DUALITY IN NONCOMMUTATIVE ALGEBRA AND GEOMETRY

AMNON YEKUTIELI

Lecture at Ben Gurion University
December 2000

Commutative Duality

Duality is one of the fundamental concepts in mathematics.

The most basic duality is that of linear algebra. We take a vector space V over a field $\mathbb{K}$ and assign to it
$$V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}).$$
If V is finite dimensional then $V \cong V^{**}$.

This can be generalized in many ways. For instance we can make V infinite, and then often we impose a topology to retain reflexivity (e.g. a Banach space).

Now change the ring of coefficients: instead of the field $\mathbb{K}$ take a commutative ring A, and instead of a vector space V we consider an A-module M. Immediately we run into difficulties.

Example: Abelian Groups. Let us look at $A = \mathbb{Z}$ the ring of integers. A $\mathbb{Z}$-module is just an abelian group. Here there are two distinct dualities for finitely generated groups. For a finite group H the dual is
$$H^* = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Q}/\mathbb{Z}),$$
whereas for a free group G the dual is
$$G^* = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z}).$$
(We want to stay within finitely generated groups, so the Ponryagin dual $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{R}/\mathbb{Z})$ is ruled out.) We know that $G^{**} \cong G$ and $H^{**} \cong H$. Is it possible to unite these two dualities into one? What about more complicated rings?

The Derived Category. This is where the *derived category* enters. The idea of Grothendieck and Verdier was to work with complexes of modules [RD]. A *complex* consists of
$$M = \left(\cdots \to M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \to \cdots \right)$$
where the M^i are A-modules, the d^i are A-linear maps and $d^i d^{i-1} = 0$. The derived category $\mathsf{D}(\mathsf{Mod}\, A)$ has complexes as objects. Morphisms in $\mathsf{D}(\mathsf{Mod}\, A)$ are more

Date: 3 Dec 00.

complicated to explain. Let us just say that the category of modules $\mathsf{Mod}\,A$ embeds as a full subcategory into $\mathsf{D}(\mathsf{Mod}\,A)$, as the complexes concentrated in degree 0.

The derived category has an internal Hom: given two complexes M and N we get a third complex $\mathrm{RHom}_A(M,N)$. This complex is related to the familiar Ext modules. Indeed, its ith cohomology (namely $\mathrm{Ker}\, d^i / \mathrm{Im}\, d^{i-1}$) is

$$\mathrm{H}^i\,\mathrm{RHom}_A(M,N) = \mathrm{Ext}^i_A(M,N).$$

Now suppose that for a certain complex R the functor

$$M \mapsto M^* := \mathrm{RHom}_A(M,R)$$

is a duality, in the sense that $M \cong M^{**}$ (with suitable finiteness restrictions on M I won't mention). Then we call R a *dualizing complex*.

Getting back to our ring $\mathbb{Z}$, one can show (without much difficulty) that the module $\mathbb{Z}$ is a dualizing complex. The two dualities, for finite groups and free groups, are included in the duality $\mathrm{RHom}_{\mathbb{Z}}(M,\mathbb{Z})$. Indeed, for G free $G^* = \mathrm{Ext}^0_{\mathbb{Z}}(G,\mathbb{Z})$; and for H finite $H^* = \mathrm{Ext}^1_{\mathbb{Z}}(H,\mathbb{Z})$.

Generally speaking, most commutative rings encountered in algebra will admit a dualizing complex.

Algebraic Geometry. Grothendieck's duality theory has geometric aspects to it, when we pass from commutative rings to algebraic varieties. It has local features (such as local duality and residues) and global features (such a finiteness). It is also a relative theory, in that it deals with maps between varieties.

Applications. Applications of Grothendieck duality in algebraic geometry are numerous: moduli problems, resolution of singularities, enumerative geometry....

Example: The Line. Consider the ring $A = \mathbb{K}[x]$ of polynomials over the field $\mathbb{K}$. It is the ring of functions on the affine line $\mathbf{A}^1_{\mathbb{K}}$. The module A is a dualizing complex. However the "correct" dualizing complex over A is the module of differential 1-forms

$$\Omega^1_A = \{f(x)\mathrm{d}x\}$$

(shifted by 1, but let's suppress this). One reason why this is "correct" is that we can integrate 1-forms on the plane.

