RIGID DUALIZING COMPLEXES AND REPRESENTATIONS OF LIE ALGEBRAS

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0. Introduction

In the lecture I will mention a few results in Lie theory that can be obtained using rigid dualizing complexes. Some of these results are new; others were already proved using different methods.

I will discuss the main features of rigid dualizing complexes and explain how apply to Lie theory.

Part of the work is joint with James J. Zhang (Seattle).

1. Results in Lie Theory

Let $\mathbb{K}$ be a field and $\mathfrak{g}$ an n-dimensional Lie algebra over it. Recall that the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ acts on $\mathfrak{g}$ from the left by the adjoint action, defined by $\mathrm{ad}(x)(y) = [x, y]$ for $x \in \mathfrak{g} \subset \mathrm{U}(\mathfrak{g})$ and $y \in \mathfrak{g}$.

If M is a left $\mathrm{U}(\mathfrak{g})$-module then $M^* := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ is a right $\mathrm{U}(\mathfrak{g})$-module. If N is another left $\mathrm{U}(\mathfrak{g})$-module then $M \otimes N$ is a left module by $x \cdot (u \otimes v) = (x \cdot u) \otimes v + v \otimes (x \cdot u)$. There is a similar formula for the exterior powers $\bigwedge^i M$, and likewise for right modules.

Thus $\bigwedge^n \mathfrak{g}$ is a one dimensional left $\mathrm{U}(\mathfrak{g})$-module with the adjoint action $\mathrm{ad}_{\bigwedge^n \mathfrak{g}}$. We view the exterior power $\bigwedge^n \mathfrak{g}^*$ as a $\mathrm{U}(\mathfrak{g})$-bimodule with coadjoint action from the right and trivial action from the left.

Theorem 1. *Let M be a $\mathbb{K}$-central $\mathrm{U}(\mathfrak{g})$-bimodule with $\dim_{\mathbb{K}} M < \infty$. Then*

$$\mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})}(M, \mathrm{U}(\mathfrak{g})) \cong M^* \otimes \bigwedge^n \mathfrak{g}^* \tag{2}$$

as $\mathrm{U}(\mathfrak{g})$-bimodules, functorially in M.

If M is just a left $\mathrm{U}(\mathfrak{g})$-module then (2) is an isomorphism of right modules. This isomorphism of right modules was proved by Brown-Levasseur [BL] (1985) for $\mathfrak{g}$ solvable and M simple. They used earlier work of Brown on Ore localization in such enveloping algebras. The same result was proved by Kempf.

Chemla [Ch] (1994) proved the isomorphism (2) for any left module M and any Lie algebra $\mathfrak{g}$ – indeed even for Lie superalgebras and other generalizations. Upon close inspection one sees that Chemla's proof (based on an idea of Duflo) can be easily extended to the case of a bimodule.

As we shall see the theorem is quite easy to prove using rigid dualizing complexes.

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Let me remark that if $\mathfrak{g}$ is abelian or semi-simple then the representation $\bigwedge^n \mathfrak{g}^*$ is trivial, so

$$M^* \otimes \bigwedge^n \mathfrak{g}^* \cong M^*.$$

However if $\mathfrak{g}$ is solvable (e.g. a Borel subalgebra) then the representation $\bigwedge^n \mathfrak{g}^*$ is often nontrivial.

There is an immediate implication (already in [BL]) for Verma modules. Here is a brief resumé about Verma modules. Assume $\mathbb{K} = \mathbb{C}$, and let $\mathfrak{g}$ be a semi-simple Lie algebra, with Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Given a weight $\lambda \in \mathfrak{h}^*$ there is a one dimensional $\mathrm{U}(\mathfrak{b})$-module denoted $\mathbb{C}_\lambda$. The Verma module is

$$M(\lambda) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho},$$

where $\rho \in \mathfrak{h}^*$ is half the weight of the $\mathrm{U}(\mathfrak{h})$-module $\bigwedge^m \mathfrak{b}$, $m = \dim_{\mathbb{K}} \mathfrak{b}$.

Now

$$\mathbb{C}^*_{\lambda-\rho} \otimes \bigwedge^m \mathfrak{b}^* \cong \mathbb{C}_{\lambda+\rho}$$

as right $\mathrm{U}(\mathfrak{b})$-modules. So from Theorem 1 applied to the left $\mathrm{U}(\mathfrak{b})$-module $\mathbb{C}_{\lambda-\rho}$ one obtains directly:

Corollary 3. *For $\mathfrak{g}$ semi-simple,*

$$\mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})}(M(\lambda), \mathrm{U}(\mathfrak{g})) \cong \mathbb{C}_{\lambda+\rho} \otimes_{\mathrm{U}(\mathfrak{b})} \mathrm{U}(\mathfrak{g});$$

i.e. the dual of a left Verma module is a right Verma module.

