Filtrations and localization of noncommutative rings

Lecture at Ben Gurion University
December 2001

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This talk is about material in the paper [YZ3], which is joint work with James Zhang.

1 Filtrations of rings

In this lecture $\Bbbk$ denotes a base field. By $\Bbbk$-algebra we mean an associative unital one. By a filtration F of a $\Bbbk$-algebra A we mean an ascending sequence of $\Bbbk$-modules

$$\cdots F_i A \subset F_{i+1} A \cdots \subset A$$

such that $F_i A \cdot F_j A \subset F_{i+j} A$. F is called exhaustive if $A = \bigcup F_i A$, separated if $\bigcap F_i A = 0$, and positive if $F_{-1} A = 0$.

The graded ring $\operatorname{gr}^F A$ is

$$\operatorname{gr}^F A := \bigoplus_i \frac{F_i A}{F_{i-1} A}.$$

For an element $a \in F_i A - F_{i-1} A$ its symbol is the nonzero element $\bar{a} \in (\operatorname{gr}^F A)_i$.

We say F is a noetherian filtration if $\operatorname{gr}^F A$ is noetherian. F is called connected if $\operatorname{gr}^F A$ is a connected graded $\Bbbk$-algebra, namely $(\operatorname{gr}^F A)_i = 0$ for $i < 0$, $(\operatorname{gr}^F A)_0 = \Bbbk$, and each $(\operatorname{gr}^F A)_i$ is finite over $\Bbbk$.

Example 1.1. The n-Weyl algebra: here $\operatorname{char} \Bbbk = 0$. A is generated by elements $x_1, \ldots, x_n, y_1, \ldots, y_n$. The relations are

$$[x_i, y_i] = 1$$

and

$$[x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 0$$

for $i \neq j$.

A is a noetherian simple ring. It comes with two natural filtrations.

The first filtration F is:

$$F_i A := \bigoplus_{|\boldsymbol{p}|+|\boldsymbol{q}| \leq i} \Bbbk \cdot \boldsymbol{x}^{\boldsymbol{p}} \boldsymbol{y}^{\boldsymbol{q}}$$

in multi-index notation. $\operatorname{gr}^F A$ is a commutative polynomial ring in the $2n$ variables $\overline{x}_1, \ldots, \overline{x}_n, \overline{y}_1, \ldots, \overline{y}_n$, each of degree 1. So F is a noetherian connected filtration.

The second filtration G is:

$$G_i A := \bigoplus_{|\boldsymbol{q}| \leq i} \Bbbk[\boldsymbol{x}] \cdot \boldsymbol{y}^{\boldsymbol{q}}.$$

$\mathrm{gr}^G A$ is also a commutative polynomial ring in the same $2n$ variables, but $\deg x_i = 0$, $\deg y_i = 1$. This is a noetherian filtration, but not connected.

Given a filtration F on A we can form the Rees ring

$$\widetilde{A} := \bigoplus_i F_i A \cdot t^i \subset A[t, t^{-1}]$$

where t is a new central variable of degree 1. Then

$$\mathrm{gr}^F A \cong \widetilde{A}/(t)$$

as graded rings, and, if F is exhaustive and separated,

$$A \cong \widetilde{A}/(t - \lambda)$$

for any $\lambda \in \Bbbk$, $\lambda \neq 0$. The filtration F can be recovered as

$$F_i A \cong \mathrm{im}\big(\bigoplus\nolimits_{p \leq i} \widetilde{A}_p \to A\big).$$

By default we assume filtrations are exhaustive and separated.

Proposition 1.2. *Let A be a ring and F a positive filtration of A. Then the graded ring $\mathrm{gr}^F A$ is noetherian iff the Rees ring $\widetilde{A}$ is noetherian. If $\mathrm{gr}^F A$ is noetherian then so is A.*

2 Filtrations as compactifications

In algebraic geometry it is often important to embed a variety V inside a projective variety $\overline{V}$. $\overline{V}$ is called a compactification of V. The good global properties of $\overline{V}$ (such as finiteness) can be used to study V.

Consider the following example. (Thanks to Steve Shnider for pointing out an earlier error in it.)

