Perverse Sheaves and Dualizing Complexes on Noncommutative Ringed Schemes

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Here is the plan of my lecture:

1. Some background on dualizing complexes
2. Quasi-coherent ringed schemes
3. Dualizing complexes on noncommutative ringed schemes
4. Perverse coherent sheaves
5. The Problem of Gluing

The lecture notes also contain:

6. Differential quasi-coherent ringed schemes
7. An example
8. Applications to commutative geometry

If time permits I will talk about some of these.

This talk is about joint work with James Zhang (Seattle).

1. Some Background on Dualizing Complexes

Dualizing complexes over schemes were introduced by Grothendieck in the 1960's (see [RD]), as a vast generalization of Serre duality.

Suppose X is a noetherian scheme. We denote by $\mathsf{Mod}\,\mathcal{O}_X$ the category of sheaves of $\mathcal{O}_X$-modules, and by $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$ its derived category. The full subcategory of bounded complexes with coherent cohomologies is $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$. It is equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathcal{O}_X)$.

A complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ is called a *dualizing complex* if the functor

$$\mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{R})$$

is an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$; plus some technical requirements.

When $X = \operatorname{Spec} A$ is affine the complex $R := \mathrm{R}\Gamma(X, \mathcal{R}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$ is a dualizing complex over A, in the sense that

$$M \mapsto \mathrm{RHom}_A(M, R)$$

is an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$.

In case X is of finite type over a nice base ring $\Bbbk$ (e.g. a field or $\mathbb{Z}$) with structural morphism $\pi : X \to \operatorname{Spec} \Bbbk$, then there is a special dualizing complex $\pi^{!}\Bbbk$ over X, which is called the Grothendieck dualizing complex.

So far for the classical commutative picture. From now on $\Bbbk$ will be a field, and A will denote a noetherian, unital, associative $\Bbbk$-algebra (not necessarily commutative).

By convention $\mathsf{Mod}\,A$ is the category of left A-modules. We shall write A^{op} for the opposite ring and $A^{\mathrm{e}} := A \otimes_{\Bbbk} A^{\mathrm{op}}$. So $\mathsf{Mod}\,A^{\mathrm{e}}$ is the category of A-bimodules.

A complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A^{\mathrm{e}})$ is called dualizing if the functor

$$\mathrm{D} := \mathrm{RHom}_A(-, R)$$

is a duality

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})$$

with adjoint

$$\mathrm{D}^{\mathrm{op}} := \mathrm{RHom}_{A^{\mathrm{op}}}(-, R).$$

Again I'm omitting some details; see [Ye1].

Van den Bergh [VdB1] discovered the following condition on a dualizing complex R that turns out to be extremely powerful. Suppose there is an isomorphism

$$\rho : R \xrightarrow{\simeq} \mathrm{RHom}_{A^{\mathrm{e}}}(A, R \otimes_{\Bbbk} R)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$. Then R is called a *rigid dualizing complex* and ρ is a *rigidifying isomorphism.*

By [VdB1] and [YZ2] the pair (R, ρ) is unique up to a unique isomorphism in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

When A is commutative and $X = \operatorname{Spec} A$, the Grothendieck dualizing complex $R := \mathrm{R}\Gamma(X, \pi^{!}\Bbbk)$ is rigid. So indeed we have a pretty good generalization of duality to the noncommutative setup.

The question of existence of rigid dualizing complexes is much harder. The best existence criterion we know is also due to Van den Bergh. Here is a brief description.

Suppose A admits a nonnegative exhaustive filtration $F = \{F_i A\}_{i \in \mathbb{Z}}$ such that the graded algebra $\bar{A} := \operatorname{gr}^F A$ is a connected graded, commutative, finitely generated $\Bbbk$-algebra.

Let

$$\widetilde{A} := \bigoplus\nolimits_i (F_i A) t^i \subset A[t]$$

be the Rees algebra, where t is a central indeterminate of degree 1. So $\bar{A} \cong \widetilde{A}/(t)$ and $A \cong \widetilde{A}/(t-1)$.

