Differential Graded Algebras

Lectures at Fudan University, Shanghai [1]
September 2001

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Forward

In these talks I will discuss several aspects of differential graded algebras (DGAs).

Unfortunately there are no comprehensive references. My main sources on DGAs are the articles [Ke2] and [Ke3] of Bernhard Keller, who also explained some points to me in person. I also learned some facts about DGAs from Vladimir Hinich. On the happy side there are a few good sources for derived categories, such as [RD] and [KS].

A couple of applications will be mentioned that were used in my own work; this will give the lectures a more personal flavor.

Here is the list of topics:

Contents

[1] This version of the notes includes some corrections.

1 Basic Properties of DGAs and DG Modules

1.1 DGAs

Throughout these lectures $\mathbb{K}$ denotes a commutative base ring. Unless stated otherwise, all rings will be $\mathbb{K}$-algebras, all modules will be $\mathbb{K}$-modules and all module homomorphisms will be $\mathbb{K}$-linear. $\otimes$ will denote $\otimes_{\mathbb{K}}$. Those who prefer it can take $\mathbb{K}$ to be a field of characteristic 0, say $\mathbb{R}$ or $\mathbb{C}$.

Let $A = \bigoplus_{i\in\mathbb{Z}} A^i$ be an associative unital graded $\mathbb{K}$-algebra. Thus

$$A^i \cdot A^j \subset A^{i+j}.$$

Recall that a graded left A-module is an A-module M with a decomposition

$$M = \bigoplus_{i\in\mathbb{Z}} M^i$$

such that

$$A^i \cdot M^j \subset M^{i+j}.$$

M^i is the homogeneous component of degree i. A homomorphism $\phi : M \to N$, where N is also graded, is said to be homogenous of degree j if

$$\phi(M^i) \subset N^{i+j}$$

for all i. Let us write

$$\mathrm{Hom}_A^{\mathrm{gr}}(M,N)^j := \{\phi \in \mathrm{Hom}_A(M,N) \text{ homogeneous of degree j }\}$$

and

$$\mathrm{Hom}_A^{\mathrm{gr}}(M,N) := \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}^{\mathrm{gr}}(M,N)^j,$$

which is a graded $\mathbb{K}$-module.

Definition 1.1.1. A *differential graded algebra*, or DGA, is an associative unital graded $\mathbb{K}$-algebra $A = \bigoplus_{i\in\mathbb{Z}} A^i$, endowed with a homomorphism $\mathrm{d} : A \to A$ of degree 1 satisfying $\mathrm{d}^2 = 0$ and the graded Leibnitz rule:

$$\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^i a \cdot \mathrm{d}(b)$$

for $a \in A^i$, $b \in A$.

Here are some examples.

Example 1.1.2. Any ring A is a DGA, with $A^i = 0$ for $i \neq 0$ and $\mathrm{d} = 0$.

Example 1.1.3. The algebraic de Rham complex of the line. Define A using generators and relations as follows:

$$A := \frac{\mathbb{K}\langle x, \xi \rangle}{(x\xi - \xi x, \xi^2)}.$$

The degrees are $\deg x = 0$, $\deg \xi = 1$. The differential is defined on generators by

$$\mathrm{d}x := \xi, \quad \mathrm{d}\xi := 0.$$

It is not hard to verify that d extends by additivity and the Leibnitz rule to all of A.

I will show less trivial examples later.

1.2 DG Modules

Here A is a fixed DGA.

Definition 1.2.1. A left DG A-module is a graded A-module

$$M = \bigoplus\nolimits_{i \in \mathbb{Z}} M^i$$

together with a homomorphism $\mathrm{d} : M \to M$ of degree 1 satisfying

$$\mathrm{d}(a \cdot m) = \mathrm{d}(a) \cdot m + (-1)^i a \cdot \mathrm{d}(m)$$

for $a \in A^i$ and $m \in M$.

We will usually work with left modules. A right DG module is defined similarly.

Given two DG A-modules M and N we define the DG $\mathbb{K}$-module $\mathrm{Hom}_A^{\mathrm{dg}}(M, N)$ as follows. As graded module,

$$\mathrm{Hom}_A^{\mathrm{dg}}(M, N) := \mathrm{Hom}_A^{\mathrm{gr}}(M, N) = \bigoplus\nolimits_{j \in \mathbb{Z}} \mathrm{Hom}_A^{\mathrm{gr}}(M, N)^j.$$

The differential is

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^j \phi \circ \mathrm{d}_M$$

for $\phi \in \mathrm{Hom}_A^{\mathrm{gr}}(M, N)^j$.

