Grothendieck Duality via Rigid Dualizing Complexes and Differential Graded Algebras [1]

Amnon Yekutieli
Ben Gurion University, ISRAEL

http://www.math.bgu.ac.il/~amyekut

Here is the plan of my lecture:

1. Dualizing Complexes: Overview
2. Rigid Dualizing Complexes – over a Base Field
3. The Arithmetic Situation
4. Application: the Residue Symbol

1. Dualizing Complexes: Overview

Let X be a noetherian scheme. Denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ the derived category of bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology sheaves.

Definition 1. (Grothendieck [RD]) A *dualizing complex* on X is a complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ satisfying the two conditions:

(i) $\mathcal{R}$ has finite injective dimension.
(ii) The canonical morphism $\mathcal{O}_X \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{R})$ is an isomorphism.

Some explanations: condition (i) says that there is a quasi-isomorphism $\mathcal{R} \to \mathcal{J}$, where $\mathcal{J}$ is a bounded complex of injective $\mathcal{O}_X$-modules. Condition (ii) then says that the homomorphism of complexes $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{J})$ is a quasi-isomorphism.

For an affine scheme $X = \operatorname{Spec} A$ there is an equivalence of categories

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \approx \mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathcal{O}_X) \approx \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A).$$

This translates to the following definition of dualizing complex over A: it is a complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$ satisfying the conditions

(i) R has finite injective dimension.
(ii) The canonical morphism $A \to \mathrm{RHom}_A(R, R)$ is an isomorphism.

Example 2. If $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension (say a field, or the ring of integers $\mathbb{Z}$) then $R_{\mathbb{K}} := \mathbb{K} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,\mathbb{K})$ is a dualizing complex.

The main results of the book [RD] can be summarized in the theorem below, in the case of schemes of finite type over a "nice" ground ring $\mathbb{K}$. By "nice" I

[1] printed: 21 Feb 2004

mean that $\mathbb{K}$ is noetherian and has some dualizing complex $R_{\mathbb{K}}$. Let us denote by $\mathsf{FTSch}/\mathbb{K}$ the category of finite type schemes over $\mathbb{K}$.

Theorem 3. [RD] *For each morphism $f : X \to Y$ in $\mathsf{FTSch}/\mathbb{K}$ there is a functor*

$$f^! : \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Y) \to \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$$

called the twisted inverse image functor*, satisfying:*

(i) *$f \mapsto f^!$ is a contravariant 2-functor, namely for composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathsf{FTSch}/\mathbb{K}$ one has an isomorphism $c_{f,g} : (g \circ f)^! \xrightarrow{\simeq} f^! \circ g^!$, and the $c_{f,g}$ satisfy a suitable compatibility condition.*
(ii) *If $\mathcal{R}$ is a dualizing complex on Y then $f^!\mathcal{R}$ is a dualizing complex on X.*
(iii) *For f finite or smooth one has specific formulas for $f^!$.*
(iv) *Flat base change.*
(v) *If f is proper then there is a nondegenerate trace morphism*

$$\mathrm{Tr}_f : \mathrm{R}f_* f^! \mathcal{M} \to \mathcal{M}.$$

Proving this theorem occupies hundreds of pages in [RD]. And even then there are omissions (verifications of complicated commutative diagrams are skipped etc.). Complementary works include [AK], [Li], [HK], [HS], [Ye1], [Co]..., but none can really replace the original treatment in [RD].

There do exist "quick" proofs (Deligne's appendix to [RD], and Neeman's [Ne]). But these shortcuts miss some important features of the full duality theory.

Dualizing sheaves and complexes have for years been key ingredients in classification of varieties, construction of moduli spaces and resolution of singularities. Recently they reappeared in theoretical physics, as the possible setting for generalized D-branes in string theory. It is therefore unfortunate that there did not exist a straightforward approach to Grothendieck duality theory.

In this lecture I will explain a new approach due to James Zhang and myself (see [YZ3]) to Grothendieck duality, which we believe solves the problem: it is quick, yet yields all the main features of the theory.

Let me outline the strategy. Suppose that the base ring $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension. From Theorem 3 and Example 2 we see that for every scheme X there is a special dualizing complex $\mathcal{R}_X := \pi_X^! \mathbb{K}$, where $\pi_X : X \to \operatorname{Spec} \mathbb{K}$ is the structural morphism. This is the *Grothendieck dualizing complex of X*.

Conversely, if we know these dualizing complexes, and how they vary (their functorial properties), then we can recover the twisted inverse image functor by the formula

$$f^! \mathcal{M} \cong \mathrm{D}_X \, \mathrm{L}f^* \, \mathrm{D}_Y \mathcal{M}.$$

Here

$$\mathrm{D}_Y \mathcal{M} := \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{R}_Y)$$

etc.