By local duality for noncommutative graded algebras one can show that $\widetilde{A}$ has a balanced dualizing complex, and hence A has a rigid dualizing complex R_A.

One should think of the filtration F as a "compactification of $\operatorname{Spec} A$". Indeed if A is commutative then $\operatorname{Proj} \widetilde{A}$ is a projective $\Bbbk$-scheme, $\{t = 0\}$ is an ample divisor, and its complement is isomorphic to $\operatorname{Spec} A$.

The goal of the work I'll discuss today is to try to combine the two duality theories recalled above, namely to find a *theory of duality for noncommutative spaces.*

2. Quasi-Coherent Ringed Schemes

We do not know a general definition of noncommutative space, in the framework of noncommutative algebraic geometry. There are several attempts in current literature; see [AZ], [Ro], [VdB2] and [KR].

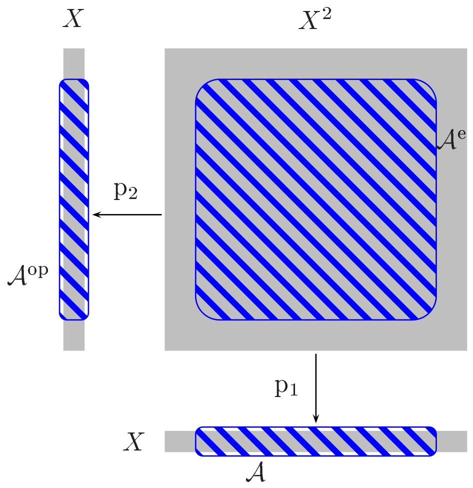
We shall concentrate on the following setup: X is a $\Bbbk$-scheme, $\mathcal{A}$ is a sheaf of rings on X, and there is ring homomorphism $\mathcal{O}_X \to \mathcal{A}$ making $\mathcal{A}$ into a quasi-coherent $\mathcal{O}_X$-module on both sides. We call $(X, \mathcal{A})$ a *quasi-coherent ringed scheme over* $\Bbbk$.

Such ringed schemes are abundant; two prototypical examples are:

(1) X is smooth, $\operatorname{char} \Bbbk = 0$ and $\mathcal{A} := \mathcal{D}_X$.
(2) X is arbitrary and $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra.

As we saw previously noncommutative dualizing complexes are complexes of bimodules. So geometrically they should live on the product, over $\Bbbk$, of $(X, \mathcal{A})$ with the opposite ringed scheme $(X, \mathcal{A}^{\mathrm{op}})$.

The product is a quasi-coherent ringed scheme which we denote by $(X^2, \mathcal{A}^{\mathrm{e}})$. The definition is pretty obvious.



It is not hard to show uniqueness of $(X^2, \mathcal{A}^{\mathrm{e}})$.

Surprisingly existence is not automatic! In [YZ4] we proved that the product exists iff certain Ore conditions are satisfied. (There are counterexamples.)

3. Dualizing Complexes on Noncommutative Ringed Schemes

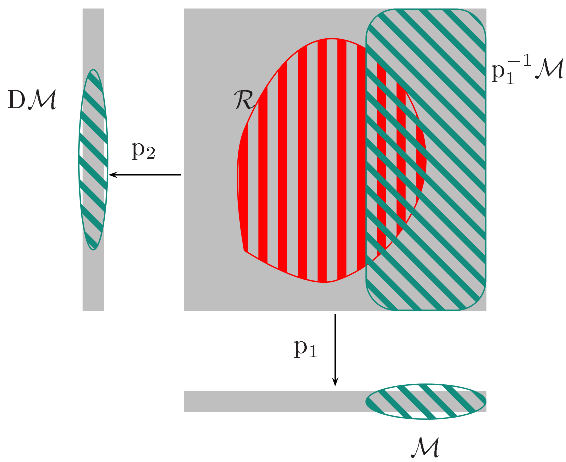
Let $(X, \mathcal{A})$ be a noetherian quasi-coherent ringed scheme. By noetherian I mean that X is noetherian, and for any affine open set U the ring $A := \Gamma(U, \mathcal{A})$ is noetherian. Assume the product $(X^2, \mathcal{A}^{\mathrm{e}})$ exists and is noetherian too.

