An Averaging Process for Unipotent Group Actions

Lecture Notes [1]

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1. Introduction

In this talk I will explain a geometric result that is a by-product of research on deformation quantization [Ye1]. This result seems to be new, and could be useful for other applications. Full details can be found in [Ye2].

Throughout $\mathbb{K}$ is a field of characteristic 0. For a linear algebraic group G over $\mathbb{K}$, and a commutative $\mathbb{K}$-algebra A, we write $G(A)$ for the group of A-valued points of G. For instance if $G = \mathrm{GL}_{n,\mathbb{K}}$ then $G(A) = \mathrm{GL}_n(A)$, the group of invertible matrices with entries in A.

A *torsor* under $G(\mathbb{K})$ – naive definition – is a set Z endowed with an action of $G(\mathbb{K})$, which is transitive and has trivial stabilizers. Thus for every point $z \in Z$ the map $G(\mathbb{K}) \to Z$, $g \mapsto g \cdot z$, is bijective.

By a *weight sequence* we mean a sequence $\boldsymbol{w} = (w_0, \ldots, w_q) \in \mathbb{K}^{q+1}$ such that $\sum_{i=0}^{q} w_i = 1$.

Suppose G is a vector space, i.e. $G \cong \mathbf{A}^n_{\mathbb{K}}$ with its additive group structure. Let Z be a torsor under $G(\mathbb{K})$. Given a sequence $\boldsymbol{z} = (z_0, \ldots, z_q) \in Z^{q+1}$ and a weight sequence $\boldsymbol{w}$ we have the usual weighted average $\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) \in Z$. Let me recall the formula. To simplify notation, in this case we will use addition for the operation in G and the action on Z. Choose any base point $z \in Z$, and define $g_i \in G(\mathbb{K})$ to be the unique element such that $z_i = g_i + z$. Using the isomorphism $G \cong \mathbf{A}^n_{\mathbb{K}}$ we get elements $w_i g_i \in G(\mathbb{K})$, and then we let

$$\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) := \Big(\sum_{i=0}^{q} w_i g_i \Big) + z.$$

Since $\sum_{i=0}^{q} w_i = 1$ this is independent of the base point z.

We want to generalize this construction.

Let $\bar{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. An element $g \in G(\bar{\mathbb{K}})$ is called *unipotent* if for any morphism of groups $\rho : G \to \mathrm{GL}_{n,\mathbb{K}}$ the matrix $\rho(g) \in \mathrm{GL}_n(\bar{\mathbb{K}})$ is unipotent (i.e. its only eigenvalue is 1). The group G is called unipotent of all elements of $G(\bar{\mathbb{K}})$ are unipotent.

Here is equivalent characterization: G is unipotent iff it is isomorphic to a closed subgroup of the triangular matrix group

$$\begin{bmatrix} 1 & \mathbb{K} & \cdots & \mathbb{K} \\ 0 & 1 & \cdots & \mathbb{K} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \subset \mathrm{GL}_{m,\mathbb{K}}$$

for some m.

Example 1. $G := \mathbf{A}^n_{\mathbb{K}}$ is unipotent.

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Example 2. Consider the polynomial algebra $\mathbb{K}[t_1, \ldots, t_n] = \mathbb{K}[\boldsymbol{t}]$ and its maximal ideal $(\boldsymbol{t})$. Fix a number $l \geq 0$, and define

$$B := \mathbb{K}[\boldsymbol{t}]/(\boldsymbol{t})^{l+1},$$

which is an artinian local algebra with maximal ideal $\mathfrak{m} = (\boldsymbol{t}) \cdot B$. Let G be the algebraic group whose group of $\mathbb{K}$-rational points $G(\mathbb{K})$ is the group of $\mathbb{K}$-algebra automorphisms of B which are trivial modulo $\mathfrak{m}^2$. Any element $g \in G(\mathbb{K})$ is of the form $t_i \mapsto t_i + f_i(\boldsymbol{t})$, where $f_i(\boldsymbol{t}) \in \mathfrak{m}^2$. This implies that G is unipotent. Note that the full automorphism group of B is the semi-direct product $\mathrm{GL}_n(\mathbb{K}) \ltimes G(\mathbb{K})$.

