Derived Picard Groups of Finite Dimensional Algebras

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Amnon Yekutieli
Ben Gurion University, ISRAEL
http://www.math.bgu.ac.il/∼amyekut

Some Background

Suppose A is a (unital, associative) $\mathbb{K}$-algebra. Let $\mathsf{Mod}\, A$ be the category of left A-modules.

Recall that Morita Theory tells us what are all $\mathbb{K}$-linear auto-equivalences

$$F : \mathsf{Mod}\, A \to \mathsf{Mod}\, A.$$

They are (up to isomorphism)

$$F M = P \otimes_A M,$$

where P is a $\mathbb{K}$-central invertible bimodule. By definition P is invertible if

$$A \cong P \otimes_A Q \cong Q \otimes_A P$$

for some bimodule Q.

The group of isomorphism classes of invertible bimodules P (with operation $P \otimes_A Q$) is the (noncommutative) Picard group of A, denoted $\mathrm{Pic}_{\mathbb{K}}(A)$.

If A is commutative then $\mathrm{Pic}_{\mathbb{K}}(A)$ contains the automorphism group $\mathrm{Aut}_{\mathbb{K}}(A)$ and the usual (commutative) Picard group $\mathrm{Pic}_A(A)$ as subgroups. In fact

$$\mathrm{Pic}_{\mathbb{K}}(A) \cong \mathrm{Aut}_{\mathbb{K}}(A) \ltimes \mathrm{Pic}_A(A).$$

Let us briefly recall a few things about the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$.

The objects of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ are the bounded complexes of A-modules

$$M = \big(\cdots \to 0 \to M^p \xrightarrow{\partial} M^{p+1} \to \cdots \to M^q \to 0 \to \cdots \big)$$

where $\partial \circ \partial = 0$.

Two complexes M and N are isomorphic if there is a homomorphism of complexes $M \to N$ that induces isomorphism in cohomology.

The category $\mathsf{Mod}\, A$ embeds in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ as the complexes concentrated in degree 0. And $M[n]$ is the complex shifted by n.

Denote by A^{op} the opposite algebra and by $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ the enveloping algebra, where $\otimes = \otimes_{\mathbb{K}}$. So an A^{e}-module is a $\mathbb{K}$-central A-bimodule.

For two complexes $T, S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ one can define the derived tensor product $T \otimes^{\mathrm{L}}_A S$.

We call $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ a two-sided tilting complex if

$$T \otimes^{\mathrm{L}}_A S \cong S \otimes^{\mathrm{L}}_A T \cong A$$

for some other complex of bimodules $S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$.

Two-sided tilting complexes were first considered by Rickard and Keller.

Example. If P is an invertible bimodule then $P[n]$ is a tilting complex.

Definition. The derived Picard group is

$$\mathrm{DPic}_{\mathbb{K}}(A) := \frac{\left\{ \begin{array}{l} \text{two-sided tilting comp-} \\ \text{lexes } T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}}) \end{array} \right\}}{\text{isomorphism}}$$

with operation $T \otimes^{\mathrm{L}}_A S$.

One reason to study $\mathrm{DPic}_{\mathbb{K}}(A)$ is:

Theorem. (Y.) $\mathrm{DPic}_{\mathbb{K}}(A)$ parameterizes the isomorphism classes of dualizing complexes over A.

We note that $\mathrm{DPic}_{\mathbb{K}}(A)$ contains two natural subgroups: a copy of $\mathbb{Z}$ (represented by the complexes $A[n]$), and the classical Picard group $\mathrm{Pic}_{\mathbb{K}}(A)$.

Theorem 1. (Y., Rouquier-Zimmermann) If A is either local, or commutative with $\mathrm{Spec}\, A$ connected, then

$$\mathrm{DPic}_{\mathbb{K}}(A) = \mathrm{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}.$$

This however is not the general situation, as the next example shows.