The integration we need is actually "contour integration." Suppose $\mathbb{K} = \mathbb{C}$, $f(x)\mathrm{d}x$ is a meromorphic differential form on the projective line $\mathbf{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and P is a point. We get the residue

$$\mathrm{Res}_P\, f(x)\mathrm{d}x := \frac{1}{2\pi i} \oint_P f(x)\mathrm{d}x.$$

An indication that the residue has geometric significance is the Residue Theorem:

$$\sum_{P \in \mathbf{P}^1_{\mathbb{C}}} \mathrm{Res}_P\, f(x)\mathrm{d}x = 0.$$

The residue $\mathrm{Res}_P\, f(x)\mathrm{d}x$ can be defined on any algebraic curve (over any field $\mathbb{K}$), and there is a Residue Theorem.

Noncommutative Duality

Consider a noncommutative ring A. What is to prevent us from seeking a duality of the form $\mathrm{RHom}_A(-, R)$? Nothing really, as long as we take R to be a complex of A-A-*bimodules*. We remind that a bimodule has two commuting actions by A, from the left and from the right. Then we actually have two functors:

$$\mathrm{RHom}_A(-, R) : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}})$$

and

$$\mathrm{RHom}_{A^{\mathrm{op}}}(-, R) : \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Mod}\, A).$$

Here $\mathsf{Mod}\, A$ is the category of left A-modules, A^{op} is the opposite ring, and $\mathsf{Mod}\, A^{\mathrm{op}}$ is the category of right A-modules. If the two functors are equivalences then R is called a dualizing complex. Again I am suppressing finiteness conditions.

Motivation. Why should we want to study dualizing complexes over noncommutative rings?

It turns out that many interesting noncommutative rings admit dualizing complexes. Moreover unlike other methods of commutative algebra and geometry, the theory of duality survives the "deformation" to the noncommutative setting.

The Rigid Dualizing Complex. One of the first observations when passing to noncommutative rings is that the notion of "correct" dualizing complex becomes sharper. By considerations of *local cohomology* (for graded rings) or the *rigidity equation* of Van den Bergh

$$R \cong \mathrm{RHom}_{A \otimes_{\mathbb{K}} A^{\mathrm{op}}}(A, R \otimes_{\mathbb{K}} R),$$

a particular dualizing complex is chosen. This is the *balanced*, respectively *rigid*, dualizing complex. See [Ye1] and [VdB1].

Example. Take the "quantum plane"

$$A := \mathbb{K}\langle x, y\rangle / (yx - qxy)$$

where $q \in \mathbb{K}, q \neq 0$. The elements of A are polynomials $f(x, y)$ in the noncommuting variables x and y that satisfy $yx = qxy$. The rigid dualizing complex is

$$\Omega^2_A = \{f(x, y) \cdot \mathrm{d}x \wedge \mathrm{d}y\}$$

(shifted), where

$$\mathrm{d}x \wedge \mathrm{d}y \cdot x = qx \cdot \mathrm{d}x \wedge \mathrm{d}y.$$

Indeed Ω^2_A is the only bimodule satisfying

$$\mathrm{Ext}^2_{A \otimes_{\mathbb{K}} A^{\mathrm{op}}}(A, \Omega^2_A \otimes_{\mathbb{K}} \Omega^2_A) \cong \Omega^2_A.$$

The experts will notice that in the commutative case ($q = 1$) the rigidity equation is just the fundamental class of the diagonal $X \subset X \times X$ where $X = \mathbf{A}^2_{\mathbb{K}} = \mathrm{Spec}\, A$.

The rigid dualizing complex enjoys functorial properties, similar to the commutative situation. In the example, the rigid dualizing complex Ω^2_A restricts to "curves" and "points" in the quantum plane. As we shall see this has nice applications.