Example 2.1. Take the affine line $\mathbf{A}^1$. It is an affine variety with coordinate ring $A = \Bbbk[x]$, where x is a variable. Suppose we give A the connected filtration

$$F_i A := \bigoplus\nolimits_{p \leq i} \Bbbk \cdot x^p.$$

Then the Rees ring is

$$\widetilde{A} \cong \Bbbk[t, tx] \subset \Bbbk[t, x].$$

Localizing and taking degree 0 components we get $\widetilde{A}[t^{-1}]_0 = \Bbbk[\frac{x}{t}]$ and $\widetilde{A}[(tx)^{-1}]_0 = \Bbbk[\frac{t}{x}]$. Hence the projective spectrum is

$$\operatorname{Proj} \widetilde{A} = (\operatorname{Spec} \Bbbk[\tfrac{x}{t}]) \cup (\operatorname{Spec} \Bbbk[\tfrac{t}{x}]) = \mathbf{P}^1,$$

the projective line. The open set $\{t \neq 0\} = \operatorname{Spec} \Bbbk[\frac{x}{t}] \subset \mathbf{P}^1$ is precisely $\mathbf{A}^1$.

More generally, if A is a finitely generated commutative $\Bbbk$-algebra, F is any connected noetherian filtration of A and $\widetilde{A}$ is the Rees ring, then $\operatorname{Proj} \widetilde{A}$ is a compactification of $\operatorname{Spec} A$.

A noetherian filtration that is not connected will give rise to a "partial compactification".

For some reason connected filtrations are important even for noncommutative rings, though the notion of compactification is not at all clear. This was observed by Michel Van den Bergh (among others). The next result combines work of Van den Bergh [VdB] and of Zhang and myself.

Theorem 2.2. ([YZ1]) *Suppose A admits a filtration F such that $\operatorname{gr}^F A$ is a noetherian connected graded $\Bbbk$-algebra that's finite over its center. Then A has an Auslander rigid dualizing complex R. For any finite A-module M one has*

$$\operatorname{GKdim} M = \operatorname{Cdim} M,$$

where GKdim *is Gelfand-Kirillov dimension, and* Cdim *is the canonical dimension*

$$\operatorname{Cdim} M := -\inf\{p \mid \operatorname{Ext}^p_A(M, R) \neq 0\}.$$

The rigid dualizing complex R, if it exists, is unique up to isomorphism (in the derived category of bimodules). The role of the filtration in the theorem is to ensure existence of R, and that it has these special properties (Auslander property and the comparison to GKdim).

3 Differential $\Bbbk$-algebras of finite type

Here is an important class of rings, which are similar to Weyl algebras.

Definition 3.1. Let A be a $\Bbbk$-algebra. A *differential filtration of finite type* on A is a filtration $G = \{G_i A\}$ such that the graded ring $\operatorname{gr}^G A$ is a finite module over its center $\mathrm{Z}(\operatorname{gr}^G A)$, and $\mathrm{Z}(\operatorname{gr}^G A)$ is a finitely generated commutative $\Bbbk$-algebra. We say A is a *differential $\Bbbk$-algebra of finite type* if it admits some differential filtration of finite type.

Such rings can be "very noncommutative," yet homologically they are quite nice. The reason is the next theorem, combined with Theorem 2.2. J.C. McConnell and J.T. Stafford also considered such filtrations, and our result extends their [MS] Corollary 1.7. The basic idea is attributed in [MS] to J. Bernstein.

Theorem 3.2. ([YZ3]) *Let A be a differential $\Bbbk$-algebra of finite type. Then there is a filtration $F = \{F_i A\}$ of A that is both a differential $\Bbbk$-filtration of finite type and a connected $\Bbbk$-filtration.*

Proof. **Step 1.** Let G be a given differential $\Bbbk$-filtration of finite type of A. Write $\overline{A} := \operatorname{gr}^G A$. Then the center $\mathrm{Z}(\overline{A})$ is a graded, finitely generated, commutative $\Bbbk$-algebra, and $\overline{A}$ is a finite $\mathrm{Z}(\overline{A})$-module.

For any nonzero element $a \in A$ we denote by $o(a)$ its G-order, namely the integer satisfying $a \in G_{o(a)}A - G_{o(a)-1}A$. By definition the symbol $\bar{a}$ of a is a nonzero element of $\operatorname{gr}^G_{o(a)} A$.