Example. Consider the smallest nonlocal, noncommutative $\mathbb{K}$-algebra

$$A = \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix}.$$

The bimodule

$$A^* := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$$

turns out to be a two-sided tilting complex.

Let τ be the class of $A^*[-1]$ in $\mathrm{DPic}(A)$. Then one can show that $\tau \notin \mathrm{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}$.

This raises:

Problem. Study the structure of the group $\mathrm{DPic}_{\mathbb{K}}(A)$ and its relation to the geometry of the noncommutative ring A.

I wish to say some more about the structure of $\mathrm{DPic}_{\mathbb{K}}(A)$ in the commutative case.

Let F be element of $\mathrm{DPic}(A)$. Then F induces an auto-equivalence of $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$, the derived category of coherent sheaves on the scheme $X := \mathrm{Spec}\, A$.

Theorem 1 says that (up to an integer shift) F is equivalent to a line bundle $\mathcal{L}$ supported on the graph of an automorphism $\sigma : X \to X$.

The formula for the action of F is

$$F\mathcal{M} \cong \mathrm{R}p_{2*}(\mathcal{L} \otimes p_1^* \mathcal{M})$$

for an $\mathcal{O}_X$-module $\mathcal{M}$. See Figure 1.

This should be compared to an integral transform F with kernel $\mathcal{L}$ acting on a function f:

$$F(f)(x_2) = \int f(x_1)\mathcal{L}(x_1, x_2)\mathrm{d}x_1.$$

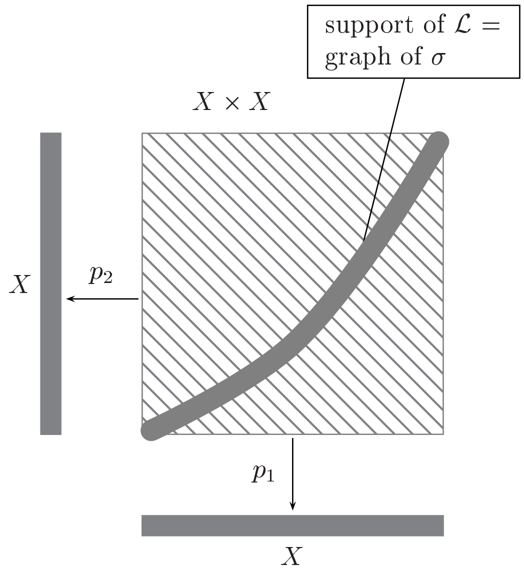


FIGURE 1

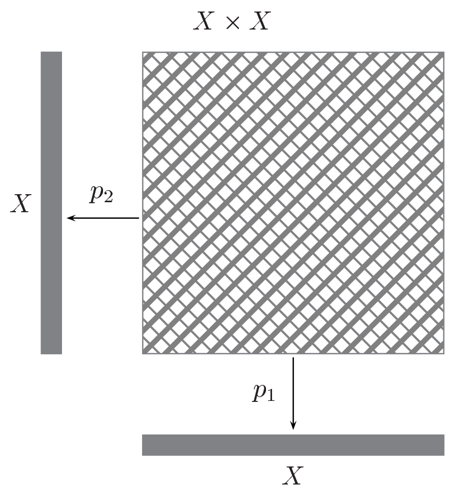


FIGURE 2

By results of Bondal-Orlov, if X is a smooth projective variety with ample canonical or anti-canonical bundle (e.g. projective space $\mathbf{P}^n$) then every auto-equivalence F of $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$ is also of the form $(\sigma, \mathcal{L})$.

However when X is a Calabi-Yau manifold things become much more complicated.

For instance, if X is an elliptic curve then the Fourier-Mukai transform F is an auto- equivalence of $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$ with kernel $\mathcal{L}$ that is supported on all of $X \times X$. See Figure 2.

Such "complications" are ingredients of the Homological Mirror Conjecture of Kontsevich.