Another construction is the tensor product. If M is a right DG A-module and N is a left one then

$$M \otimes_A N = \bigoplus\nolimits_{i \in \mathbb{Z}} (M \otimes_A N)^i$$

where $(M \otimes_A N)^i$ is the additive subgroup generated by the tensors $m \otimes n$, $m \in M^p$, $n \in N^q$, $p + q = i$. The differential is

$$\mathrm{d}(m \otimes n) = \mathrm{d}(m) \otimes n + (-1)^p m \otimes \mathrm{d}(n).$$

Definition 1.2.2. The category $\mathsf{DGMod}\, A$ of DG A-modules has as its objects the class of all DG A-modules. A morphism $\phi : M \to N$ is an A-linear homomorphism of degree 0 that commutes with d.

In other terms,

$$\mathrm{Hom}_{\mathsf{DGMod}\, A}(M, N) = \mathrm{Z}^0\, \mathrm{Hom}_A^{\mathrm{dg}}(M, N),$$

where

$$\mathrm{Z}^i K := \{k \in K^i \mid \mathrm{d}(k) = 0\}$$

is the group of degree i coboundaries of a DG module K.

$\mathsf{DGMod}\, A$ is an abelian category. Kernels and cokernels are same as in the underlying graded module category $\mathsf{GrMod}\, A$, and they are endowed with induced differentials.

Example 1.2.3. If A is a ring we make it into a DGA with $A = A^0$ and $\mathrm{d}_A = 0$. Then a DG A-module M will satisfy

$$\mathrm{d}_M(a \cdot m) = \mathrm{d}_A(a) \cdot m + (-1)^i a \cdot \mathrm{d}_M(m) = a \cdot \mathrm{d}_M(m)$$

for $a \in A$ and $m \in M$. We see that a DG A-module is precisely a complex of A-modules. The notation for the category of complexes of A-modules is

$$\mathsf{C}(\mathsf{Mod}\, A) = \mathsf{DGMod}\, A.$$

Example 1.2.4. We can create new DGAs from old ones as endomorphisms of DG modules. Suppose A is a DGA and M is a DG A-module. Then

$$B := \mathrm{Hom}_A^{\mathrm{dg}}(M, M)$$

is a DGA.

Example 1.2.5. Special case of the previous example. Take $A := \mathbb{K}[x]$ the polynomial ring and let M be the DG A-module

$$M^{-1} := A,\ M^0 := A \text{ and } M^i := 0 \text{ otherwise}$$

with differential

$$\mathrm{d}(m) := x \cdot m, \quad m \in M^{-1}.$$

I leave it as an exercise to compute the structure of the DGA $B = \mathrm{Hom}_A^{\mathrm{dg}}(M, M)$.

1.3 Examples of DGAs with Geometric Origins

Here are some interesting DGAs.

Example 1.3.1. Here is a generalization of the Example 1.1.3 to n-space. Take

$$A := \frac{\mathbb{K}\langle x_1, \ldots, x_n, \xi_1, \ldots, \xi_n \rangle}{(x_i x_j - x_j x_i, x_i \xi_j - \xi_j x_i, \xi_i \xi_j + \xi_j \xi_i, \xi_i^2)}.$$

The degrees are $\deg x_i = 0$, $\deg \xi_i = 1$. The differential is defined on generators by

$$\mathrm{d}x_i := \xi_i, \quad \mathrm{d}\xi_i := 0,$$

and extended to all of A by the Leibnitz rule.

Example 1.3.2. More generally, suppose C is a commutative $\mathbb{K}$-algebra. Let

$$I := \ker(\mu : C \otimes C \to C), \quad \mu(a \otimes b) := ab$$

and

$$\Omega^1_{C/\mathbb{K}} := I/I^2.$$

Let

$$\Omega^p_{C/\mathbb{K}} := \bigwedge\nolimits^p_C \Omega^1_{C/\mathbb{K}},$$

the pth exterior power of $\Omega^1_{C/\mathbb{K}}$ as C-module. Then

$$\Omega^{\cdot}_{C/\mathbb{K}} := \bigoplus\nolimits_p \Omega^p_{C/\mathbb{K}}$$

is a DGA. The rule for

$$\mathrm{d} : C = \Omega^0_{C/\mathbb{K}} \to I/I^2 = \Omega^1_{C/\mathbb{K}}$$

is

$$\mathrm{d}(c) := c \otimes 1 - 1 \otimes c.$$

It extends to all of $\Omega^{\cdot}_{C/\mathbb{K}}$ by the Leibnitz rule.

$\Omega^{\cdot}_{C/\mathbb{K}}$ is called the de Rham complex of C. The elements of $\Omega^p_{C/\mathbb{K}}$ are called the Kähler differentials of degree p, and d is called the exterior derivative.

