DERIVED PICARD GROUPS OF FINITE DIMENSIONAL ALGEBRAS

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0. Introduction

Let $\mathbb{K}$ be a field and A a $\mathbb{K}$-algebra. The derived Picard group $\mathrm{DPic}_{\mathbb{K}}(A)$ is the group of isomorphism classes of two-sided tilting complexes over A.

The group $\mathrm{DPic}_{\mathbb{K}}(A)$ becomes interesting when A is nonlocal and noncommutative.

Today I will talk about the case when A is a finite dimensional $\mathbb{K}$-algebra. In this case the group $\mathrm{DPic}_{\mathbb{K}}(A)$ has a geometric structure. For a hereditary algebra A we can completely describe the structure of $\mathrm{DPic}_{\mathbb{K}}(A)$.

1. Some background

Suppose A is a (unital, associative) $\mathbb{K}$-algebra. Let $\mathsf{Mod}\, A$ be the category of left A-modules. Recall that Morita Theory tells us what are all $\mathbb{K}$-linear auto-equivalences $F : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$. They are (up to isomorphism) $FM = P \otimes_A M$, where P is a $\mathbb{K}$-central invertible bimodule (i.e. $A \cong P \otimes_A Q \cong Q \otimes_A P$ for some bimodule Q). The group of isomorphism classes of invertible bimodules P (with operation $P \otimes_A Q$) is the (noncommutative) Picard group of A, denoted $\mathrm{Pic}_{\mathbb{K}}(A)$.

Remark 1.1. If A is commutative then $\mathrm{Pic}_{\mathbb{K}}(A)$ contains the automorphism group $\mathrm{Aut}_{\mathbb{K}}(A)$ and the usual (commutative) Picard group $\mathrm{Pic}_A(A)$ as subgroups.

Let briefly recall a few things about the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. The objects of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ are the bounded complexes of A-modules

$$M = \Big(\cdots \to 0 \to M^p \xrightarrow{\delta} M^{p+1} \xrightarrow{\delta} \cdots \to M^q \to 0 \to \cdots \Big)$$

where $\delta \circ \delta = 0$. The category $\mathsf{Mod}\, A$ embeds in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ as the complexes concentrated in degree 0. And $M[n]$ is the complex shifted by n.

Denote by A^{op} the opposite algebra and by $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ the enveloping algebra, where $\otimes = \otimes_{\mathbb{K}}$. So an A^{e}-module is a $\mathbb{K}$-central A-bimodule. For two complexes $T, S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ one can define the derived tensor product $T \otimes^{\mathrm{L}}_A S$. We call $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ a *two-sided tilting complex* if $T \otimes^{\mathrm{L}}_A S \cong S \otimes^{\mathrm{L}}_A T \cong A$ for some complex of bimodules $S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$. Two-sided tilting complexes were first considered by Rickard and Keller.

Example 1.2. If P is an invertible bimodule then $P[n]$ is a tilting complex.

Definition 1.3. The derived Picard group is

$$\mathrm{DPic}_{\mathbb{K}}(A) := \frac{\{\text{two-sided tilting complexes } T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})\}}{\text{isomorphism}}$$

with operation $T \otimes^{\mathrm{L}}_{A} S$.

One reason to study $\mathrm{DPic}_{\mathbb{K}}(A)$ is:

Theorem 1.4 (Y.). *$\mathrm{DPic}_{\mathbb{K}}(A)$ parameterizes the isomorphism classes of dualizing complexes over A.*

We note that $\mathrm{DPic}_{\mathbb{K}}(A)$ contains two natural subgroups: a copy of $\mathbb{Z}$ (represented by the complexes $A[n]$), and $\mathrm{Pic}_{\mathbb{K}}(A)$.

Theorem 1.5 (Y., Rouquier-Zimmermann). *If A is either local, or commutative with* $\operatorname{Spec} A$ *connected, then* $\mathrm{DPic}_{\mathbb{K}}(A) = \mathrm{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}$.

This however is not the general situation:

Example 1.6 (Y). Consider the smallest nonlocal, noncommutative $\mathbb{K}$-algebra $A = \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix}$. The bimodule $A^* := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ turns out to be a tilting complex. Let τ be the class of $A^*[-1]$ in $\mathrm{DPic}(A)$. Then one can show that $\tau \notin \mathrm{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}$.

This raises:

Problem 1.7. Study the structure of the group $\mathrm{DPic}_{\mathbb{K}}(A)$ and its relation to the geometry of the noncommutative ring A.