Let d be large enough such that $\mathrm{Z}(\overline{A})$ is generated as $\mathrm{Z}(\overline{A})_0$-algebra by elements of degrees $\leq d$, and $\overline{A}$ is generated as $\mathrm{Z}(\overline{A})$-module by elements if degrees $\leq d$.

Choose nonzero elements $a_1, \ldots, a_p \in G_0 A = \overline{A}_0$ such that their symbols $\bar{a}_1, \ldots, \bar{a}_p$ are in $\mathrm{Z}(\overline{A})_0$, and they generate $\mathrm{Z}(\overline{A})_0$ as $\Bbbk$-algebra. Next choose nonzero $b_1, \ldots, b_m \in G_d A$ such that their symbols $\bar{b}_1, \ldots, \bar{b}_m$ are in $\mathrm{Z}(\overline{A})$, and they generate $\bigoplus_{j=0}^{d} \mathrm{Z}(\overline{A})_j$ it as $\mathrm{Z}(\overline{A})_0$-module. Finally choose nonzero $c_1, \ldots, c_n \in G_d A$ such that their symbols $\bar{c}_1, \ldots, \bar{c}_n$ generate $\bigoplus_{j=0}^{d} \overline{A}_j$ as $\mathrm{Z}(\overline{A})_0$-module.

Let B be the subalgebra of A generated by $a_1, \ldots, a_p, b_1, \ldots, b_m$. It has an induced filtration G, and

$$\operatorname{gr}^G B = \mathrm{Z}(\overline{A}) \subset \overline{A} = \operatorname{gr}^G A.$$

For any i, j the elements a_i and a_j commute in A. Since $\operatorname{gr}^G B$ is commutative, the commutators satisfy

$$[a_i, b_j] \in G_{o(b_j)-1} B.$$

Hence there are polynomials $f^{[a,b]}_{i,j,k}(\boldsymbol{x})$ in the commutative polynomial ring $\Bbbk[\boldsymbol{x}] := \Bbbk[x_1, \ldots, x_p]$ such that

$$[a_i, b_j] = \sum_{k \mid o(b_k) \leq o(b_j)-1} f^{[a,b]}_{i,j,k}(\boldsymbol{a}) \cdot b_k. \tag{1}$$

Likewise

$$[b_i, b_j] \in G_{o(b_i)+o(b_j)-1} B,$$

so there are polynomials $f^{[b,b]}_{i,j,k,l}(\boldsymbol{x})$ satisfying

$$[b_i, b_j] = \sum_{k,l \mid o(b_k)+o(b_l) \leq o(b_i)+o(b_j)-1} f^{[b,b]}_{i,j,k,l}(\boldsymbol{a}) \cdot b_k \cdot b_l.$$

Since $\bar{a}_i$ is in the center of $\overline{A}$, we have

$$[a_i, c_j] \in G_{o(c_j)-1} A,$$

and thus there are polynomials $f^{[a,c]}_{i,j,k}(\boldsymbol{x})$ such that

$$[a_i, c_j] = \sum_{k \mid o(c_k) \le o(c_j)-1} f^{[a,c]}_{i,j,k}(\boldsymbol{a}) \cdot c_k.$$

Similarly there are polynomials $f^{[b,c]}_{i,j,k,l}(\boldsymbol{x})$, satisfying

$$[b_i, c_j] = \sum_{k,l \mid o(b_k)+o(c_l) \le o(b_i)+o(c_j)-1} f^{[b,c]}_{i,j,k,l}(\boldsymbol{a}) \cdot b_k \cdot c_l.$$

Because

$$\overline{A} = \sum \mathrm{Z}(\overline{A}) \cdot \overline{c}_i$$

it follows that

$$G_j A = \sum_i (G_{j-o(c_i)} B) \cdot c_i.$$

Hence there are polynomials $f^{c\cdot c}_{i,j,k,l}(\boldsymbol{x})$ satisfying

$$c_i \cdot c_j = \sum_{k,l \mid o(b_k)+o(c_l) \le o(c_i)+o(c_j)} f^{c\cdot c}_{i,j,k,l}(\boldsymbol{a}) \cdot b_k \cdot c_l.$$