Before stating the next result, let us observe that there is a bimodule isomorphism:

$$\mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g} \cong \mathrm{U}(\mathfrak{g})^\gamma,$$

where the right action of $\mathrm{U}(\mathfrak{g})$ on $\mathrm{U}(\mathfrak{g})^\gamma$ is twisted by an automorphism $\gamma \in \mathrm{Aut}(\mathrm{U}(\mathfrak{g}))$, which we call the canonical automorphism. . The formula for γ is

$$\gamma(x) = x - \mathrm{tr}\,\mathrm{ad}_{\bigwedge^n \mathfrak{g}}(x),\ \ x \in \mathfrak{g}.$$

Theorem 4 (Y, 1998)**.** *Let M be a finitely generated left $\mathrm{U}(\mathfrak{g})$-module. Then the annihilators satisfy*

$$\gamma(\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})} M) \subset \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})^{\mathrm{op}}} \mathrm{Ext}^i_{\mathrm{U}(\mathfrak{g})}(M, \mathrm{U}(\mathfrak{g}))$$

for all i. If M is pure of Gelfand-Kirillov dimension $\mathrm{GKdim} = m$ and $i = n - m$ then there is equality.

As far as I know this result is totally new (for $m > 0$).

It is interesting to compare Theorem 4 to something else proved by Brown-Levasseur. They show that for $\mathfrak{g}$ solvable and a prime ideal $P \subset \mathrm{U}(\mathfrak{g})$ there is an element $\lambda_P \in (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$ depending on P with the following property. Define $\gamma_P \in \mathrm{Aut}(\mathrm{U}(\mathfrak{g}))$ by $\gamma_P(x) = x + \lambda_P(x)$, $x \in \mathfrak{g}$. Let $m := \mathrm{GKdim}\,\mathrm{U}(\mathfrak{g})/P$. Then

$$\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})^{\mathrm{op}}} \mathrm{Ext}^{n-m}_{\mathrm{U}(\mathfrak{g})}(\mathrm{U}(\mathfrak{g})/P, \mathrm{U}(\mathfrak{g})) = \gamma_P(P).$$

We obtain the following curious corollary.

Corollary 5.

$$\gamma_P(P) = \gamma(P).$$

The mystery goes away when P has finite codimension ($m = 0$), since then

$$\lambda_P = -\operatorname{tr}\operatorname{ad}_{\bigwedge^n \mathfrak{g}}$$

and we are back to Theorem 1

If $\mathfrak{g}$ is semisimple then $\bigwedge^n \mathfrak{g}$ is trivial and hence γ is the identity. Since a Verma module has pure GK dimension we obtain directly from Theorem 4:

Corollary 6. *Let $\mathfrak{g}$ be semisimple and $M(\lambda)$ a Verma module. Then*

$$\operatorname{Ann}_{\mathrm{U}(\mathfrak{g})^{\mathrm{op}}} \operatorname{Ext}^{n-m}_{\mathrm{U}(\mathfrak{g})}(M(\lambda), \mathrm{U}(\mathfrak{g})) = \operatorname{Ann}_{\mathrm{U}(\mathfrak{g})} M(\lambda).$$

This was already proved by Brown-Levasseur from Corollary 3, and relying on earlier work of Duflo [Du].

About Rigid Dualizing Complexes

Let me begin by recalling a bit of Grothendieck duality in (commutative) algebraic geometry [RD]. Let $\mathbb{K}$ be a field and X a finite type $\mathbb{K}$-scheme. Then there is a canonical object $\mathcal{R}_X$ in the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ which is dualizing: $\mathrm{R}\mathcal{H}om_X(-, \mathcal{R}_X)$ is a duality of the category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. Also if $f : X \to Y$ is a proper map then there is a nondegenerate trace $\operatorname{Tr}_f : \mathrm{R}f_* \mathcal{R}_X \to \mathcal{R}_Y$. Here is how to obtain $\mathcal{R}_X$: if $\pi : X \to \operatorname{Spec} \mathbb{K}$ is the structural morphism, then $\mathcal{R}_X = \pi^! \mathbb{K}$.

We want to do something like this for noncommutative $\mathbb{K}$-algebras. We will denote by A and B noetherian associative unital $\mathbb{K}$-algebras. A^{op} is the opposite algebra and $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$. Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ be the bounded derived category of complexes of left A-modules with finitely generated cohomologies.

Definition 7 (**sketch**; Y. 1992)**.** A *dualizing complex* over A is (omitting details) a complex of bimodules $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ such that the functors $\mathrm{R}\operatorname{Hom}_A(-, R)$ and $\mathrm{R}\operatorname{Hom}_{A^{\mathrm{op}}}(-, R)$ are a duality between $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})$.

The complex $R \otimes R$ has four A-module structures.

Definition 8 (Van den Bergh 1997)**.** A dualizing complex R is *rigid* if there is an isomorphism

$$\phi : R \xrightarrow{\simeq} \mathrm{R}\operatorname{Hom}_{A^{\mathrm{e}}}(A, R \otimes R)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

In [VdB1] it is proved that a rigid dualizing complex is unique up to isomorphism.

A ring homomorphism is *finite* if B is a finitely generated A-module on both sides.