Quivers and Path Algebras

We will concentrate on a finite dimensional $\mathbb{K}$-algebra A. Let us assume that $\mathbb{K}$ is algebraically closed, and that A is basic (i.e. $A/\mathfrak{m} = \mathbb{K}$ for every maximal ideal). Since every $\mathbb{K}$-algebra is Morita equivalent to a basic one the latter is really no restriction.

The geometry of A is described by it's quiver (directed graph) $\vec{\Delta}(A)$.

Conversely, given a finite quiver $\vec{\Delta}$ one can construct an algebra $\mathbb{K}\vec{\Delta}$, called the path algebra of $\vec{\Delta}$.

There is an isomorphism of algebras

$$A \cong \mathbb{K}\vec{\Delta}(A)/I$$

for some ideal I; and $I = 0$ iff A is hereditary.

Here is a nonorthodox construction of $\vec{\Delta}(A)$. The set of vertices of $\vec{\Delta}(A)$ is $\operatorname{Spec} A$, namely the set of maximal ideals.

Let $\mathfrak{r}$ be the Jacobson radical of A, and for each $\mathfrak{m}_i \in \operatorname{Spec} A$ let $e_i \in A/\mathfrak{r}$ be the corresponding central idempotent. Then

$$\mathfrak{r}/\mathfrak{r}^2 = \bigoplus_{i,j} e_i(\mathfrak{r}/\mathfrak{r}^2)e_j \cong \bigoplus_{i,j} \frac{\mathfrak{m}_i \cap \mathfrak{m}_j}{\mathfrak{m}_i \mathfrak{m}_j}$$

as A-A-bimodules.

Define

$$d_{i,j} := \dim_{\mathbb{K}} \frac{\mathfrak{m}_i \cap \mathfrak{m}_j}{\mathfrak{m}_i \mathfrak{m}_j}.$$

Then one puts $d_{i,j}$ arrows $\mathfrak{m}_i \to \mathfrak{m}_j$.

The connected components of $\vec{\Delta}(A)$ are called cliques, and they control Öre localization.

Example. Take

$$A := \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix}.$$

The maximal ideals are

$$\mathfrak{m}_1 = \begin{bmatrix} 0 & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix} \quad \text{and} \quad \mathfrak{m}_2 = \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & 0 \end{bmatrix}.$$

Clearly

$$\frac{\mathfrak{m}_1 \cap \mathfrak{m}_2}{\mathfrak{m}_1 \mathfrak{m}_2} = \begin{bmatrix} 0 & \mathbb{K} \\ 0 & 0 \end{bmatrix}$$

so the quiver $\vec{\Delta}(A)$ is

$$\underset{\mathfrak{m}_1}{\bullet} \longrightarrow \underset{\mathfrak{m}_2}{\bullet}$$

a Dynkin quiver of type A_2.

The Auslander-Reiten Quiver

In this section $A = \mathbb{K}\vec{\Delta}$, a finite dimensional path algebra. An A-module is also called a representation of $\vec{\Delta}$. By convention one writes $\mathsf{mod}\, A$ for the category of finitely generated A-modules.

One of the important invariants of the algebra A is the set of isomorphism classes of indecomposable modules. If this set is finite A is said to have finite representation type.

By a famous theorem of Gabriel, A has finite representation type iff $\vec{\Delta}(A)$ is a Dynkin quiver of type A_n, D_n, E_6, E_7 or E_8.

Happel showed how to define the Auslander-Reiten quiver of the derived category

$$\vec{\Gamma} = \vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)).$$

This is an infinite quiver. Usually it is very complicated, but it contains a nice subquiver denoted $\vec{\mathbb{Z}}\vec{\Delta}$ that is easy to describe (the quiver $\vec{\mathbb{Z}}\vec{\Delta}$ was introduced by Riedtmann).