In the previous examples we had the special case $C = \mathbb{K}[x_1, \ldots, x_n]$, the polynomial ring.

References: [EGA IV] and [Ye1].

Example 1.3.3. Now let X be a smooth (i.e. C^∞) real manifold. For any p one has the module of degree p smooth differential forms on X, denoted by

$\Omega^p(X)$. It is a module over the ring $\mathrm{C}^\infty(X) = \Omega^0(X)$. In local coordinates $t_1, \ldots, t_n$ we can write such a form as

$$\sum_{i_1 < i_2 < \cdots < i_p} f_{i_1, \ldots, i_p}(x) \mathrm{d}t_{i_1} \wedge \cdots \wedge \mathrm{d}t_{i_p}$$

where the $f_{i_1, \ldots, i_p}(x)$ are C^∞ functions. The exterior derivative $\mathrm{d} : \Omega^p(X) \to \Omega^{p+1}(X)$ is expressed in local coordinates as

$$\mathrm{d}(f_{i_1, \ldots, i_p}(x) \mathrm{d}t_{i_1} \wedge \cdots \wedge \mathrm{d}t_{i_p}) := \mathrm{d}(f_{i_1, \ldots, i_p}) \wedge \mathrm{d}t_{i_1} \wedge \cdots \wedge \mathrm{d}t_{i_p}$$

where

$$\mathrm{d}(f) := \sum_j \frac{\partial f}{\partial t_j}(x) \mathrm{d}t_j.$$

The DGA

$$\Omega^{\cdot}(X) := \bigoplus \Omega^p(X)$$

is called the de Rham complex of X.

Example 1.3.4. Let X be a topological space. For any p let $S_p(X)$ denote the set of p-dimensional singular chains in X, namely the set of continuous function $\sigma : \Delta^p \to X$. Here Δ^p is the standard p-dimensional simplex

$$\Delta^p := \{(a_0, \ldots, a_p) \in \mathbb{R}^{p+1} \mid a_i \geq 0, \ \sum a_i = 1\}.$$

Define

$$C^p(X) := \mathrm{Hom}_{\mathsf{Sets}}(S^p(X), \mathbb{K}).$$

Then

$$C^{\cdot}(X) := \bigoplus\nolimits_p C^p(X)$$

is a DGA. It's called the ring of simplicial cochains on X. The multiplication is called the Alexander-Whitney product; it is not commutative. More about that later.

1.4 The Cohomology of a DG Algebra or Module

Let A be a DGA. Recall that the group of p-cocycles of A is

$$\mathrm{Z}^p A := \{a \in A^p \mid \mathrm{d}a = 0\}.$$

The group of p-coboundaries is

$$\mathrm{B}^p A := \{\mathrm{d}a \mid a \in A^{p-1}\}.$$

It's easy to see that $\mathrm{Z}A := \bigoplus_p \mathrm{Z}^p A$ is a graded subring of A and $\mathrm{B}A := \bigoplus_p \mathrm{B}^p A$ is a graded ideal of $\mathrm{Z}A$. Thus the cohomology

$$\mathrm{H}A := \mathrm{Z}A/\mathrm{B}A = \bigoplus_p \mathrm{H}^p A$$

is a graded ring.

Example 1.4.1. In the setup of Example 1.2.5 we have $\mathrm{H}B \cong \mathbb{K}[\epsilon]$ where $\epsilon^2 = 0$ and $\deg \epsilon = 1$.

Example 1.4.2. Suppose $\mathbb{K}$ is a field and X is a smooth affine variety over $\mathbb{K}$. (If $\operatorname{char} \mathbb{K} = 0$ this is the same as a nonsingular variety.) Let $C := \mathcal{O}(X)$, the ring of regular function on X. Consider the DGA $\Omega^{\cdot}_{C/\mathbb{K}}$ from Example 1.3.2. By definition the de Rham cohomology of X is

$$\mathrm{H}^p_{\mathrm{DR}}(X) := \mathrm{H}^p \Omega^{\cdot}_{C/\mathbb{K}}.$$

The multiplication in $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X)$ is called the cup product.

Example 1.4.3. Take a C^∞ real manifold X. The de Rham cohomology of X is

$$\mathrm{H}^p_{\mathrm{DR}}(X) := \mathrm{H}^p \Omega^{\cdot}(X).$$

The multiplication in $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X)$ is called the cup product.

Example 1.4.4. Let X be a topological space and $C^{\cdot}(X)$ the DGA of simplicial cochains. The cohomology

$$\mathrm{H}^p(X, \mathbb{K}) := \mathrm{H}^p C^{\cdot}(X)$$

is the pth singular cohomology of X with coefficients in $\mathbb{K}$.