Let G be a unipotent group and Z a torsor under $G(\mathbb{K})$. Again we are given a sequence of points $\boldsymbol{z} = (z_0, \ldots, z_q)$ in Z and a weight sequence $\boldsymbol{w}$. We want to construct a weighted average $\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) \in Z$. A couple of obvious requirements of such a construction are:

(1) Functoriality: suppose $G \to G'$ is a morphism of groups, with G' also unipotent; Z' is a torsor under $G'(\mathbb{K})$; $f : Z \to Z'$ is a $G(\mathbb{K})$-equivariant function; and $\boldsymbol{z}' := (f(z_0), \ldots, f(z_q))$. Then

$$f(\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z})) = \mathrm{wav}_{G',\boldsymbol{w}}(\boldsymbol{z}').$$

(2) Simpliciality. This will be defined later, but here is an instance: suppose $w_q = 0$. Then

$$\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) = \mathrm{wav}_{G,(w_0,\ldots,w_{q-1})}(z_0, \ldots, z_{q-1}).$$

2. Exponential Map

As before G is a unipotent linear algebraic group over $\mathbb{K}$. Let $\mathfrak{g}$ be the Lie algebra of G. This is a Lie algebra over $\mathbb{K}$, and also a scheme: $\mathfrak{g} \cong \mathbf{A}^n_{\mathbb{K}}$, where $n = \dim G$. There is an isomorphism of schemes

$$\exp_G : \mathfrak{g} \xrightarrow{\simeq} G$$

called the *exponential map*. Its inverse is $\log_G$. The exponential map is functorial; in fact it is an equivalence between the category of unipotent groups and the category of nilpotent Lie algebras over $\mathbb{K}$.

Example 3. For a unipotent closed subgroup $G \subset \mathrm{GL}_{n,\mathbb{K}}$ the Lie algebra $\mathfrak{g}(\mathbb{K})$ is a subalgebra of $\mathfrak{gl}_n(\mathbb{K})$ consisting of nilpotent matrices, and for $\gamma \in \mathfrak{g}(\mathbb{K})$ the formula is

$$\exp_G(\gamma) = \sum_{i \geq 0} \tfrac{1}{i!}\gamma^i \in G(\mathbb{K}) \subset \mathrm{GL}_n(\mathbb{K}).$$

The multiplication in G is expressible via the CBH formula:

$$\exp_G(\gamma_1) \cdot \exp_G(\gamma_2) = \exp_G(F(\gamma_1, \gamma_2)),$$

where

$$F(\gamma_1, \gamma_2) = \gamma_1 + \gamma_2 + \tfrac{1}{2}[\gamma_1, \gamma_2] + \cdots$$

is a universal "commutator power series".

3. First Approximation of the Average

With G a unipotent group as before let Z be a torsor under $G(\mathbb{K})$. Let $\boldsymbol{w} = (w_0, \ldots, w_q)$ be a weight sequence and let $\boldsymbol{z} = (z_0, \ldots, z_q)$ be sequence of points in Z. Choose some base point $z \in Z$, and define elements $g_i \in G(\mathbb{K})$ by the rule $z_i = g_i \cdot z$. For any i let $\gamma_i := \log_G(g_i)$. Using the vector space structure of $\mathfrak{g}(\mathbb{K})$ define

$$\gamma := \sum_{i=0}^{q} w_i \gamma_i \in \mathfrak{g}(\mathbb{K}).$$

Next let $g := \exp_G(\gamma) \in G(\mathbb{K})$, and define

$$\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) := g \cdot z \in Z.$$

But there is a problem here: this "average" depends on the choice of base point z! A calculation shows that this dependence is in the form of commutators.

Observe that in the abelian case $G \cong \mathbf{A}^n_{\mathbb{K}}$ the exponential map is actually an isomorphism of groups

$$\exp_G : (\mathfrak{g}, +) \xrightarrow{\simeq} (G, \cdot),$$

and the problem of defining the average does not arise.

4. Improved Formula for the Average

Continuing with the previous setup, we will do something a little bit strange. There are $q+1$ preferred base points $z_0, \ldots, z_q \in Z$. We shall perform the averaging process with respect to each of these base points. Thus, for fixed weight sequence $\boldsymbol{w}$, we will obtain $q+1$ possibly distinct "averages" $z'_0, \ldots, z'_q$. The explicit formulas are

$$z'_i := \exp_G\Big(\sum_{j=0}^{q} w_j \log_G(g_{i,j})\Big) \cdot z_i,$$

where

$$g_{i,j} \cdot z_i = z_j.$$

This can be viewed as an operation

$$\mathrm{wsym}_{G,\boldsymbol{w}} : \begin{cases} Z^{q+1} \to Z^{q+1} \\ (z_0, \ldots, z_q) \mapsto (z'_0, \ldots, z'_q). \end{cases}$$

I claim that if we iterate the operation $\mathrm{wsym}_{G,\boldsymbol{w}}$ enough times we end up with a constant sequence. The reason is that the group G has a filtration by closed normal subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_d = \{1\}$$

such that G_i/G_{i+1} is abelian. Therefore, if the points $z_0, \ldots, z_q$ are all in the same G_i-orbit for some i, then $z'_0, \ldots, z'_q$ are in the same G_{i+1}-orbit.