As customary we write $\mathsf{mod}\, A$ for the category of finitely generated A-modules (for A noetherian). Another problem, essentially raised by Rickard:

Problem 1.8. Does $\mathrm{DPic}_{\mathbb{K}}(A)$ coincide with the group of $\mathbb{K}$-linear triangle auto-equivalences of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$?

There is an obvious inclusion in one direction. When A is hereditary (i.e. has global dimension 1) it is known that the two groups are equal.

2. Quivers and Path Algebras

We will concentrate on a finite dimensional $\mathbb{K}$-algebra A. Let us assume that $\mathbb{K}$ is algebraically closed, and that A is basic (i.e. $A/\mathfrak{m} = \mathbb{K}$ for every maximal ideal). Since every $\mathbb{K}$-algebra is Morita equivalent to a basic one the latter is really no restriction.

The geometry of A is described by a quiver $\vec{\Delta}$. Here is a nonorthodox definition of $\vec{\Delta}$. The set of vertices of $\vec{\Delta}$ is $\operatorname{Spec} A$, namely the set of maximal ideals. Let $\mathfrak{r}$ be the Jacobson radical of A, and for each $\mathfrak{m}_i \in \operatorname{Spec} A$ let $e_i \in A/\mathfrak{r}$ be the corresponding central idempotent. So

$$\mathfrak{r}/\mathfrak{r}^2 = \bigoplus_{i,j} e_i(\mathfrak{r}/\mathfrak{r}^2)e_j \cong \bigoplus_{i,j} \frac{\mathfrak{m}_i \cap \mathfrak{m}_j}{\mathfrak{m}_i \mathfrak{m}_j}$$

as A-A-bimodules. Define $d_{i,j} := \dim_{\mathbb{K}} e_i(\mathfrak{r}/\mathfrak{r}^2)e_j$. Then there are $d_{i,j}$ arrows $\mathfrak{m}_i \to \mathfrak{m}_j$. The connected components of $\vec{\Delta}$ are called cliques, and they control Öre localization.

Example 2.1. Take $A = \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix}$. The maximal ideals are $\mathfrak{m}_1 = \begin{bmatrix} 0 & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix}$ and $\mathfrak{m}_2 = \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & 0 \end{bmatrix}$. Clearly $\frac{\mathfrak{m}_1 \cap \mathfrak{m}_2}{\mathfrak{m}_1 \mathfrak{m}_2} = \begin{bmatrix} 0 & \mathbb{K} \\ 0 & 0 \end{bmatrix}$. So the quiver $\vec{\Delta}$ is $\underset{\mathfrak{m}_1}{\bullet} \longrightarrow \underset{\mathfrak{m}_2}{\bullet}$, a Dynkin quiver of type A_2.

On the other hand, given a finite quiver $\vec{\Delta}$ one can construct an algebra $\mathbb{K}\vec{\Delta}$, called the path algebra.

If A is a basic finite dimensional $\mathbb{K}$-algebra then $A \cong \mathbb{K}\vec{\Delta}/I$ for some ideal I; and $I = 0$ iff A is hereditary.

Remark 2.2. The surjection $\mathbb{K}\vec{\Delta} \twoheadrightarrow A$ should be viewed as a noncommutative generalization of the Cohen structure theorem for commutative complete local rings: $\mathbb{K}[[t_1, \ldots, t_n]] \twoheadrightarrow A$. Indeed the arrows in $\mathbb{K}\vec{\Delta}$ are lifts of a basis of $\mathfrak{r}/\mathfrak{r}^2$.

3. The Auslander-Reiten Quiver

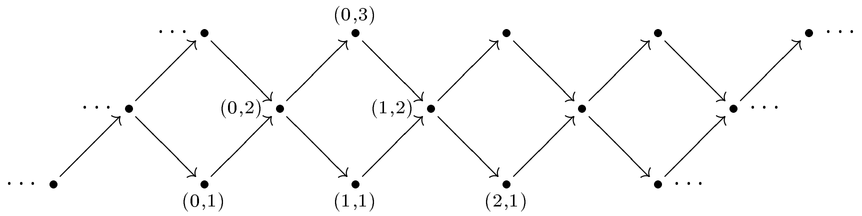
In this section $A = \mathbb{K}\vec{\Delta}$, a finite dimensional path algebra. An A-module is also called a representation of $\vec{\Delta}$.

