence the name "sheaves".

Theorem 5.4. ([YZ2]) *Let $U = \operatorname{Spec} A$ be an affine scheme, R_A the rigid dualizing complex of A and $\mathcal{R}_U$ its sheafication. Then $\mathcal{R}_U \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_U)^{0}$.*

Combining the last two theorems we obtain:

Corollary 5.5. *There is a dualizing complex $\mathcal{R}_X$ on X such that for each affine open set U the complex $\mathrm{R}\Gamma(U, \mathcal{R}_X)$ is a rigid dualizing complex over $\Gamma(U, \mathcal{O}_X)$.*

Let $f : X \to Y$ be a finite morphism. It is not hard to show that $\mathrm{R}f_* \mathcal{R}_X$ is a perverse coherent sheaf on Y. Hence the affine traces can be glued to a global trace morphism

$$ \mathrm{Tr}_f : \mathrm{R}f_* \mathcal{R}_X \to \mathcal{R}_Y $$

in ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Y)^{0}$.

6. Concluding Remarks

What is it good for? The Cousin method becomes difficult to apply for certain situations, notably base change. I think some complications can be removed with the perverse + rigid method.

Perverse coherent sheaves are becoming relevant in algebraic geometry for other purposes (cf. talk by Abramovich).

Research problem: develop theory of rigid dualizing complexes over a base ring $\mathbb{k}$ (e.g. over $\mathbb{Z}$). The hard technical issue is that the $\mathbb{k}$-algebra A will no longer be flat; a solution shall most likely involve DGAs.

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