Let $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{e}})$ be some complex. We may define a functor

$$\mathrm{D} : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}) \to \mathsf{D}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{op}})$$

as follows:

$$\mathrm{D}\mathcal{M} := \mathrm{R}\mathrm{p}_{2*}\mathrm{R}\mathcal{H}om_{\mathrm{p}_1^{-1}\mathcal{A}}(\mathrm{p}_1^{-1}\mathcal{M}, \mathcal{R}).$$



This is a contravariant Fourier-Mukai transform.

There is an opposite version of this functor:

$$\mathrm{D}^{\mathrm{op}} : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Mod}\,\mathcal{A}).$$

By definition $\mathcal{R}$ is a dualizing complex over $(X, \mathcal{A})$ if the adjunction morphisms $\mathbf{1} \to \mathrm{D}^{\mathrm{op}}\mathrm{D}$ and $\mathbf{1} \to \mathrm{D}\mathrm{D}^{\mathrm{op}}$ are both isomorphisms. (I am suppressing some details.)

I would like to remark that there are nontrivial technical complications hidden behind this definition. One of them is that unlike the commutative situation, here an injective object in the category $\mathsf{QCoh}\,\mathcal{A}$ of quasi-coherent $\mathcal{A}$-modules need not be injective in the bigger category $\mathsf{Mod}\,\mathcal{A}$. Recently Kashiwara found a counterexample [Ka].

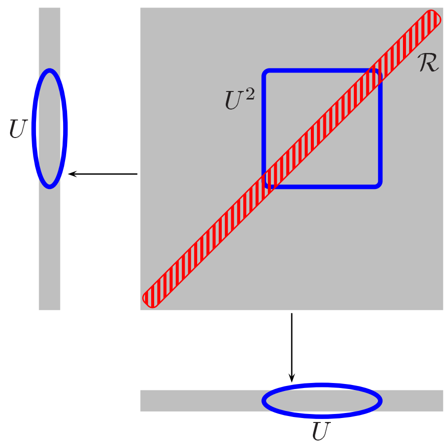
This phenomenon seems to be related somehow to the following fact peculiar to noncommutative geometry: given a pair of coherent $\mathcal{A}$-modules the sheaf $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ is sometimes constant (rather than coherent). E.g. $\operatorname{char} \Bbbk = 0$, $X := \mathbb{A}^1$, $\mathcal{A} := \mathcal{D}_X$ and $\mathcal{M}, \mathcal{N} := \mathcal{O}_X$.

The definition I just gave allows for all kinds of exotic dualizing complexes. For instance, say X is an elliptic curve and take $\mathcal{A} := \mathcal{O}_X$. Then the product is $(X^2, \mathcal{A}^{\mathrm{e}}) = (X^2, \mathcal{O}_{X^2})$. It turns out that the Poincaré bundle $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{e}})$ is a dualizing complex over $(X, \mathcal{O}_X)$ in the noncommutative sense.

We are interested in dualizing complexes $\mathcal{R}$ that behave similarly to Grothendieck's dualizing complex $\pi^! \Bbbk$. Hence the definition below.

A *rigid dualizing complex over* $(X, \mathcal{A})$ is a dualizing complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{e}})$ supported on the diagonal in X^2, together with a collection $\boldsymbol{\rho} = \{\rho_U\}$ of rigidifying isomorphisms indexed by the affine open sets of X.

For any such U, letting $A := \Gamma(U, \mathcal{A})$ and $R := \mathrm{R}\Gamma(U^2, \mathcal{R})$, the pair (R, ρ_U) is a rigid dualizing complex over A. We require the collection $\boldsymbol{\rho}$ to satisfy a suitable compatibility condition.



Let us check the commutative picture, i.e. a separated finite type scheme X. If one looks carefully at the variance properties of $\pi^! \Bbbk$ that are worked out in [RD], one sees that this is in fact a rigid dualizing over $(X, \mathcal{O}_X)$. So we are still on track.