We break down the task of constructing the dualizing complexes $\mathcal{R}_X$ into two main parts: one local and the other global.

In the local part we consider affine schemes (i.e. finitely generated $\mathbb{K}$-algebras), and prove existence and uniqueness of *rigid dualizing complexes*. This is what I will talk about today.

The global part handles gluing of dualizing complexes, and there we use *perverse coherent sheaves*. That's a subject for a separate lecture.

2. Rigid Dualizing Complexes – over a Base Field

Now $\mathbb{K}$ is a field. We consider finitely generated commutative $\mathbb{K}$-algebras.

Existence of dualizing complexes is easy, due to the following facts:

Theorem 4. [RD] *Suppose R_A is a dualizing complex over A, and $A \to B$ is a homomorphism of $\mathbb{K}$-algebras.*

(i) *If B is finite over A then $R_B := \mathrm{RHom}_A(B, R_A)$ is a dualizing complex over B.*
(ii) *If B is smooth over A of relative dimension n then $R_B := \Omega^n_{B/A}[n] \otimes_A R_A$ is a dualizing complex over B.*

There is also a "uniqueness" result:

Theorem 5. [RD] *Suppose R and R' are two dualizing complexes over A. Then $R' \cong R \otimes_A L[n]$ for some integer n and invertible A-module L.*

This mild-looking indeterminacy is actually a source of tremendous difficulty. This is because we need to single out a particular dualizing complex, the Grothendieck dualizing complex R_A, with its specific functorial properties, in order to proceed. Overcoming this indeterminacy occupies the bulk of [RD].

Enter Van den Bergh's idea of *rigid dualizing complex* from [VdB].

For a complex $M \in \mathsf{D}(\mathsf{Mod}\, A)$ we write

$$\mathrm{Sq}_{A/\mathbb{K}}\, M := \mathrm{RHom}_{A \otimes_{\mathbb{K}} A}(A, M \otimes_{\mathbb{K}} M).$$

It is called the *square of M*. We view $\mathrm{Sq}_{A/\mathbb{K}}\, M$ as an object of $\mathsf{D}(\mathsf{Mod}\, A)$ via A in the first argument of RHom.

This is a functor

$$\mathrm{Sq}_{A/\mathbb{K}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A),$$

which is not linear. In fact, given a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, A)$ and an element $a \in A$ one has

$$\begin{aligned} \mathrm{Sq}_{A/\mathbb{K}}(a\phi) = a^2\, \mathrm{Sq}_{A/\mathbb{K}}(\phi) : \\ \mathrm{Sq}_{A/\mathbb{K}}\, M \to \mathrm{Sq}_{A/\mathbb{K}}\, N. \end{aligned} \tag{1}$$

Definition 6. Let $M \in \mathsf{D}(\mathsf{Mod}\, A)$. Suppose there exists an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}}\, M$$

in $\mathsf{D}(\mathsf{Mod}\, A)$. Then ρ is called a *rigidifying isomorphism for M*, and the pair (M, ρ) is called a *rigid complex over A relative to $\mathbb{K}$*.

Definition 7. Say (M, ρ_M) and (N, ρ_N) are rigid complexes over A relative to $\mathbb{K}$. A morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, A)$ is called a *rigid morphism relative to* $\mathbb{K}$ if the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & \operatorname{Sq}_{A/\mathbb{K}} M \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle \operatorname{Sq}_{A/\mathbb{K}}(\phi)} \\ N & \xrightarrow{\rho_N} & \operatorname{Sq}_{A/\mathbb{K}} N \end{array}$$

is commutative.

Definition 8. A *rigid dualizing complex over A relative to* $\mathbb{K}$ is a rigid complex (R, ρ), such that R is a dualizing complex over A.

Example 9. Take $A := \mathbb{K}$ and $R := \mathbb{K}$. We know that $\mathbb{K}$ is a dualizing complex. Also

$$\operatorname{Sq}_{\mathbb{K}/\mathbb{K}} \mathbb{K} = \operatorname{RHom}_{\mathbb{K}\otimes_{\mathbb{K}}\mathbb{K}}(\mathbb{K}, \mathbb{K} \otimes_{\mathbb{K}} \mathbb{K}) = \mathbb{K},$$

so we have the *tautological rigidifying isomorphism* $\rho_{\text{tau}} : \mathbb{K} \xrightarrow{\simeq} \operatorname{Sq}_{\mathbb{K}/\mathbb{K}} \mathbb{K}$, and $(\mathbb{K}, \rho_{\text{tau}})$ is a rigid dualizing complex.