In case X is a C^∞ real manifold there is a canonical isomorphism of graded $\mathbb{R}$-algebras (de Rham Theorem)

$$\mathrm{H}^{\cdot}_{\mathrm{DR}}(X) \cong \mathrm{H}^{\cdot}(X, \mathbb{R}).$$

Let A be a DGA and M a DG A-module. Then the group of cocycles $\mathrm{Z}M$ and the group of coboundaries $\mathrm{B}M$ are graded $\mathrm{Z}A$-submodules of M, and thus the cohomology

$$\mathrm{H}M := \mathrm{Z}M/\mathrm{B}M$$

is a graded $\mathrm{H}A$-module.

1.5 Quasi-Isomorphisms of DG Modules

Definition 1.5.1. Let A be a DGA and $\phi : M \to N$ a morphism in $\mathsf{DGMod}\, A$. ϕ is called a quasi-isomorphism if for all i, $\mathrm{H}^i(\phi) : \mathrm{H}^i M \to \mathrm{H}^i N$ is an isomorphism. In other words, $\mathrm{H}(\phi) : \mathrm{H}M \to \mathrm{H}N$ is an isomorphism of graded $\mathbb{K}$-modules.

Example 1.5.2. Let A be a ring and M an A-module. Choose a projective resolution of M

$$\cdots \to P^{-2} \xrightarrow{\delta} P^{-1} \xrightarrow{\delta} P^0 \xrightarrow{\phi^0} M \to 0.$$

Now consider A as a DGA concentrated in degree 0, with 0 differential. Then M is a DG A-module concentrated in degree 0. Defining $P^i := 0$ for $i > 0$, we can view $P := \bigoplus P^i$ as a DG A-module, with differential δ. The homomorphism $\phi^0 : P^0 \to M$ extends to a morphism $\phi : P \to M$ in $\mathsf{DGMod}\, A$. Since

$$\mathrm{H}^0(\phi) : \mathrm{H}^0 P \to \mathrm{H}^0 M = M$$

is bijective and $\mathrm{H}^i P = 0$ for all $i \neq 0$ it follows that ϕ is a quasi-isomorphism.

Example 1.5.3. Continuing with the previous example let N be any other A-module. Then

$$\mathrm{H}^i \operatorname{Hom}_A^{\mathrm{dg}}(P, N) = \operatorname{Ext}_A^i(M, N).$$

As we saw before

$$B := \operatorname{Hom}_A^{\mathrm{dg}}(P, P)$$

is a DGA. A calculation shows that the morphism

$$B = \operatorname{Hom}_A^{\mathrm{dg}}(P, P) \to \operatorname{Hom}_A^{\mathrm{dg}}(P, M)$$

induced by $P \to M$ is a quasi-isomorphism. Hence

$$\mathrm{H}^i B = \operatorname{Ext}_A^i(M, M).$$

The multiplication in the graded ring $\mathrm{H}B$ is called the Yoneda product.

2 The Derived Category of a DGA

2.1 Triangulated Categories

The construction we shall discuss now will take us out of the world of abelian categories. Instead of exact sequences we will have triangles. In what follows all additive categories and functors are assumed to be $\mathbb{K}$-linear.

A triangulated category C is an additive category with an automorphism $T : \mathsf{C} \to \mathsf{C}$ called the translation, and a class of sequences of the form

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} TL \tag{1}$$

called the triangles in C. The list of axioms satisfied by the class of triangles can be found in [RD] or [KS]. I will only give a sketch of their properties.

The class of triangles is closed under isomorphisms: given a commutative diagram

$$\begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & TL \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & {\scriptstyle h}\big\downarrow & & {\scriptstyle T(f)}\big\downarrow \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & TL' \end{array} \tag{2}$$

such that the top arrow is a triangle and the vertical arrows are isomorphisms, the bottom row is also a triangle.

Every morphism $\alpha : L \to M$ can be imbedded in a triangle (1).

The composition of consecutive morphisms in a triangle is zero.

Given a commutative diagram (2), but without the morphism h, such the rows are triangles, a morphism h exists to make the diagram commutative.

Given a commutative diagram (2), such the rows are triangles, if f and g are isomorphisms then so is h.

A functor $H : \mathsf{C} \to \mathsf{Mod}\,\mathbb{K}$ is called a cohomological functor if for every distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} TL$$

in C the resulting long sequence

$$\cdots \to HL \xrightarrow{H(\alpha)} HM \xrightarrow{H(\beta)} HN \xrightarrow{H(\gamma)} HTL \xrightarrow{HT(\alpha)} HTM \to \cdots$$

is exact.