Our problem can now be stated precisely: *prove existence of a rigid dualizing complex over* $(X, \mathcal{A})$.

4. The Problem of Gluing

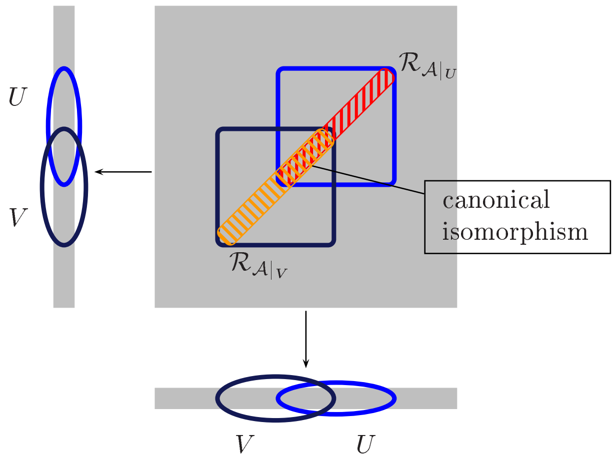
Let $(X, \mathcal{A})$ be a ringed scheme as before. Suppose that for every affine open set $U \subset X$ the ring $A := \Gamma(U, \mathcal{A})$ admits a rigid dualizing complex R_A that's supported on the diagonal $\Delta(U) \subset U^2$.

The fact that R_A is supported on the diagonal implies that it sheafifies to a complex $\mathcal{R}_{\mathcal{A}|_U} \in \mathsf{D}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}}|_{U^2})$, which is a dualizing complex over the affine ringed scheme $(U, \mathcal{A}|_U)$.

Because of the uniqueness of rigid dualizing complexes we obtain canonical isomorphisms

$$\mathcal{R}_{\mathcal{A}|_U}|_{U^2 \cap V^2} \cong \mathcal{R}_{\mathcal{A}|_V}|_{U^2 \cap V^2}$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}}|_{U^2 \cap V^2})$ for any two affine open sets U and V.



We would like to glue the affine dualizing complexes $\mathcal{R}_{\mathcal{A}|_U}$ into a global complex $\mathcal{R}_{\mathcal{A}} \in \mathsf{D}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}})$. But here we encounter a genuine problem: *usually objects in derived categories cannot be glued!*

Grothendieck's solution in the commutative case, in [RD], was to use Cousin complexes. However, as explained in [YZ3], this solution seldom applies in the noncommutative context.

The main discovery in [YZ5] is that *perverse coherent sheaves can be used instead of Cousin complexes to glue dualizing complexes.*

5. Perverse Coherent Sheaves

T-structures and perverse sheaves were introduced by Beilinson, Bernstein and Deligne [BBD] around 1980. This was in the context of intersection cohomology on singular spaces. For such a space X they were interested in t-structures on subcategories of $\mathsf{D}(\mathsf{Mod}\, \Bbbk_X)$, where $\Bbbk_X$ is a constant sheaf of rings on X.

Perverse coherent sheaves came into the scene only very recently, independently in the work of Bezrukavnikov (after Deligne) [Bz], Bridgeland [Br], Kashiwara [Ka] and our paper [YZ5].

Let me recall what is a t-structure on a triangulated category D. It consists of the datum of two full subcategories $\mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 0}$ satisfying the axioms below, where $\mathsf{D}^{\leq n} := \mathsf{D}^{\leq 0}[-n]$ and $\mathsf{D}^{\geq n} := \mathsf{D}^{\geq 0}[-n]$.

(i) $\mathsf{D}^{\leq -1} \subset \mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 1} \subset \mathsf{D}^{\geq 0}$.
(ii) $\mathrm{Hom}_{\mathsf{D}}(M, N) = 0$ for $M \in \mathsf{D}^{\leq 0}$ and $N \in \mathsf{D}^{\geq 1}$.
(iii) For any $M \in \mathsf{D}$ there is a distinguished triangle

$$M' \to M \to M'' \to M'[1]$$

in D with $M' \in \mathsf{D}^{\leq 0}$ and $M'' \in \mathsf{D}^{\geq 1}$.