Definition 9 (Y.-Zhang)**.** Suppose $A \to B$ is a finite homomorphism of $\mathbb{K}$-algebras and (R_A, ϕ_A) and (R_B, ϕ_B) are rigid dualizing complexes. Then there is at most one morphism $\operatorname{Tr}_{B/A} : R_B \to R_A$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ such that

(i) (Nondegenerate) $\operatorname{Tr}_{B/A}$ induces isomorphisms

$$R_B \cong \mathrm{R}\operatorname{Hom}_A(B, R_A) \cong \mathrm{R}\operatorname{Hom}_{A^{\mathrm{op}}}(B, R_A)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

(ii) (Rigid) The diagram

$$\begin{tikzcd} R_B \arrow[r, "\phi_B"] \arrow[d, "\operatorname{Tr}"'] & \mathrm{R}\operatorname{Hom}_{B^{\mathrm{e}}}(B, R_B \otimes R_B) \arrow[d, "\operatorname{Tr} \otimes \operatorname{Tr}"'] \\ R_A \arrow[r, "\phi_A"] & \mathrm{R}\operatorname{Hom}_{A^{\mathrm{e}}}(A, R_A \otimes R_A) \end{tikzcd}$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ is commutative.

This morphism is called the *rigid trace.*

Remark 10. If A is commutative and $X = \operatorname{Spec} A$ then the complex $R_A := \Gamma(X, \mathcal{R}_X)$ is rigid. This corresponds to the fundamental class of the diagonal $X \subset X \times X$. For a finite homomorphism $A \to B$ the rigid trace coincides with the trace of [RD].

To prove existence of rigid dualizing complexes we rely on work of Van den Bergh. A filtration $F = \{F_n A\}$ of A is called *noetherian connected* if the graded algebra $\operatorname{gr}^F A$ is a noetherian connected graded $\mathbb{K}$-algebra.

Theorem 11 (Van den Bergh)**.** *Suppose A has a noetherian connected filtration such that $\operatorname{gr} A$ has a balanced dualizing complex (e.g. $\operatorname{gr} A$ is commutative). Then A has a rigid dualizing complex R_A.*

Theorem 12 (Y.-Zhang)**.** *Let A be as above and $A \to B$ a finite centralizing homomorphism (e.g. $B = A/I$). Then B has a rigid dualizing complex R_B and the rigid trace $\operatorname{Tr}_{B/A}$ exists.*

2. Application to Enveloping Algebras

Evidently Theorems 11 and 12 apply when $A = \mathrm{U}(\mathfrak{g})$ is an enveloping algebra and $B = \mathrm{U}(\mathfrak{g})/I$ is any quotient.

The next theorem was conjectured by Van den Bergh.

Theorem 13 (Y. 1998)**.** *The rigid dualizing complex of* $\mathrm{U}(\mathfrak{g})$ *is*

$$R_{\mathrm{U}(\mathfrak{g})} = \left(\mathrm{U}(\mathfrak{g}) \otimes \bigwedge\nolimits^{n} \mathfrak{g}\right)[n].$$

Remark 14. This is to be expected from deformation considerations. If $\mathfrak{g}$ is abelian then $A := \mathrm{U}(\mathfrak{g})$ is commutative and $A \otimes \bigwedge^n \mathfrak{g} = \Omega^n_{A/\mathbb{K}}$.

Proof of Theorem 1. We are given a finite dimensional left $\mathrm{U}(\mathfrak{g})$-module M. Let $I := \operatorname{Ann} M$ and $B := \mathrm{U}(\mathfrak{g})/I$. Then $\mathbb{K} \to B$ is a finite homomorphism and the rigid dualizing complex is $R_B = \operatorname{Hom}_{\mathbb{K}}(B, \mathbb{K}) = B^*$. Because the rigid trace

$$R_B = B^* \to R_A = \left(\mathrm{U}(\mathfrak{g}) \otimes \bigwedge\nolimits^{n} \mathfrak{g}\right)[n]$$

exists in the derived category we have isomorphisms of right modules

$$M^* = \operatorname{Hom}_B(M, B^*) \xrightarrow{\simeq} \operatorname{Ext}^n_{\mathrm{U}(\mathfrak{g})}(M, (\mathrm{U}(\mathfrak{g}) \otimes \bigwedge\nolimits^{n} \mathfrak{g})).$$

Twisting by $\otimes \bigwedge^n \mathfrak{g}^*$ we get the desired isomorphism. The same proof works when M is a bimodule.

Proof of Theorem 4. Taking $I := \operatorname{Ann} M$ and $B := \mathrm{U}(\mathfrak{g})/I$ we get like before

$$\operatorname{Ext}^i_{\mathrm{U}(\mathfrak{g})}(M, \mathrm{U}(\mathfrak{g})) \cong \operatorname{Ext}^i_B(M, R_B)^{\gamma^{-1}}.$$

By definition R_B is a complex of B-bimodules. Hence the right module $\operatorname{Ext}^i_B(M, R_B)^{\gamma^{-1}}$ is annihilator by $\gamma(I)$. For the second statement one needs to use the Auslander property of R_B which we don't have time to discuss today.

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