An important example of cohomological functor is the functor

$$\mathrm{Hom}_{\mathsf{C}}(M, -) : \mathsf{C} \to \mathsf{Mod}\,\mathbb{K}$$

for any object $M \in \mathsf{C}$.

Given another triangulated category D, an triangle functor (or exact functor, or ∂ functor) $F : \mathsf{C} \to \mathsf{D}$ is an additive functor that commutes with the translation T and sends triangles to triangles.

2.2 The Homotopy Category

Let A be a DGA. Given two DG A-modules M, N and two morphisms $\phi_0, \phi_1 : M \to N$ in $\mathsf{DGMod}\,A$, we say that ϕ_0 and ϕ_1 are homotopic if there is an A-linear homomorphism $h : M \to N$ of degree -1 such that

$$\phi_0 - \phi_1 = \mathrm{d}_N \circ h + h \circ \mathrm{d}_M.$$

Homotopy is an equivalence relation.

Definition 2.2.1. The homotopy category of DG A-modules, denoted by $\mathsf{K}(\mathsf{DGMod}\,A)$, has the same objects as $\mathsf{DGMod}\,A$, and the morphisms are homotopy classes of morphisms in $\mathsf{DGMod}\,A$.

Two homotopic morphisms differ by a nullhomotopic morphism ($\phi_0 - \phi_1$ is homotopic to 0). The nullhomotopic morphisms are precisely the 0-coboundaries in $\mathrm{Hom}_A^{\mathrm{dg}}(M,N)$. Hence

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{DGMod}\,A)}(M,N) = \mathrm{H}^0\,\mathrm{Hom}_A^{\mathrm{dg}}(M,N).$$

For a DG module M we let $M[1]$ be the shifted DG module with $M[1]^i := M^{i+1}$ and $\mathrm{d}_{M[1]} := -\mathrm{d}_M$.

Given a morphism $\alpha : M \to N$ in $\mathsf{DGMod}\,A$, the mapping cone of α is the DG module

$$C := N \oplus M[1] = \begin{bmatrix} N \\ M[1] \end{bmatrix}$$

with differential

$$\mathrm{d}_C := \begin{bmatrix} \mathrm{d}_N & \alpha[1] \\ 0 & \mathrm{d}_{M[1]} \end{bmatrix}.$$

Consider the DG homomorphisms of degree 0

$$\beta : N \to C, \quad \beta(n) := \begin{bmatrix} n \\ 0 \end{bmatrix},$$

and

$$\gamma : C \to M[1], \quad \gamma(\begin{bmatrix} n \\ m \end{bmatrix}) := m.$$

Clearly $\gamma \circ \beta = 0$ in $\mathsf{DGMod}\,A$. The other compositions $\beta \circ \alpha$ and $\alpha \circ \gamma$ are not necessarily 0 but they are nullhomotopic.

The mapping cylinder of α is the sequence

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} C \xrightarrow{\gamma} M[1] \tag{3}$$

in $\mathsf{K}(\mathsf{DGMod}\,A)$.

Theorem 2.2.2. *$\mathsf{K}(\mathsf{DGMod}\,A)$ is a triangulated category. The translation functor is $TM := M[1]$, and the triangles are the sequences isomorphic to mapping cylinders.*

2.3 The Derived Category

Consider the functor

$$\mathrm{H}^0 : \mathsf{DGMod}\,A \to \mathsf{Mod}\,A.$$

If $\phi : M \to N$ is nullhomotopic then $\mathrm{H}^0(\phi) = 0$. Thus there is an induced functor

$$\mathrm{H}^0 : \mathsf{K}(\mathsf{DGMod}\,A) \to \mathsf{Mod}\,A.$$

In fact it is a cohomological functor (triangles go to long exact sequences). Note that $\mathrm{H}^0(M[i]) = \mathrm{H}^i M$.