As an aside, I should say that the rigidity formula already carries a deep meaning in commutative algebraic geometry. This is related to research of Lipman on the fundamental class of a flat map.

Applications. Let us look at some applications in algebra.

1. Let $\mathfrak{g}$ be an n-dimensional Lie algebra over $\mathbb{K}$ and $\mathrm{U}(\mathfrak{g})$ its universal enveloping algebra. I proved [Ye3] that the rigid dualizing complex of $\mathrm{U}(\mathfrak{g})$ is the following twisted bimodule:

$$R = \mathrm{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \bigwedge^n \mathfrak{g}.$$

Note that this deforms the commutative picture, since for $\mathfrak{g}$ abelian $A = \mathrm{U}(\mathfrak{g})$ is a commutative polynomial ring and $\mathrm{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \bigwedge^n \mathfrak{g} = \Omega^n_A$.

The formula $R = \mathrm{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \bigwedge^n \mathfrak{g}$ implies results of Duflo and Brown-Levasseur [BL] on duals of Verma modules, as well as generalizations of their results. The implication is obtained using the functorial properties of the rigid dualizing complex, as studied in [YZ2].

2. Let C be a smooth commutative $\mathbb{K}$-algebra (in characteristic 0) and $\mathcal{D}(C)$ the ring of differential operators. The duality for $\mathcal{D}(C)$-modules, as developed by Bernstein and Kashiwara, is perhaps the most famous noncommutative duality. One can show [Ye3] that the ring $\mathcal{D}(C)$ itself is the rigid dualizing complex (there is no twist in this case).

3. Wu and Zhang [WZ] have proved that a Hopf algebra finite over its center is a Gorenstein ring. This partly solves a conjecture of K.A. Brown. The proof relies on dualizing complexes and their properties.

4. The rigid dualizing complex makes it possible to define Cohen-Macaulay modules over a noncommutative ring A. This was used recently by Etingof-Ginzburg [EG].

Noncommutative Algebraic Geometry

There are a few current theories going under the name "noncommutative geometry." I will concentrate on the noncommutative geometry that is closely related to commutative algebraic geometry. It was developed by Artin, Van den Bergh and Rosenberg with contributions by many others [ATV1], [AZ], [Ro], [VdB2], [SV].

Noncommutative Projective Varieties. A key construction is the projective spectrum $\operatorname{Proj} A$ of a noncommutative graded algebra A. Recall Serre's fundamental theorem relating sheaves on a projective variety X to graded modules over the homogeneous coordinate algebra A of X.

Artin and Zhang showed that a meaningful geometric object $\operatorname{Proj} A$ can be constructed by reversing Serre's theorem, i.e. describing $\operatorname{Proj} A$ entirely in terms of graded A-modules. This only works if the algebra A satisfies a mysterious condition called χ (satisfied if A is commutative, but also in many other interesting cases).

In [YZ1] Zhang and I we proved that if the graded algebra A has a balanced dualizing complex then condition χ holds, and moreover $\operatorname{Proj} A$ has Serre duality. The converse was proved by Van den Bergh [VdB1].

Noncommutative Schemes. Van den Bergh's more general theory of noncommutative schemes (or quasi-schemes) is based on ideas of Gabriel and Rosenberg. It is rich enough to include features such as blow ups.

Duality for general noncommutative schemes is not understood yet.

Regular Graded Algebras. A remarkable class of noncommutative algebras is the 3-dimensional Artin-Schelter regular algebras [ASc]. These were classified Artin, Tate and Van den Bergh [ATV1], [ATV2] by studying moduli of families of graded modules. The classification is in terms of elliptic curves and additional data.

Bondal and Polishchuk [BP] studied the same algebras using a totally different method – through their derived categories, with ideas of deformation related to Kontsevich's homological mirror symmetry.

If A is a regular graded algebra then $\operatorname{Proj} A$ is a noncommutative surface.