When $\vec{\Delta}$ is a Dynkin quiver, then in fact

$$\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)) = \vec{\mathbb{Z}}\vec{\Delta}.$$

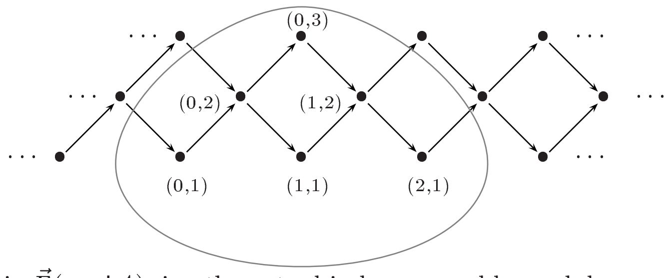
Example. Take $\vec{\Delta}$ a Dynkin quiver of type A_3:

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$$

Then the Auslander-Reiten quiver

$$\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)) = \vec{\mathbb{Z}}\vec{\Delta}$$

is



The vertices in $\vec{\Gamma}(\mathsf{mod}\, A)$, i.e. the actual indecomposable modules, are labelled.

The group $\mathrm{DPic}_{\mathbb{K}}(A)$ acts naturally on the quiver $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$, by quiver automorphisms. Using this action we can describe the structure of $\mathrm{DPic}_{\mathbb{K}}(A)$.

The element $\tau \in \mathrm{DPic}_{\mathbb{K}}(A)$, represented by the tilting complex $A^*[-1]$, is called the translation. In the example above it is a shift by 1 to the left.

Let $\mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})$ denote the group of quiver automorphisms, and $\mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})^{\langle\tau\rangle}$ the subgroup of elements commuting with τ.

Theorem 2. (Miyachi and Y.) Suppose $A = \mathbb{K}\vec{\Delta}$ has finite representation type. Then

$$\mathrm{DPic}_{\mathbb{K}}(A) \cong \mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})^{\langle\tau\rangle}.$$

Example. Let $\sigma \in \mathrm{DPic}_{\mathbb{K}}(A)$ be represented by $A[1]$. If $\vec{\Delta}$ is of Dynkin type A_n then $\mathrm{DPic}_{\mathbb{K}}(A)$ is abelian, generated by τ and σ, with one relation $\tau^{n+1} = \sigma^{-2}$.

Let us denote by $\mathrm{Aut}((\mathbb{Z}\vec{\Delta})_0; d)$ the group of permutations of the vertex set $(\mathbb{Z}\vec{\Delta})_0$ that preserve the arrow-multiplicity d. Let $\mathrm{Out}_{\mathbb{K}}(A)$ be the group of outer automorphisms of A. It embeds naturally into $\mathrm{Pic}_{\mathbb{K}}(A)$. $\mathrm{Out}_{\mathbb{K}}(A)$ is a linear algebraic group, and we denote by $\mathrm{Out}^0_{\mathbb{K}}(A)$ its identity component.

Theorem 3. (Miyachi and Y.) If $A = \mathbb{K}\vec{\Delta}$ has infinite representation type then there is an isomorphism of groups

$$\begin{aligned} &\mathrm{DPic}_{\mathbb{K}}(A) \\ &\quad \cong \big(\mathrm{Aut}((\mathbb{Z}\vec{\Delta})_0; d)^{\langle\tau\rangle} \ltimes \mathrm{Out}^0_{\mathbb{K}}(A)\big) \times \mathbb{Z}. \end{aligned}$$

$\mathrm{Out}^0_{\mathbb{K}}(A)$ is trivial when $\vec{\Delta}$ is a tree.

The Geometry of $\mathrm{DPic}_{\mathbb{K}}(A)$

In Theorem 3 one of the factors of $\mathrm{DPic}_{\mathbb{K}}(A)$ was an algebraic group. It is natural to ask if the whole group can be endowed with a geometric structure.