Thus, starting with any sequence of points $\boldsymbol{z}$ we get

$$\mathrm{wsym}^d_{G,\boldsymbol{w}}(\boldsymbol{z}) = (z, \ldots, z)$$

for some $z \in Z$. Now we can define the correct weighted average

$$\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) := z.$$

It is easy to see that this average enjoys the required properties (functoriality and simpliciality).

5. More Sophisticated Geometry

The same idea can be dressed up in fancier language. As before G is a unipotent group over $\mathbb{K}$. By G-torsor we mean a $\mathbb{K}$-scheme Z with action $\mu : G \times Z \to Z$ such that $\mu \times \mathrm{p}_2 : G \times Z \to Z \times Z$ is an isomorphism of schemes.

For $p, q \in \mathbb{N}$ let $\mathbf{\Delta}^q_p$ be the set of order preserving functions $\{0, \ldots, p\} \to \{0, \ldots, q\}$. Thus for $p = 0$ we have $\mathbf{\Delta}^q_0 \cong \{0, \ldots, q\}$. The set

$$Z(\mathbb{K})^{\mathbf{\Delta}^q_0} = \mathrm{Hom}_{\mathsf{Sets}}(\mathbf{\Delta}^q_0, Z(\mathbb{K}))$$

is nothing but the set of sequences $(z_0, \ldots, z_q)$ in $Z(\mathbb{K})$.

The geometric q-dimensional simplex is

$$\mathbf{\Delta}^q_{\mathbb{K}} := \mathrm{Spec}\, \mathbb{K}[t_0, \ldots, t_q]/(t_0 + \cdots + t_q - 1).$$

A point in $\mathbf{\Delta}^q_{\mathbb{K}}(\mathbb{K})$ is precisely a weight sequence. We denote by $\mathrm{Hom}_{\mathrm{Sch}/\mathbb{K}}(\mathbf{\Delta}^q_{\mathbb{K}}, Z)$ the set of scheme morphisms.

The construction in the previous section can be geometrized to yield a function

$$\mathrm{wav}_G : Z(\mathbb{K})^{\mathbf{\Delta}^q_0} \to \mathrm{Hom}_{\mathrm{Sch}/\mathbb{K}}(\mathbf{\Delta}^q_{\mathbb{K}}, Z) \tag{1}$$

such that for any $\boldsymbol{w} \in \mathbf{\Delta}^q_{\mathbb{K}}(\mathbb{K})$ and $\boldsymbol{z} \in Z(\mathbb{K})^{\mathbf{\Delta}^q_0}$ one has

$$\mathrm{wav}_G(\boldsymbol{z})(\boldsymbol{w}) = \mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) \in Z(\mathbb{K}).$$

The function wav_G enjoys the functoriality explained earlier. Regarding simpliciality: as q varies the two sets appearing in equation (1) become simplicial sets, and the function wav_G becomes a map of simplicial sets.

We can make things even more sophisticated. Suppose X is some $\mathbb{K}$-scheme and $\pi : Z \to X$ is a torsor under the group scheme $G \times X$. For any X-scheme Y the set of Y-valued sections of π is of course $\mathrm{Hom}_{\mathrm{Sch}/X}(Y, Z)$. The averaging operation now becomes a function

$$\mathrm{wav}_G : \mathrm{Hom}_{\mathrm{Sch}/X}(Y, Z)^{\mathbf{\Delta}^q_0} \to \mathrm{Hom}_{\mathrm{Sch}/X}(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z) \tag{2}$$

with functorial and simplicial behavior.

6. An Application

Here is an application. As above $\pi : Z \to X$ is a torsor under $G \times X$. Assume $X = \bigcup_{i=0}^m U_{(i)}$ is an open covering, and for every i we are given a section $\sigma_{(i)} : U_{(i)} \to Z$ of π. Consider a simplex $\boldsymbol{i} = (i_0, \ldots, i_q) \in \mathbf{\Delta}^m_q$, and let

$$U_{\boldsymbol{i}} := U_{(i_0)} \cap \cdots \cap U_{(i_q)}.$$

Each of the sections

$$\sigma_{(i_0)}, \ldots, \sigma_{(i_q)} : U_{\boldsymbol{i}} \to Z$$

is an element of $\mathrm{Hom}_{\mathrm{Sch}/X}(U_{\boldsymbol{i}}, Z)$. The averaging process of (2) gives a morphism

$$\sigma_{\boldsymbol{i}} := \mathrm{wav}_G(\sigma_{(i_0)}, \ldots, \sigma_{(i_q)}) : \mathbf{\Delta}^q_{\mathbb{K}} \times U_{\boldsymbol{i}} \to Z.$$

This is a family of sections $U_{\boldsymbol{i}} \to Z$ interpolating between the various sections $\sigma_{(i_0)}, \ldots, \sigma_{(i_q)}$. As $\boldsymbol{i}$ varies the morphisms $\sigma_{\boldsymbol{i}}$ form a *simplicial section* of $\pi : Z \to X$. See Figure 1.