When these conditions are satisfied one defines the *heart of* D to be the full subcategory $\mathsf{D}^0 := \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$. This is an abelian category.

Given a ringed scheme $(X, \mathcal{A})$ the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})$ has the *standard t-structure*, in which

$$\begin{aligned}
&\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\leq 0} := \\
&\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}) \mid \mathrm{H}^i \mathcal{M} = 0 \text{ for all } i > 0\}, \\
&\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\geq 0} := \\
&\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}) \mid \mathrm{H}^i \mathcal{M} = 0 \text{ for all } i < 0\}.
\end{aligned}$$

The heart $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^0$ is equivalent to the category $\mathsf{Coh}\,\mathcal{A}$ of coherent sheaves.

Other t-structures will be referred to as *perverse t-structures*.

Here is an observation. Suppose the ring A has a rigid dualizing complex R_A. Then the duality $\mathrm{D} := \mathrm{RHom}_A(-, R_A)$ gives rise to a perverse t-structure

$$\begin{aligned}
&{}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\leq 0} := \\
&\quad \{M \mid \mathrm{H}^i \mathrm{D} M = 0 \text{ for all } i < 0\}, \\
&{}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\geq 0} := \\
&\quad \{M \mid \mathrm{H}^i \mathrm{D} M = 0 \text{ for all } i > 0\}.
\end{aligned}$$

We call it the *rigid perverse t-structure*. The heart is denoted by ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^0$.

The next theorem was proved in [YZ5].

Theorem 1. *Let $(X, \mathcal{A})$ be a noetherian quasi-coherent ringed $\Bbbk$-scheme. Assume that for any affine open set $U \subset X$ the $\Bbbk$-algebra $A := \Gamma(U, \mathcal{A})$ has a rigid dualizing complex R_A, which is supported on the diagonal $\Delta(U) \subset U^2$. Also assume A^{e} is noetherian. Define*

$$\begin{aligned}
{}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\star} := \{&\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}) \mid \\
&\mathrm{R}\Gamma(U, \mathcal{M}) \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}\big(\mathsf{Mod}\,\Gamma(U, \mathcal{A})\big)^{\star} \\
&\textit{for all affine open sets } U\}.
\end{aligned}$$

Then:

(1) *The pair*

$$\big({}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\leq 0}, {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\geq 0}\big)$$

is a t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})$.

(2) *The assignment $V \mapsto {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}|_V)^0$, for $V \subset X$ open, is a stack of abelian categories on X.*

Item (2) says that the objects of ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^0$, which we call *perverse coherent sheaves*, can be glued. They behave like sheaves, and hence the name.

In Section 6 of the notes we explain how Theorem 1 is used to prove existence and uniqueness of a rigid dualizing complex when $(X, \mathcal{A})$ is a differential quasi-coherent ringed scheme of finite type.

To end this section (and perhaps the lecture) let's look at the commutative case. If X is a finite type $\Bbbk$-scheme and $\mathcal{A} = \mathcal{O}_X$ then the hypothesis of Theorem 1 is easily verified. Indeed, given $A = \Gamma(U, \mathcal{O}_X)$ let us choose a surjection $\Bbbk[\boldsymbol{t}] \twoheadrightarrow A$ where $\Bbbk[\boldsymbol{t}] = \Bbbk[t_1, \ldots, t_n]$ is a polynomial ring. Then

$$R_A := \operatorname{RHom}_{\Bbbk[\boldsymbol{t}]}(A, \Omega^n_{\Bbbk[\boldsymbol{t}]}[n])$$

is a rigid dualizing complex over A, and being central it is supported on the diagonal $\Delta(U) \subset U^2$.

Furthermore each local piece $\mathcal{R}_{\mathcal{A}|_U}$ is a perverse sheaf, so they can be glued.

As a consequence we obtain a totally new approach to dualizing complexes in commutative algebraic geometry. For more on the commutative case see Section 8 below.