Example 10. Suppose A is smooth over $\mathbb{K}$ of dimension n. There are canonical isomorphisms

$$\operatorname{Ext}^i_{A\otimes_{\mathbb{K}} A}(A, \Omega^{2n}_{(A\otimes_{\mathbb{K}} A)/\mathbb{K}}) \cong \begin{cases} \Omega^n_{A/\mathbb{K}} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Omega^n_{A/\mathbb{K}} \otimes_{\mathbb{K}} \Omega^n_{A/\mathbb{K}} \cong \Omega^{2n}_{(A\otimes_{\mathbb{K}} A)/\mathbb{K}}.$$

Hence $\Omega^n_{A/\mathbb{K}}[n]$ is a rigid dualizing complex over A relative to $\mathbb{K}$.

Theorem 11. ([VdB], [YZ1]) *A rigid dualizing complex over A relative to $\mathbb{K}$ is unique up to a unique rigid isomorphism.*

Here is a sketch of the proof: suppose (R, ρ) and (R', ρ') are two rigid dualizing complexes over A. Using Theorem 5 and the rigidity equation one shows that $L \cong A$ and $n = 0$, i.e. there's some isomorphism $\phi' : R \xrightarrow{\simeq} R'$. So we get a diagram

$$\begin{array}{ccc} R & \xrightarrow{\rho} & \operatorname{Sq}_{A/\mathbb{K}} R \\ {\scriptstyle\phi'}\downarrow & & \downarrow{\scriptstyle \operatorname{Sq}_{A/\mathbb{K}}(\phi')} \\ R' & \xrightarrow{\rho'} & \operatorname{Sq}_{A/\mathbb{K}} R'. \end{array}$$

This diagram might not commute; but there is a unique element $u \in A^\times$ such that

$$\rho' \circ \phi' = u \operatorname{Sq}_{A/\mathbb{K}}(\phi') \circ \rho.$$

Letting $\phi := u\phi'$, and using equation (1), we then get

$$\rho' \circ \phi = \operatorname{Sq}_{A/\mathbb{K}}(\phi) \circ \rho.$$

So $\phi : R \xrightarrow{\simeq} R'$ is the desired isomorphism.

We shall denote the rigid dualizing complex of A by (R_A, ρ_A).

Existence of a rigid dualizing complex R_A is easy to demonstrate. Simply present A as a quotient of a polynomial algebra $\mathbb{K}[\boldsymbol{t}] = \mathbb{K}[t_1, \ldots, t_n]$, and take

$$R_A := \mathrm{RHom}_{\mathbb{K}[\boldsymbol{t}]}(A, \Omega^n_{\mathbb{K}[\boldsymbol{t}]/\mathbb{K}}[n]).$$

Theorem 12. [YZ1] *Suppose $A \to B$ is a finite homomorphism. Then:*

(1) *The complex* $\mathrm{RHom}_A(B, R_A) \in \mathsf{D}(\mathsf{Mod}\, B)$ *has a rigidifying isomorphism induced from the rigidifying isomorphism ρ_A. Hence there is a canonical isomorphism*
$$R_B \cong \mathrm{RHom}_A(B, R_A).$$
(2) *There is a unique morphism*
$$\mathrm{Tr}_{B/A} : R_B \to R_A$$
which is rigid relative to $\mathbb{K}$, and is nondegenerate.

The morphism $\mathrm{Tr}_{B/A}$ is called the *rigid trace.*

Theorem 13. [YZ1] *Suppose $A \to B$ is a smooth homomorphism of relative dimension n. Then the complex*

$$\Omega^n_{B/A}[n] \otimes_A R_A \in \mathsf{D}(\mathsf{Mod}\, B)$$

has a rigidifying isomorphism induced from the rigidifying isomorphism ρ_A. Hence there is a canonical isomorphism

$$R_B \cong \Omega^n_{B/A}[n] \otimes_A R_A.$$

The proof is similar to Example 10.

Suppose X is a scheme and $V \subset U \subset X$ are affine open sets. Let $A := \Gamma(U, \mathcal{O}_X)$ and $B := \Gamma(V, \mathcal{O}_X)$, so that $A \to B$ is étale. As a special case of Theorem 13 we obtain a canonical isomorphism $R_B \cong B \otimes_A R_A$.

Using perverse sheaf theory, we can glue the rigid dualizing complexes R_A, for affine open sets $U = \operatorname{Spec} A \subset X$, into a dualizing complex $\mathcal{R}_X$. As explained before this allows us to construct the twisted inverse image functor $f^!$.