Let S be the class of quasi-isomorphisms in $\mathsf{K}(\mathsf{DGMod}\,A)$, i.e. the morphisms $s : M \to N$ in $\mathsf{K}(\mathsf{DGMod}\,A)$ such that $\mathrm{H}^i(s) : \mathrm{H}^i M \to \mathrm{H}^i N$ is an isomorphism for all i. Evidently S is closed under composition. In analogy to rings, S is a "multiplicatively closed set". It turns out that S also satisfies the "left and right Ore conditions". Therefore the localization exists:

Theorem 2.3.1. *There is a triangulated category* $\mathsf{D}(\mathsf{DGMod}\,A)$*, called the derived category of* $\mathsf{DGMod}\,A$*, and an exact functor*

$$Q : \mathsf{K}(\mathsf{DGMod}\,A) \to \mathsf{D}(\mathsf{DGMod}\,A),$$

satisfying:

(a) *For every $s \in S$ the morphism $Q(s)$ is invertible.*

(b) *Any exact functor $F : \mathsf{K}(\mathsf{DGMod}\,A) \to \mathsf{C}$ to a triangulated category C in which all the morphisms $F(s)$, $s \in S$ are invertible, factors uniquely through Q.*

If A is a ring then $\mathsf{D}(\mathsf{DGMod}\,A) = \mathsf{D}(\mathsf{Mod}\,A)$, the usual unbounded derived category of Grothendieck and Verdier, see [RD] Chapter I.

Example 2.3.2. Consider the setup of Example 1.5.2. Then $\phi : P \to M$ is an isomorphism in $\mathsf{D}(\mathsf{DGMod}\,A)$.

2.4 K-Projectives etc.

A DG module N is acyclic if $\mathrm{H}^i N = 0$ for all i.

Definition 2.4.1. A DG module $P \in \mathsf{K}(\mathsf{DGMod}\,A)$ is called K-projective if

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{DGMod}\,A)}(P, N) = 0$$

for every acyclic N.

Denote by $\mathsf{K}(\mathsf{DGMod}\,A)_{\mathrm{proj}}$ the full subcategory of $\mathsf{K}(\mathsf{DGMod}\,A)$ consisting of K-projective objects.

Theorem 2.4.2. *For every DG A-module M there is a quasi-isomorphism $P \to M$ in* $\mathsf{K}(\mathsf{DGMod}\,A)$*, with P being K-projective. Moreover the canonical functor*

$$\mathsf{K}(\mathsf{DGMod}\,A)_{\mathrm{proj}} \to \mathsf{D}(\mathsf{DGMod}\,A)$$

is an equivalence.

In this formulation the theorem was proved by Keller [Ke2]. It generalizes the original results of Grothendieck and Verdier for the bounded above category $\mathsf{K}^{-}(\mathsf{Mod}\, A)$, A a ring; and the work of Spaltenstein and Bockstedt-Neeman for the unbounded category $\mathsf{K}(\mathsf{Mod}\, A)$.

It is not hard to show that if P is K-projective and $M \to N$ is a quasi-isomorphism in $\mathsf{DGMod}\, A$ then

$$\mathrm{Hom}_A^{\mathrm{dg}}(P, M) \to \mathrm{Hom}_A^{\mathrm{dg}}(P, N)$$

is a quasi-isomorphism in $\mathsf{DGMod}\, \mathbb{K}$. Also if $P \to Q$ is a quasi-isomorphism in $\mathsf{DGMod}\, A$ with Q also K-projective then

$$\mathrm{Hom}_A^{\mathrm{dg}}(Q, M) \to \mathrm{Hom}_A^{\mathrm{dg}}(P, M)$$

is a quasi-isomorphism in $\mathsf{DGMod}\, \mathbb{K}$.

We almost obtained the following theorem.

Theorem 2.4.3. *There is a bifunctor*

$$\mathrm{R}\,\mathrm{Hom}_A^{\mathrm{dg}} : \mathsf{D}(\mathsf{DGMod}\, A)^{\mathrm{op}} \times \mathsf{D}(\mathsf{DGMod}\, A) \to \mathsf{D}(\mathsf{DGMod}\, \mathbb{K})$$

that's exact in each argument. For any two DG modules M, N one has

$$\mathrm{H}^i \mathrm{R}\,\mathrm{Hom}_A^{\mathrm{dg}}(M, N) \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{DGMod}\, A)}(M, N[i]).$$

The proof is the same as in [RD] Section I.6. Cf. also [Sp].

The formula for $\mathrm{R}\,\mathrm{Hom}_A^{\mathrm{dg}}$ is

$$\mathrm{R}\,\mathrm{Hom}_A^{\mathrm{dg}}(M, N) = \mathrm{Hom}_A^{\mathrm{dg}}(P, N)$$

where $P \to M$ is a K-projective resolution of M (i.e. $P \to M$ is a quasi-isomorphism and P is K-projective).

3 Quasi-Isomorphisms of DGAs

3.1 Invariance of the Derived Category

So far there was not any dramatic distinction between rings and DGAs, as far as the derived categories were concerned. We shall now see such a distinction.

Let A and B be two DGAs. A DGA homomorphism $f : A \to B$ is a graded ring homomorphism that commutes with d. Note that $\mathrm{H}(f) : \mathrm{H}A \to \mathrm{H}B$ is then a graded ring homomorphism. f is called a quasi-isomorphism if $\mathrm{H}(f)$ is an isomorphism.