Define

$$\begin{aligned} e := 1 + \max\{ &\deg f^{[a,b]}_{i,j,k}(\boldsymbol{x}), \deg f^{[a,c]}_{i,j,k}(\boldsymbol{x}), \\ &\deg f^{[b,b]}_{i,j,k,l}(\boldsymbol{x}), \deg f^{[b,c]}_{i,j,k,l}(\boldsymbol{x}), \deg f^{c\cdot c}_{i,j,k,l}(\boldsymbol{x}), 0\}. \end{aligned}$$

Let

$$\Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\rangle := \langle x_1, \ldots, x_p, y_1, \ldots, y_m, z_1, \ldots, z_n\rangle$$

be the free associative algebra. We have a surjection $\phi : \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\rangle \to A$ sending $x_i \mapsto a_i$, $y_i \mapsto b_i$ and $z_i \mapsto c_i$. Define a grading on $\Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\rangle$ by declaring

$$\begin{aligned} \deg x_i &:= 1 \\ \deg y_i &:= 1 + eo(b_i) \\ \deg z_i &:= 1 + eo(c_i) + e. \end{aligned}$$

We get a filtration F of $\Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\rangle$ with

$$F_i \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\rangle := \bigoplus_{j \le i} \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\rangle_j.$$

Using this we can define a filtration F on A by

$$F_i A := \phi(F_i \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\rangle).$$

Step 2. In this step we verify that the filtration F is a differential $\Bbbk$-filtration of finite type and also a connected filtration. To simplify our notation we are going to "recycle" the expressions $\overline{A}$, $o(a)$, $\overline{a}_i$, $\overline{b}_i$ and $\overline{c}_i$. From here on we define $\overline{A} := \operatorname{gr}^F A$, and $o(a)$ is the F-order of $a \in A$. When we need to refer to the G-order we shall do so by $o^G(a)$.

The graded $\Bbbk$-algebra homomorphism

$$\overline{\phi} := \operatorname{gr}^F(\phi) : \operatorname{gr}^F \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle = \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle \to \operatorname{gr}^F A = \overline{A}$$

is surjective. Define $\overline{a}_i := \overline{\phi}(x_i)$, $\overline{b}_i := \overline{\phi}(y_i)$ and $\overline{c}_i := \overline{\phi}(z_i)$. Observe that either $o(a_i) = o(x_i)$, in which case $\overline{a}_i$ is the F-symbol of a_i, and is a nonzero element of $\overline{A}_{o(a_i)}$; or $o(a_i) < o(x_i)$, and then $\overline{a}_i = 0$. Similar statements hold for $\overline{b}_i$ and $\overline{c}_i$.

Since $\overline{\phi}$ is surjective we see that $\overline{A}$ is a connected graded $\Bbbk$-algebra generated by the elements $\overline{a}_1, \ldots, \overline{a}_p, \overline{b}_1, \ldots, \overline{b}_m, \overline{c}_1, \ldots, \overline{c}_n$. We claim that the subalgebra generated by $\overline{a}_1, \ldots, \overline{a}_p, \overline{b}_1, \ldots, \overline{b}_m$, namely $\overline{B} := \operatorname{gr}^F B = \overline{\phi}(\Bbbk\langle \boldsymbol{x}, \boldsymbol{y} \rangle)$, is in the center of $\overline{A}$, and that $\overline{A} = \sum_{i=1}^{n} \overline{B} \cdot \overline{c}_i$.

Let us check that $[\overline{a}_i, \overline{b}_j] = 0$. If either $o(a_i) < o(x_i)$ or $o(b_j) < o(y_j)$ then $\overline{a}_i \overline{b}_j = \overline{b}_j \overline{a}_i = 0$. Otherwise $\overline{a}_i$ is the F-symbol of a_i, $\overline{b}_j$ is the F-symbol of b_j, and $\overline{a}_i \overline{b}_j \in \overline{A}_{o(a_i)+o(b_j)}$ is the residue of $a_i b_j$ modulo $F_{o(a_i)+o(b_j)-1} A$. Similarly for $\overline{b}_j \overline{a}_i$. Hence $[\overline{a}_i, \overline{b}_j] \in \overline{A}_{o(a_i)+o(b_j)}$ is the residue of $[a_i, b_j]$ modulo $F_{o(a_i)+o(b_j)-1} A$. But according to formula (1) we have

$$o([a_i, b_j]) \leq \max\{ o(f^{[a,b]}_{i,j,k}(\boldsymbol{a})) + o(b_k) \mid o^G(b_k) \leq o^G(b_j) - 1 \}.$$