Definition. A locally algebraic group over $\mathbb{K}$ is a group G, with a normal subgroup G^0, such that G^0 is a connected algebraic group over $\mathbb{K}$, each coset of G^0 is a variety, and multiplication and inversion are morphisms of varieties.

In some examples studied in [MY] the group $\mathrm{DPic}_{\mathbb{K}}(A)$ was a locally algebraic group.

Example. Let $\vec{\Omega}_n$ be the quiver

$$\begin{tikzcd}[column sep=huge] 1 \arrow[r, bend left=40, "\alpha_1"] \arrow[r, bend left=15, "\alpha_2"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=40, "\alpha_n"'] & 2 \end{tikzcd}$$

Let A be the path algebra $\mathbb{K}\vec{\Omega}_n$. Then

$$\mathrm{DPic}_{\mathbb{K}}(A) \cong \mathbb{Z} \times \big(\mathbb{Z} \ltimes \mathrm{PGL}_n(\mathbb{K})\big).$$

In the semi-direct product a generator T of $\mathbb{Z}$ acts on a matrix $\sigma \in \mathrm{PGL}_n(\mathbb{K})$ by $\sigma \mapsto (\sigma^{-1})^{\mathrm{t}}$.

And in fact we now have a general result:

Theorem 4. (Y.) Let A be a finite dimensional $\mathbb{K}$-algebra. Then $\mathrm{DPic}_{\mathbb{K}}(A)$ is a locally algebraic group, with identity component $\mathrm{Out}^0_{\mathbb{K}}(A)$.

To prove this theorem one needs to know that $\mathrm{Out}^0_{\mathbb{K}}(A)$ is a normal subgroup of $\mathrm{DPic}_{\mathbb{K}}(A)$, and conjugation by any tilting complex induces an automorphism of algebraic groups on $\mathrm{Out}^0_{\mathbb{K}}(A)$.

This turns out to follow quite easily from a deep theorem proved recently, by Huisgen-Zimmermann and Saorin, and independently by Rouquier.

Theorem. (H.Z.-S., R.) Suppose A and B are finite dimensional $\mathbb{K}$-algebras and $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes A^{\mathrm{op}})$ is a two-sided tilting complex. Then T induces an isomorphism of algebraic groups $\phi : \mathrm{Out}^0_{\mathbb{K}}(A) \xrightarrow{\simeq} \mathrm{Out}^0_{\mathbb{K}}(B)$ such that $T \otimes^{\mathrm{L}}_A P \cong \phi(P) \otimes^{\mathrm{L}}_B T$.

Here is an outline of Rouquier's proof. He shows that the algebraic group $\mathrm{Out}_{\mathbb{K}}(B)$ represents the functor

$$\mathsf{Varieties}\,/\mathbb{K} \to \mathsf{Groups}$$

which assigns to a variety X the group of locally free invertible $(\mathcal{O}_X \otimes B)$-bimodules.

A two-sided tilting complex $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes A^{\mathrm{op}})$ gives rise to a locally free invertible $(\mathcal{O}_G \otimes B)$-bimodule, where $G := \mathrm{Out}^0_{\mathbb{K}}(A)$; and thus to a morphism $G \to \mathrm{Out}_{\mathbb{K}}(B)$.

The similarity to Picard schemes in algebraic geometry is striking. We know that if X is a complete variety then the subgroup $\mathrm{Pic}^0(X)$ of degree 0 line bundles is represented by an algebraic group. The whole Picard group $\mathrm{Pic}(X)$ is indeed a locally algebraic group; but since $\mathrm{Pic}(X)$ is abelian this fact is not so interesting.

However there is an example due to Orlov of a nontrivial locally algebraic group arising in algebraic geometry. Orlov shows there is an abelian variety X such that the group of auto-equivalences of the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$ is isomorphic to

$$D \ltimes (X \times \widehat{X}),$$

where D is a discrete group (an extension of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$) and $\widehat{X}$ is the dual abelian variety.

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