Theorem 3.1.1. *Suppose $A \to B$ is a quasi-isomorphism of DGAs. Then the restriction of scalars functor*

$$\mathsf{D}(\mathsf{DGMod}\, B) \to \mathsf{D}(\mathsf{DGMod}\, A)$$

is an equivalence of triangulated categories. Its inverse is the functor

$$B \otimes^{\mathrm{L}}_{A} - : \mathsf{D}(\mathsf{DGMod}\, A) \to \mathsf{D}(\mathsf{DGMod}\, B).$$

To calculate $B \otimes^{\mathrm{L}}_{A} M$ we choose a K-projective resolution $M \to P$, and then

$$B \otimes^{\mathrm{L}}_{A} M = B \otimes_{A} P.$$

It's not clear who first discovered this theorem. It appears in papers of Keller; yet Hinich knew it independently.

Here is an application of the theorem to rings.

Theorem 3.1.2. *Assume $\mathbb{K}$ is a field. Let A and B be rings (i.e. $\mathbb{K}$-algebras), and let $T \in \mathsf{D}(\mathsf{Mod}\, A)$ be a complex satisfying*

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(T, T[i]) \cong \begin{cases} B^{\mathrm{op}} \text{ as rings if } i = 0; \\ 0 \text{ otherwise.} \end{cases}$$

Then there is a complex $\tilde{T} \in \mathsf{D}(\mathsf{Mod}(A \otimes B^{\mathrm{op}}))$ and an isomorphism $\tilde{T} \cong T$ in $\mathsf{D}(\mathsf{Mod}\, A)$ such that induced ring homomorphism

$$B^{\mathrm{op}} \xrightarrow{\rho} \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A)}(\tilde{T}) \cong \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A)}(T) \cong B^{\mathrm{op}}$$

is the identity, where ρ is right multiplication on $\tilde{T}$.

Proof. Choose an isomorphism $T \cong P$ in $\mathsf{D}(\mathsf{Mod}\, A) = \mathsf{D}(\mathsf{DGMod}\, A)$ with P a K-projective DG module. Define the DGA

$$C := (\mathrm{Hom}^{\mathrm{dg}}_{A}(P, P))^{\mathrm{op}}.$$

By assumption $\mathrm{H}^{i} C = 0$ for all $i \neq 0$ and $\mathrm{H}^{0} C \cong B$ as rings, and by construction P is a DG $(A \otimes C^{\mathrm{op}})$-module.

Let D be the DGA

$$D := (\bigoplus\nolimits_{i<0} C^{i}) \oplus (ZC)^{0} \subset C.$$

There is a quasi-isomorphism $D \to B$; and hence also a quasi-isomorphism

$$A \otimes D^{\mathrm{op}} \to A \otimes B^{\mathrm{op}}.$$

We have a commutative diagram arising from restriction of scalars:

$$\begin{tikzcd}
\mathsf{D}(\mathsf{DGMod}(A \otimes D^{\mathrm{op}})) & \mathsf{D}(\mathsf{DGMod}(A \otimes C^{\mathrm{op}})) \arrow[l, "G"'] \arrow[d, "E"] \\
\mathsf{D}(\mathsf{DGMod}(A \otimes B^{\mathrm{op}})) \arrow[u, "F"] \arrow[r] & \mathsf{D}(\mathsf{DGMod}\, A).
\end{tikzcd}$$

By definition of C we have $P \in \mathsf{D}(\mathsf{DGMod}(A \otimes C^{\mathrm{op}}))$, and $EP \cong T$. According to Theorem 3.1.1 F is an equivalence, and hence there is a complex $\tilde{T} \in \mathsf{D}(\mathsf{DGMod}(A \otimes B^{\mathrm{op}}))$ such that $F\tilde{T} \cong GP$. A little diagram chasing shows that $\tilde{T}$ is the complex we want. $\square$

Corollary 3.1.3. *Suppose $\mathbb{K}$ is a field and A and B are two rings that are derived Morita equivalent, namely there exists a triangle equivalence*

$$F : \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A).$$

Then there is a two-sided tilting complex $\tilde{T} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes B^{\mathrm{op}})$.

Proof. Take $T := FB \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$, and apply the Theorem 3.1.2. $\square$

This result was first proved by Rickard; the proof above was discovered by Keller [Ke1]. See [Ye2] for more details.

3.2 A_∞-Algebras and Morphisms

This is a generalization of DGAs that has regained center stage recently with the work of Kontsevich. A good reference is [Ke3]. I will sketch some features. Here we assume $\mathbb{K}$ is a field of characteristic 0.