Since for every k

$$o(b_k) \leq o(y_k) = 1 + e o^G(b_k)$$

and

$$o(f^{[a,b]}_{i,j,k}(\boldsymbol{a})) \leq e - 1,$$

and because $o(y_j) = o(b_j)$, we get

$$\begin{aligned} o([a_i, b_j]) &\leq (e-1) + (1 + e(o^G(b_j) - 1)) \\ &= o(y_j) - 1 = o(b_j) - 1 \leq o(a_i) + o(b_j) - 1, \end{aligned}$$

from which we see that $[a_i, b_j] \in F_{o(a_i)+o(b_j)-1} A$. The calculation for the other commutators is similar and we leave it to the reader.

Finally the calculation gives the amusing result that $\overline{c}_i \overline{c}_j = 0$ for all i, j. This implies that $\overline{A} = \sum \overline{B} \cdot \overline{c}_i$. $\square$

4 Localization and cohomology

In this section we consider Ore localization of rings. Recall that if A is a commutative ring and $S \subset A$ is a multiplicatively closed set, then we can form the ring of fractions $S^{-1} A$. For a noncommutative ring one has to assume more in order that the ring of fractions will exist: S must be a denominator set. This means it has to satisfy the Ore condition and the torsion condition (there are left and right variants).

Suppose A is a commutative noetherian ring and M, N are two finite A-modules. Then the Ext modules localize well. Namely if $A' := S^{-1}A$ is an Ore localization then

$$A' \otimes_A \mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^i_{A'}(A' \otimes_A M, A' \otimes_A N).$$

This fails terribly for noncommutative rings. The module $\mathrm{Ext}^i_A(M, N)$ is usually only a $\Bbbk$-module, and localization breaks down completely.

Example 4.1. Take A to be the first Weyl algebra, with generators x, y. Let $M := A/(A \cdot y) \cong \Bbbk[x]$ and $N := \mathrm{H}^1_{(x)} M$. Then $\mathrm{Hom}_A(M, M) = \Bbbk$ and $\mathrm{Hom}_A(N, N) = \Bbbk$. If we invert x, i.e. pass to $A' := A[x^{-1}]$, then

$$\mathrm{Hom}_{A'}(A' \otimes_A M, A' \otimes_A M) = \Bbbk$$

but

$$\mathrm{Hom}_{A'}(A' \otimes_A N, A' \otimes_A N) = 0.$$

Actually we are interested in cohomology with supports. Denote by M_p the class of A-modules M with $\dim M \leq p$. Here dim stands for some exact dimension function on modules, like GKdim. We should think of it as in commutative algebra, i.e. $\dim M$ represents the dimension of the support of the module M.

For a module M we let $\Gamma_{\mathsf{M}_p} M$ be the biggest submodule of dimension p. The functor Γ_{M_p} has derived functors $\mathrm{H}^i_{\mathsf{M}_p}$ called the cohomologies with supports in M_p. They are a limits of Exts:

$$\mathrm{H}^i_{\mathsf{M}_p} M \cong \lim_{\rightarrow} \mathrm{Ext}^i_A(A/L, M)$$

where L runs over the set of left ideals such that $\dim A/L \leq p$.

The cohomology $\mathrm{H}^i_{\mathsf{M}_p} M$ is an A-module, and we want to know sufficient conditions for having an isomorphism

$$A' \otimes_A \mathrm{H}^i_{\mathsf{M}_p} M \cong \mathrm{H}^i_{\mathsf{M}_p(A')}(A' \otimes_A M).$$

Here A' is some Ore localization of A and $\mathrm{H}^i_{\mathsf{M}_p(A')}$ is the cohomology with supports functor for A'-modules.

The reason such a localization formula is important is for perverse sheaves, where one of the conditions is the vanishing of $\mathrm{H}^i_{\mathsf{M}_p} M$ for certain i and p. Actually we need M to be a bounded complex of A-modules with finite cohomologies, but this poses no extra problem.