I would like to mention that rigid dualizing complexes make sense (and in fact were introduced in the context of) noncommutative $\mathbb{K}$-algebras.

3. The Arithmetic Situation

Now consider the case of a ground ring $\mathbb{K}$ that is noetherian, but is not a field; e.g. $\mathbb{K} = \mathbb{Z}$. We wish to define rigid dualizing complexes over a finitely generated $\mathbb{K}$-algebra A.

The naive attempt fails immediately: the squaring operation

$$\mathrm{RHom}_{A \otimes_{\mathbb{K}} A}(A, R \otimes_{\mathbb{K}} R)$$

makes no sense.

Indeed, since R is an object of $\mathsf{D}(\mathsf{Mod}\, A)$ we have to replace $R \otimes_{\mathbb{K}} R$ by the derived tensor product $R \otimes^{\mathrm{L}}_{\mathbb{K}} R \in \mathsf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} A)$. The obvious way to do this is to find a resolution $P \to R$ by a complex of A-modules that are flat $\mathbb{K}$-modules, and then to define

$$R \otimes^{\mathrm{L}}_{\mathbb{K}} R := P \otimes_{\mathbb{K}} P.$$

That may be impossible.

Example 14. Take $\mathbb{K} := \mathbb{Z}$ and $A := \mathbb{Z}/(6)$. There does not exist an A-module that's flat over $\mathbb{K}$.

This is where *differential graded algebras* (DG algebras) enter the picture.

A DG algebra over $\mathbb{K}$ is a graded $\mathbb{K}$-algebra $A = \bigoplus_{i \in \mathbb{Z}} A^i$, together with a $\mathbb{K}$-linear coboundary operator $\mathrm{d} : A \to A$ of degree $+1$, satisfying $\mathrm{d} \circ \mathrm{d} = 0$ and

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^i a\mathrm{d}(b)$$

for $a \in A^i$.

A DG algebra quasi-isomorphism is a homomorphism $f : A \to B$ respecting degrees, multiplications and coboundaries, and such that $\mathrm{H}(f) : \mathrm{H}A \to \mathrm{H}B$ is an isomorphism (of graded algebras).

We shall only consider *super-commutative* DG algebras, namely those satisfying $ab = (-1)^{ij} ba$ and $c^2 = 0$ for $a \in A^i$, $b \in A^j$ and $c \in A^{2i+1}$.

We view a commutative $\mathbb{K}$-algebra A as a super-commutative DG algebra, concentrated in degree 0.

Suppose we are given a set X of variables, partitioned into $X = \coprod_{i \in \mathbb{Z}} X_i$; the variables in X_i are those of degree i. Let $X_{\mathrm{ev}} := \coprod_{i \in \mathbb{Z}} X_{2i}$, and likewise define X_{odd}. The super-polynomial algebra on X is

$$\mathbb{K}[X_{\mathrm{ev}} | X_{\mathrm{odd}}] := \mathbb{K}[X_{\mathrm{ev}} | \emptyset] \otimes_{\mathbb{K}} \mathbb{K}[\emptyset | X_{\mathrm{odd}}],$$

where $\mathbb{K}[X_{\mathrm{ev}} | \emptyset]$ is a polynomial algebra and $\mathbb{K}[\emptyset | X_{\mathrm{odd}}]$ is an exterior algebra.

A *semi-free non-positive* DG $\mathbb{K}$-algebra is a DG algebra A, such that after forgetting the coboundary it is isomorphic, as graded algebra, to a super-polynomial algebra $\mathbb{K}[X_{\mathrm{ev}} | X_{\mathrm{odd}}]$ for some graded set $X = \coprod_{i \leq 0} X_i$.

Definition 15. Let A be a $\mathbb{K}$-algebra. A *semi-free non-positive resolution of A relative to* $\mathbb{K}$ is a quasi-isomorphism of DG $\mathbb{K}$-algebras $\widetilde{A} \to A$, where $\widetilde{A}$ is a semi-free non-positive DG $\mathbb{K}$-algebra.

Such resolutions always exist, and they are unique up to quasi-isomorphism.

Example 16. In the setup of Example 14, where we had $\mathbb{K} = \mathbb{Z}$ and $A = \mathbb{Z}/(6)$, we can take $\widetilde{A} := \mathbb{Z} \oplus \mathbb{Z}\xi$, where we put $\deg(\xi) := -1$, $\xi^2 = 0$ and $\mathrm{d}(\xi) := 6$.