One of the important invariants of the algebra A is the set of isomorphism classes of indecomposable modules. If this set is finite A is said to have finite representation type.

By a famous theorem of Gabriel, A has finite representation type iff $\vec{\Delta}$ is a Dynkin quiver of type A_n, D_n, E_6, E_7 or E_8.

Happel showed how to define the *Auslander-Reiten quiver* of the derived category $\vec{\Gamma} = \vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$. This is an infinite quiver. Usually it is very complicated, but it contains a nice subquiver denoted $\vec{\mathbb{Z}}\vec{\Delta}$ that is easy to describe (the quiver $\vec{\mathbb{Z}}\vec{\Delta}$ was introduced by Riedtmann). When $\vec{\Delta}$ is a Dynkin quiver, then in fact $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)) = \vec{\mathbb{Z}}\vec{\Delta}$.

Example 3.1. Take $\vec{\Delta} = \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$. Then $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)) = \vec{\mathbb{Z}}\vec{\Delta}$ is



The vertices in $\vec{\Gamma}(\mathsf{mod}\, A)$ are labeled.

The group $\mathrm{DPic}_{\mathbb{K}}(A)$ acts naturally on the quiver $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$ (by quiver automorphisms). The element $\tau \in \mathrm{DPic}_{\mathbb{K}}(A)$, represented by the tilting complex $A^*[-1]$, is called the translation. In the example above it is a shift by 1 to the left. Using this action we proved the following theorems. Let $\mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})$ denote the group of quiver automorphisms and $\mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})^{\langle\tau\rangle}$ the subgroup of elements commuting with τ.

Theorem 3.2 (Miyachi and Y.)**.** *Suppose A has finite representation type. Then*

$$\mathrm{DPic}_{\mathbb{K}}(A) \cong \mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})^{\langle\tau\rangle}.$$

Example 3.3. Let $\sigma \in \mathrm{DPic}_{\mathbb{K}}(A)$ be represented by $A[1]$. If $\vec{\Delta}$ is of Dynkin type A_n then $\mathrm{DPic}_{\mathbb{K}}(A)$ is abelian, generated by τ and σ, with one relation $\tau^{n+1} = \sigma^{-2}$.

We denote by $\mathrm{Aut}((\vec{\mathbb{Z}}\vec{\Delta})_0; d)$ the group of permutations of the vertex set $(\vec{\mathbb{Z}}\vec{\Delta})_0$ that preserve the arrow-multiplicity d. Let $\mathrm{Out}_{\mathbb{K}}(A)$ be the group of outer automorphisms of A. It embeds naturally into $\mathrm{Pic}_{\mathbb{K}}(A)$. $\mathrm{Out}_{\mathbb{K}}(A)$ is a linear algebraic group, and we denote by $\mathrm{Out}^0_{\mathbb{K}}(A)$ its identity component.

Theorem 3.4 (Miyachi and Y.). *If A has infinite representation type then there is an isomorphism of groups*

$$\mathrm{DPic}_{\mathbb{K}}(A) \cong \big(\mathrm{Aut}((\mathbb{Z}\vec{\Delta})_0; d)^{\langle \tau \rangle} \ltimes \mathrm{Out}^0_{\mathbb{K}}(A)\big) \times \mathbb{Z}.$$

$\mathrm{Out}^0_{\mathbb{K}}(A)$ *is trivial when $\vec{\Delta}$ is a tree.*

Problem 3.5. Can this analysis be carried out for other finite dimensional algebras A? We note that the quiver $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$ exists whenever $\mathrm{gl.dim}\, A < \infty$.

4. The Geometry of $\mathrm{DPic}_{\mathbb{K}}(A)$

In Theorem 3.4 one of the factors of $\mathrm{DPic}_{\mathbb{K}}(A)$ was an algebraic group. It is natural to ask if the whole group can be endowed with a geometric structure.

A *locally algebraic group* over $\mathbb{K}$ is a group G, with a normal subgroup G^0, such that G^0 is a connected algebraic group over $\mathbb{K}$, each coset of G^0 is a variety, and multiplication and inversion are morphisms of varieties. A morphism $\phi : G \to H$ of locally algebraic groups is a group homomorphism such that $\phi(G^0) \subset H^0$ and the restriction $\phi^0 : G^0 \to H^0$ is a morphism of varieties. We call ϕ an open immersion if ϕ is injective and ϕ^0 is an isomorphism.