If we knew that injective modules localize our task would be easy. Indeed

$$\mathrm{H}^i_{\mathsf{M}_p} M = \mathrm{H}^i \Gamma_{\mathsf{M}_p} I^{\cdot}$$

where $I^{\cdot} = (I^0 \to I^1 \to \cdots)$ is an injective resolution of M. If we knew that $A' \otimes_A I^{\cdot}$ is a complex of injective A'-module, then this, combined with the flatness of $A \to A'$, would finish the story. Unfortunately we have no idea whether injectives localize for noncommutative rings.

5 A solution in some interesting cases

Let us go back to differential $\Bbbk$-algebras of finite type, and modify the definition slightly. It will bring us closer to the case of the Weyl algebras.

Let C be a ring. A C-ring is a ring A with a homomorphism $C \to A$.

Definition 5.1. Suppose C is a finitely generated commutative $\Bbbk$-algebra and A is a C-ring. A *differential C-filtration of finite type* on A is a filtration $G = \{G_i A\}$ of A with the following properties:

(i) $G_{-1}A = 0$ and $A = \bigcup G_i A$.

(ii) The image of C is in $G_0 A$.

(iii) The graded ring $\mathrm{gr}^G A$ is a C-algebra.

(iv) $\mathrm{gr}^G A$ is finite over its center $\mathrm{Z}(\mathrm{gr}^G A)$, and $\mathrm{Z}(\mathrm{gr}^G A)$ is a finitely generated C-algebra.

A *differential C-ring of finite type* is a C-ring A admitting some differential C-filtration of finite type.

Below are two examples of such rings.

Example 5.2. $\operatorname{char} \Bbbk = 0$, C is a smooth commutative finitely generated $\Bbbk$-algebra and $A := \mathcal{D}(C)$ is its ring of differential operators. The filtration G is by order of operator. When C is a polynomial ring, A is a Weyl algebra.

Example 5.3. A is a finite C-algebra. The filtration G is trivial.

Definition 5.1 includes Definition 3.1, if we take $C = \Bbbk$. The idea of the new definition is that now we can localize on $\operatorname{Spec} C$. Indeed any multiplicatively closed set $S \subset C$ becomes a denominator set in A. The localized ring $A' := S^{-1}A$ is a differential C'-ring of finite type, where $C' := S^{-1}C$. Geometrically what we obtain is a sheaf $\mathcal{A}$ of quasi-coherent $\mathcal{O}_X$-rings where we put $X := \operatorname{Spec} C$.

Theorem 5.4. ([YZ3]) *Let C be a finitely generated commutative $\Bbbk$-algebra and let A be a differential C-ring of finite type. Let $S \subset C$ be a multiplicatively closed set and $A' := S^{-1}A$. Then for any finite A-module M and any integers i, p there is an isomorphism*

$$A' \otimes_A \mathrm{H}^i_{\mathsf{M}_p(A)} M \cong \mathrm{H}^i_{\mathsf{M}_p(A')}(A' \otimes_A M).$$

Proof. (sketch) We begin by localizing a complex of bimodules. That is easier since there is more structure. The Auslander rigid dualizing complexes R_A and $R_{A'}$ of A and A' respectively satisfy

$$R_{A'} \cong A' \otimes_A R_A \cong R_A \otimes_A A'.$$

Using this fact, plus the known behavior of Gelfand-Kirillov dimension with respect to this kind of localization, we deduce that

$$A' \otimes_A \mathrm{R}\Gamma_{\mathsf{M}_p(A)} R_A \cong \mathrm{R}\Gamma_{\mathsf{M}_p(A')} R_{A'}. \tag{2}$$

The next step is to use the Local Duality Theorem from [YZ]. It gives

$$\mathrm{R}\Gamma_{\mathsf{M}_p(A)} M \cong \mathrm{RHom}_{^{\circ\mathrm{p}}}(\mathrm{RHom}_A(M, R), \mathrm{R}\Gamma_{\mathsf{M}_p(A)} R)).$$

Applying $A' \otimes_A -$, doing some manipulations, plugging in equation (2) and using Local Duality once more gives an isomorphism

$$A' \otimes_A \mathrm{R}\Gamma_{\mathsf{M}_p(A)} M \cong \mathrm{R}\Gamma_{\mathsf{M}_p(A')}(A' \otimes_A M).$$

It remains to take cohomology on both sides. □

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