An A_∞-algebra is a graded $\mathbb{K}$-module $A = \bigoplus_{p \in \mathbb{Z}} A^p$ together with $\mathbb{K}$-linear homomorphisms

$$\mu_n : A^{\otimes n} \to A, \quad n \geq 1$$

homogenous of degree $2 - n$, satisfying the equations:

$$\sum_{r+s+t=n} (-1)^{r+st} \mu_{r+1+t}(\mathbf{1}^{\otimes r} \otimes \mu_s \otimes \mathbf{1}^{\otimes t}) = 0$$

as functions $A^{\otimes n} \to A$, for all n.

For $n = 1$ the homomorphism of degree 1

$$\mu_1 : A \to A$$

is nothing but a differential, so that (A, μ_1) is a DG $\mathbb{K}$-module. The map of degree 0

$$\mu_2 : A^{\otimes 2} \to A$$

is a multiplication, and (A, μ_1, μ_2) is a DGA – but not an associative one. The product μ_2 is associative only up to a homotopy coming from

$$\mu_3 : A^{\otimes 3} \to A.$$

Etc. This is why another name for A_∞-algebra is "strongly homotopy associative algebra".

The cohomology $\mathrm{H}A$, w.r.t. the differential μ_1, is an associative graded algebra, with product induced by μ_2.

Example 3.2.1. Any DGA A is an A_∞-algebra, with $\mu_n = 0$ for $n \geq 3$.

Given two A_∞-algebras A and B one can talk about an A_∞-morphism $f : A \to B$. f consists of a sequence of $\mathbb{K}$-linear homomorphisms

$$f_n : A^{\otimes n} \to B$$

of degree $1 - n$, for $n \geq 1$. These satisfy:

$$\begin{aligned} &\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(\mathbf{1}^{\otimes r} \otimes \mu_s \otimes \mathbf{1}^{\otimes t}) \\ &\qquad = \sum_{1 \leq r \leq n} \quad \sum_{m_1+\cdots+m_r=n} (\pm 1) \mu_r(f_{m_1} \otimes \cdots \otimes f_{m_r}) \end{aligned}$$

as functions $A^{\otimes n} \to B$, for all n.

We say $f : A \to B$ is a quasi-isomorphism if the induced graded ring homomorphism $\mathrm{H}(f) : \mathrm{H}A \to \mathrm{H}B$ is an isomorphism.

Example 3.2.2. If A and B are two DGAs and $f : A \to B$ is a DGA homomorphism then it is also an A_∞-morphism.

Here is a remarkable occurrence of an A_∞-quasi-isomorphism between DGAs. Recall the DGA $C^{\cdot}(X)$ of simplicial cochains on a topological space X (Example 1.3.4). The Alexander-Whitney product is noncommutative; indeed the center of $C^{\cdot}(X)$ is very small. Nonetheless, as we know from algebraic topology, the cohomology ring $\mathrm{H}C^{\cdot}(X) = \mathrm{H}(X, \mathbb{K})$ is graded-commutative!

The "commutative cochain problem" was to find another DGA, this one commutative, that will also compute the cohomology $\mathrm{H}(X, \mathbb{K})$.

If $\mathbb{K} = \mathbb{R}$ and X is a smooth manifold we can use the de Rham complex $\Omega^{\cdot}(X)$, see Example 1.3.3.

The general solution, by Thom and Sullivan, was to start with the simplicial set $S(X) = \{S_p(X)\}$ of singular cochains of X, but to replace the combinatorial construction of the DGA $C^{\cdot}(X)$ with another construction of a DGA $\tilde{C}^{\cdot}(X)$. This second construction mixes singular chains on X and differential forms on the standard simplices Δ^p. The DGA $\tilde{C}^{\cdot}(X)$ is commutative. There is a canonical $\mathbb{K}$-linear quasi-isomorphism

$$\int_{\Delta} : \tilde{C}^{\cdot}(X) \to C^{\cdot}(X)$$

that induces an isomorphism of graded rings on cohomology. Yet $\int_{\Delta}$ is far from being multiplicative. Indeed $\int_{\Delta}$ is surjective, $\tilde{C}^{\cdot}(X)$ is commutative and $C^{\cdot}(X)$ is very noncommutative...

It turns out however that $f_1 := \int_{\Delta}$ extends to a canonical A_{∞}-quasi-isomorphism

$$f : \tilde{C}^{\cdot}(X) \to C^{\cdot}(X).$$

See [BG].

The Thom-Sullivan construction was used in the paper [HY] to deal with adelic Chern forms in algebraic geometry.

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