In some examples studied in [MY] the group $\mathrm{DPic}_{\mathbb{K}}(A)$ was a locally algebraic group.

Example 4.1. Let $\vec{\Omega}_n$ be the quiver

$$\vec{\Omega}_n \qquad \begin{tikzcd} 1 \bullet \arrow[rr, bend left=40, "\alpha_1"] \arrow[rr, bend left=20, "\alpha_2"] \arrow[rr, phantom, "\vdots"] \arrow[rr, bend right=30, "\alpha_n"'] & & \bullet 2 \end{tikzcd} .$$

Let A be the path algebra $\mathbb{K}\vec{\Omega}_n$. Then

$$\mathrm{DPic}_{\mathbb{K}}(A) \cong \mathbb{Z} \times \big(\mathbb{Z} \ltimes \mathrm{PGL}_n(\mathbb{K})\big).$$

In the semi-direct product a generator T of $\mathbb{Z}$ acts by conjugation on a matrix $\sigma \in \mathrm{PGL}_n(\mathbb{K})$ by $\sigma \mapsto (\sigma^{-1})^{\mathrm{t}}$.

And in fact we now have a general result:

Theorem 4.2 (Y.). *Let A be a finite dimensional $\mathbb{K}$-algebra. Then* $\mathrm{DPic}_{\mathbb{K}}(A)$ *is a locally algebraic group, with identity component* $\mathrm{Out}^0_{\mathbb{K}}(A)$.

To prove this theorem one needs to know that $\mathrm{Out}^0_{\mathbb{K}}(A)$ is a normal subgroup of $\mathrm{DPic}_{\mathbb{K}}(A)$, and conjugation by any tilting complex induces an automorphism of algebraic groups on $\mathrm{Out}^0_{\mathbb{K}}(A)$. This turns out to follow quite easily from a deep theorem proved recently, by Huisgen-Zimmermann and Saorin, and independently by Rouquier.

Theorem 4.3. *Suppose A and B are finite dimensional $\mathbb{K}$-algebras and $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes A^{\mathrm{op}})$ is a two-sided tilting complex. Then T induces an isomorphism of algebraic groups $\phi : \mathrm{Out}^0_{\mathbb{K}}(A) \xrightarrow{\simeq} \mathrm{Out}^0_{\mathbb{K}}(B)$ such that $T \otimes_A P \cong \phi(P) \otimes_B T$.*

I wish to give an outline of Rouquier's proof. Let X be a scheme over $\mathbb{K}$. An invertible $\mathcal{O}_X \otimes A$-bimodule is a coherent sheaf $\mathcal{M}$ of $\mathcal{O}_X \otimes A^{\mathrm{e}}$-modules such that $\mathcal{M} \otimes_{\mathcal{O}_X \otimes A} \mathcal{N} \cong \mathcal{N} \otimes_{\mathcal{O}_X \otimes A} \mathcal{M} \cong \mathcal{O}_X \otimes A$ for some other bimodule $\mathcal{N}$. Define

$$\mathrm{Out}_{\mathcal{O}_X}(\mathcal{O}_X \otimes A) := \frac{\left\{ \begin{array}{l} \text{invertible } \mathcal{O}_X \otimes A\text{-bimodules } \mathcal{M}, \text{ locally} \\ \text{free of rank 1 on the right} \end{array} \right\}}{\text{isomorphism}}.$$

This is a functor $X \mapsto \mathrm{Out}_{\mathcal{O}_X}(\mathcal{O}_X \otimes A)$, $\mathsf{Sch}/\mathbb{K} \to \mathsf{Groups}$. Rouquier shows the functor is represented by the algebraic group $\mathrm{Out}_{\mathbb{K}}(A)$. This relies on arithmetic properties of a certain subgroup of $\mathrm{Aut}_{\mathbb{K}}(A)$, which imply that the morphism $\mathrm{Aut}_{\mathbb{K}}(A) \twoheadrightarrow \mathrm{Out}_{\mathbb{K}}(A)$ is locally split for the Zariski topology. Finally a two-sided tilting complex T gives rise to an object of $\mathrm{Out}_{\mathcal{O}_G}(\mathcal{O}_G \otimes B)$ where G is the scheme $\mathrm{Out}^0_{\mathbb{K}}(A)$.

Remark 4.4. Locally algebraic groups arise also in (commutative) algebraic geometry. There are examples due to Orlov of abelian varieties X such that the group of triangle auto-equivalences of the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$ is a nontrivial locally algebraic group.

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