Application: Theoretical Physics. I want to conclude with a recent paper in theoretical physics by Kapustin, Kuznetsov and Orlov [KKO] that uses noncommutative projective varieties and dualizing complexes on them. Recall that an instanton is a solution of a differential equation on the space of connections on a vector bundle on a Riemannian 4-manifold. Nekrasov and Schwarz proposed a construction of instantons on noncommutative $\mathbb{R}^4$ (this is a deformation in the Riemannian geometry realm).

Kapustin, Kuznetsov and Orlov show how to interpret these noncommutative instantons in terms of vector bundles on a noncommutative projective surface X. This surface X is precisely $\operatorname{Proj} A$ where A is a 3-dimensional Artin-Schelter regular algebra. Moreover, the definition of a vector bundle relies on the existence and properties of the dualizing complex of X, as worked out in [YZ1].

References

[ASc] M. Artin and W. Schelter, Graded algebras of global dimension 3, Adv. in Math. **66** (1987), 17–216.

[ATV1] M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, in: "The Grothendieck Festschrift," vol. 1, Birkhäuser, Boston, 1990.

[ATV2] M. Artin, J. Tate and M. Van den Bergh, Modules over regular algebras of dimension 3, Invent. Math. **106** (1991), 335-388.

[AZ] M. Artin and J.J. Zhang, Noncommutative projective schemes, Adv. Math. **109** (1994), 228-287.

[Jo] P. Jorgensen, Gorenstein homomorphisms of noncommutative rings, J. Algebra **211** (1999), 240-267.

[BL] K. A. Brown and T. Levasseur, Cohomology of bimodules over enveloping algebras, Math. Z. **189** (1985), 493-413.

[BP] A.I. Bondal and A. E. Polishchuk, Homological properties of associative algebras: the method of helices. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **57** (1993), 3-50; translation in Russian Acad. Sci. Izv. Math. **42** (1994), 219-260.

[EG] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero-Moser system, and deformed Harish-Chandra homomorphism, eprint: math.AG/0011114.

[KKO] A. Kapustin, A. Kuznetsov and D. Orlov, Noncommutative Instantons and Twistor Transform, eprint: hep-th/0002193.

[KR] M. Kontsevich and A. Rosenberg, Noncommutative smooth spaces, eprint: math/9812158.

[Li] J. Lipman, Dualizing sheaves, differentials and residues on algebraic varieties, Asterisque **117** (1984).

[LS] J. Lipman and P. Sastry, Regular differential and equidimensional scheme-maps. J. Algebraic Geom. **1** (1992), 101-130.

[RD] R. Hartshorne, "Residues and Duality", Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[Ro] A.L. Rosenberg. "Noncommutative Algebraic Geometry and Representations of Quantized Algebras," Kluwer, Dordrecht, 1995.

[SV] J.T. Stafford and M. Van den Bergh, Noncommutative curves and noncommutative surfaces, eprint: math.RA/9910082.

[VdB1] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, J. Algebra **195** (1997), no. 2, 662-679.

[VdB2] M. Van den Bergh, "Blowing up of non-commutative smooth surfaces," to appear in Memoirs of the AMS.

[WZ] Q. Wu and J.J. Zhang, Homological identities for noncommutative rings, preprint.

[Ye1] A. Yekutieli, Dualizing complexes over noncommutative graded algebras, J. Algebra **153** (1992), 41-84.

[Ye2] A. Yekutieli, "An Explicit Construction of the Grothendieck Residue Complex" (with an appendix by P. Sastry), Astérisque **208** (1992).

[Ye3] A. Yekutieli, Residues and differential operators on schemes, Duke Math. J. **95** (1998), 305-341.

[Ye4] A. Yekutieli, The rigid dualizing complex of an enveloping algebra, J. Pure Appl. Algebra **150** (2000), 85-93. eprint: math.RA/9810016.

[YZ1] A. Yekutieli and J.J. Zhang, Serre duality for noncommutative projective schemes, Proc. Amer. Math. Soc. **125** (1997), 697-707.

[YZ2] A. Yekutieli and J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra **213** (1999), 1-51.

Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, ISRAEL

E-mail address: amyekut@math.bgu.ac.il
WWW: http://www.math.bgu.ac.il/~amyekut