6. Differential Quasi-Coherent Ringed Schemes

All quasi-coherent ringed schemes $(X, \mathcal{A})$ that "occur naturally" are differential. This means that there is a filtration $G = \{G_i \mathcal{A}\}$ on the sheaf of rings $\mathcal{A}$ such that $\operatorname{gr}^G \mathcal{A}$ is a coherent module over its center $\operatorname{Z}(\operatorname{gr}^G \mathcal{A})$, and $\operatorname{Z}(\operatorname{gr}^G \mathcal{A})$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite type.

For instance when X is smooth in characteristic 0 and $\mathcal{A} = \mathcal{D}_X$ the ring of differential operators, then we have the order filtration on $\mathcal{A}$.

If $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra (e.g. an Azumaya algebra) then we can use the trivial filtration.

We call the next result the "Theorem on the Two Filtrations". It is proved in [YZ4], and generalizes the familiar filtrations of the Weyl algebras. A slightly weaker result appeared in [MS].

Theorem 2. *Assume the ring A has a filtration G such that $\operatorname{gr}^G A$ is finite over its center $\operatorname{Z}(\operatorname{gr}^G A)$, and the latter is a finitely generated $\Bbbk$-algebra. Then there exists a filtration F on A such that the graded algebra $\operatorname{gr}^F A$ is a connected graded, commutative, finitely generated $\Bbbk$-algebra.*

As explained in Section 1, Van den Bergh's existence criterion implies that for any affine open set U the ring $A := \Gamma(U, \mathcal{A})$ has a rigid dualizing complex R_A.

Furthermore we proved in [YZ4] that R_A is supported on the diagonal in U^2. Therefore Theorem 1 can be applied, and we deduce that the rigid perverse t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})$ exists.

Now recall that noncommutative dualizing complexes live on the product $(X^2, \mathcal{A}^{\mathrm{e}})$. The fact that $(X, \mathcal{A})$ is differential implies that the product exists. It is not hard to show that the product itself is a differential quasi-coherent ringed scheme of finite type. So we have at our disposal the stack ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{e}})^0$ of perverse bimodules.

The last piece in the puzzle is the fact that the affine dualizing complexes $\mathcal{R}_{\mathcal{A}|_U}$ are perverse bimodules, namely they lie in ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{e}}|_{U^2})^0$. Therefore the gluing data (arising from uniqueness of rigid dualizing complexes) becomes effective. We thus obtain:

Theorem 3. *Let $(X, \mathcal{A})$ be a separated differential quasi-coherent ringed $\Bbbk$-scheme of finite type. Then there exists a rigid dualizing complex $(\mathcal{R}_{\mathcal{A}}, \boldsymbol{\rho})$ over $\mathcal{A}$. It is unique up to a unique isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}})$.*

7. An Example

Let $\operatorname{char} \Bbbk = 0$. Consider the first Weyl algebra

$$A := \Bbbk\langle x, y\rangle / (yx - xy - 1).$$

It is of course isomorphic to the ring of differential operators on the affine line $\mathbf{A}^1 = \operatorname{Spec} \Bbbk[x]$, via $y \mapsto \frac{\partial}{\partial x}$.

A has a filtration F in which $\deg^F(x) = \deg^F(y) = 1$. The Rees algebra $\widetilde{A} := \bigoplus_i (F_i A) t^i$ has three homogeneous generators of degree 1, namely t, $u := xt$ and $v := yt$.

Passing to $\mathbf{P}^1 = \operatorname{Proj}\, \Bbbk[t, u]$ we obtain a differential quasi-coherent ringed scheme of finite type $(\mathbf{P}^1, \mathcal{A})$.

On the open set $\mathbf{A}^1 = \{t \neq 0\}$ we get $\mathcal{A}|_{\mathbf{A}^1} \cong \mathcal{D}_{\mathbf{A}^1}$ and

$$\Gamma(\mathbf{A}^1, \mathcal{A}) \cong \widetilde{A}/(t - 1) \cong A.$$

However the ringed schemes $(\mathbf{P}^1, \mathcal{A})$ and $(\mathbf{P}^1, \mathcal{D}_{\mathbf{P}^1})$ are not isomorphic.