This construction is used in my paper [Ye1]. There X is a smooth n-dimensional scheme over $\mathbb{K}$, and $\pi : Z \to X$ is the *bundle of formal coordinate systems* (cf. [Ko]). The bundle Z is infinite dimensional, and it turns out to be a torsor under the group scheme $(\mathrm{GL}_{n,\mathbb{K}} \ltimes G) \times X$. Here $\mathrm{GL}_{n,\mathbb{K}} \ltimes G$ is the automorphism group of the

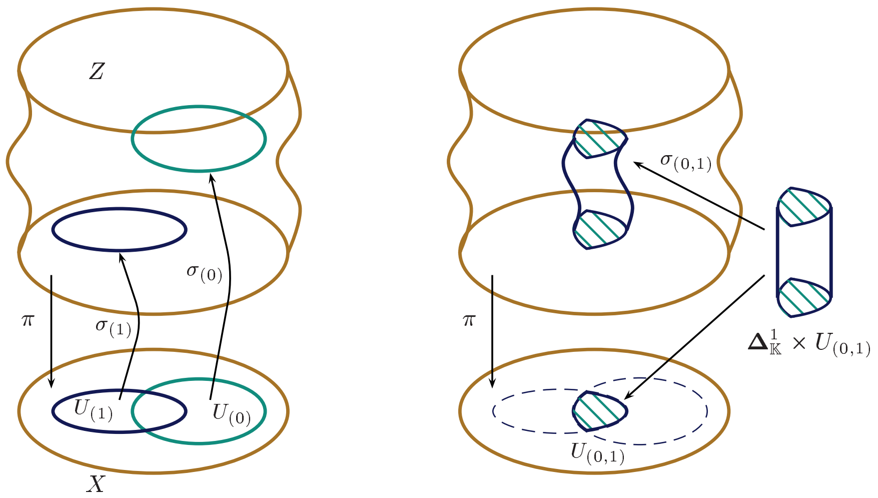

FIGURE 1. Simplicial section, $q = 1$. We start with sections over two open sets $U_{(0)}$ and $U_{(1)}$ in the left diagram; and we pass to a simplicial section $\sigma_{(0,1)}$ on the right.

power series algebra $\mathbb{K}[[t_1, \ldots, t_n]]$, and the normal subgroup G is pro-unipotent; cf. Example 2. Let $\bar{Z}$ be the quotient bundle $Z/\mathrm{GL}_{n,\mathbb{K}}$. Locally the bundle $\bar{Z}$ has sections, and a modification of the averaging process described above yields a global simplicial section of $\bar{Z}$. Using this simplicial section we can prove an algebro-geometric analogue of Kontsevich's result on deformation quantization.

7. Another Application

The last result I want to talk about is due to Z. Reichstein. Recall that a linear algebraic group G is called *special* if any G-torsor Z over $\mathbb{K}$ has a $\mathbb{K}$-rational point. Our averaging process shows that a unipotent group in characteristic 0 is special. (Note that this is true in any characteristic, and the standard proof is by Galois cohomology.)

So let G be a unipotent group over $\mathbb{K}$, and let Z be a G-torsor over $\mathbb{K}$. We want to show that $Z(\mathbb{K})$ is nonempty. Choose any closed point $z \in Z$. There is some finite Galois extension L of $\mathbb{K}$ such that $z \in Z(L)$. Let us denote by Γ the Galois group of $L/\mathbb{K}$, and let $z_0 = z, z_1, \ldots, z_q$ be the Γ-conjugates of z inside $Z(L)$. Consider the weight sequence $\boldsymbol{w} := (\frac{1}{q+1}, \ldots, \frac{1}{q+1})$. Let

$$z' := \mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) \in Z(L).$$

It is not hard to check, using the functoriality of $\mathrm{wav}_{G,\boldsymbol{w}}$, that z' is Γ-invariant, implying that $z' \in Z(\mathbb{K})$ as desired.

References

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