Let $\operatorname{Proj} \widetilde{A}$ be the "noncommutative complete surface" defined by Artin-Zhang [AZ]. This is the geometric entity such that "$\mathsf{QCoh} \operatorname{Proj} \widetilde{A}$" is the quotient category $\mathsf{GrMod}\, \widetilde{A}/(\text{torsion})$.

It is not hard to see that $(\mathbf{P}^1, \mathcal{A})$ is isomorphic to an open subspace of $\operatorname{Proj} \widetilde{A}$, whose complement is one point.

In [KKO] there is a duality between $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh} \operatorname{Proj} \widetilde{A})$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh} \operatorname{Proj} \widetilde{A}^{\mathrm{op}})$. Kazhdan asked recently us whether the diagram

$$\begin{tikzcd} \mathsf{D}^{\mathrm{b}}(\mathsf{Coh} \operatorname{Proj} \widetilde{A}) \arrow[r, "\approx"] \arrow[d, "\mathrm{rest}"'] & \mathsf{D}^{\mathrm{b}}(\mathsf{Coh} \operatorname{Proj} \widetilde{A}^{\mathrm{op}}) \arrow[d, "\mathrm{rest}"'] \\ \mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, \mathcal{A}) \arrow[r, "\mathrm{D}_{\mathcal{A}}"] & \mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, \mathcal{A}^{\mathrm{op}}) \end{tikzcd}$$

where $\mathrm{D}_{\mathcal{A}}$ is the duality determined by the rigid dualizing complex $\mathcal{R}_{\mathcal{A}}$, is commutative. This has been answered affirmatively in [YZ5].

8. Applications to Commutative Geometry

Even though commutative Grothendieck duality has been around for about forty year it still attracts considerable attention. See the recent papers [Ne], [Ye2], [AJL] and [Co] and their references.

In our paper [YZ5] we prove, using the same method described above, that if X is a separated finite type $\Bbbk$-scheme, then the rigid dualizing complex $\mathcal{R}_X$ can be chosen to be central – i.e. an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$. We then prove that $\mathcal{R}_X$ is canonically isomorphic to the Grothendieck dualizing complex $\pi^! \Bbbk$.

Changing points of view, see that we have obtained a new way of constructing $\pi^! \Bbbk$.

Perhaps more interesting is the next result. Recall that a complex $\mathcal{M} \in \mathsf{D}^{+}(\mathsf{Mod}\,\mathcal{O}_X)$ is called *Cohen-Macaulay* if for every point x the local cohomologies $\mathrm{H}^i_x \mathcal{M}$ all vanish except for $i = -\dim \overline{\{x\}}$. Equivalently, $\mathcal{M}$ is Cohen-Macaulay if it is isomorphic to a Cousin complex. See [RD].

Theorem 4. *Let X be a finite type scheme over $\Bbbk$, let $\mathcal{R}_X$ be the central rigid dualizing complex of X, and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$. Then the following conditions are equivalent for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.*

(i) *$\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).*
(ii) *$\mathrm{D}\mathcal{M}$ is a coherent sheaf, i.e. $\mathrm{H}^i \mathrm{D}\mathcal{M} = 0$ for all $i \neq 0$.*
(iii) *$\mathcal{M}$ is a Cohen-Macaulay complex.*

In particular this implies the Cohen-Macaulay complexes form an abelian subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$, a fact that seems to have eluded Grothendieck.

I did not say this earlier, but the rigidity condition as stated only makes sense when $\Bbbk$ is a field (more precisely, when A is a flat $\Bbbk$-algebra). I believe it should be possible to overcome this technical snag, and make everything work for any commutative base ring $\Bbbk$, using differential graded algebras. The key modification is to view dualizing complexes as objects of

$$\mathsf{D}(\mathsf{DGMod}\,(A \otimes^{\mathrm{L}}_{\Bbbk} A^{\mathrm{op}})),$$

where $A \otimes^{\mathrm{L}}_{\Bbbk} A^{\mathrm{op}}$ is a suitable DGA.

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