For a DG algebra A one has the category $\mathsf{DGMod}\, A$ of DG A-modules. It is analogous to the category of complexes of modules over a ring, and by a similar process of inverting quasi-isomorphisms we obtain the derived category $\mathsf{D}(A)$; see [Ke]. When A is concentrated in degree 0, i.e. it is a ring, then $\mathsf{D}(A) = \mathsf{D}(\mathsf{Mod}\, A)$.

It is possible to derive functors of DG modules, again in analogy to $\mathsf{D}(\mathsf{Mod}\, A)$.

An added feature is that for a quasi-isomorphism $A \to A'$ the restriction of scalars functor $\mathsf{D}(A') \to \mathsf{D}(A)$ is an equivalence.

Getting back to our initial problem, suppose A is a $\mathbb{K}$-algebra. Choose a semi-free non-positive resolution $\widetilde{A} \to A$. For $M \in \mathsf{D}(\mathsf{Mod}\, A)$ define

$$\begin{aligned} \operatorname{Sq}_{A/\mathbb{K}} M &:= \operatorname{RHom}_{\widetilde{A} \otimes_{\mathbb{K}} \widetilde{A}}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M) \\ &\in \mathsf{D}(\mathsf{Mod}\, A). \end{aligned}$$

In [YZ3] we proved that this functor is independent of the resolution $\widetilde{A} \to A$. Thus we can invoke Definition 8, namely we can define rigid dualizing complexes over A relative to $\mathbb{K}$.

If we assume $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension (e.g. $\mathbb{K} = \mathbb{Z}$) then Theorems 11, 12 and 13 hold. Therefore we can define the Grothendieck dualizing complexes $\mathcal{R}_X$ for $X \in \mathsf{FTSch}\,/\mathbb{K}$, and the functors $f^!$, with all the nice properties listed in Theorem 3.

4. Application: the Residue Symbol

Suppose A is a finitely generated $\mathbb{K}$-algebra, and B is a smooth A-algebra of relative dimension n. Also suppose $\boldsymbol{b} = (b_1, \ldots, b_n)$ is a sequence of elements in B such that $B/(\boldsymbol{b})$ is finite over A.

In this situation one can define the *residue symbol*

$$\operatorname{Res}_{B/A} \begin{bmatrix} \beta \\ \boldsymbol{b} \end{bmatrix} \in A$$

for any $\beta \in \Omega^n_{B/A}$, as follows.

Let us write

$$A \xrightarrow{f^*} B \xrightarrow{g^*} B/(\boldsymbol{b})$$

and

$$h^* := g^* \circ f^* : A \to B/(\boldsymbol{b}).$$

We can use the Koszul complex determined by $\boldsymbol{b}$ to calculate

$$E(A, B, \boldsymbol{b}) := \operatorname{Ext}^n_A(B/(\boldsymbol{b}), \Omega^n_{B/A}),$$

and this gives us an element $\left[\begin{smallmatrix} \beta \\ \boldsymbol{b} \end{smallmatrix}\right] \in E(A, B, \boldsymbol{b})$.

Now we have morphisms

$$\begin{gathered} E(A, B, \boldsymbol{b}) \cong g^! \, f^! A, \\ c_{f,g} : h^! A \xrightarrow{\simeq} g^! \, f^! A, \\ \operatorname{Tr}_{h^*} : h^! A \to A; \end{gathered}$$

and we define

$$\operatorname{Res}_{B/A} \begin{bmatrix} \beta \\ \boldsymbol{b} \end{bmatrix} := \operatorname{Tr}_{h^*} \circ \, c^{-1}_{f,g}\Big(\begin{bmatrix} \beta \\ \boldsymbol{b} \end{bmatrix} \Big).$$

In [RD] several properties of the residue symbol were listed, without proofs. Most were eventually proved by Lipman in [Li]; with the exception of property R4, the transitivity property. The latter was proved at the end of the book [Hu]. Another proof later appeared at the end of the book [Co].

We show in [YZ3] how the properties of the residue symbol can be proved easily using rigidity; and we present a detailed proof of property R4.

The key idea is as follows. The A-module A has the tautological rigidifying isomorphism relative to A, $\rho_{\mathrm{tau}} : A \xrightarrow{\simeq} \operatorname{Sq}_{A/A} A$.

As in Theorems 13 and 12 the $B/(\boldsymbol{b})$ -module $E(A, B, \boldsymbol{b})$ has a rigidifying isomorphism relative to A, induced from ρ_{tau}. We show that the residue symbol

$$\operatorname{Res}_{B/A} : E(A, B, \boldsymbol{b}) \to A$$

is the *unique* nondegenerate rigid morphism relative to A.

References

[AK] A. Altman and S. Kleiman, "Introduction to Grothendieck Duality," Lecture Notes in Math. **20**, Springer, 1970.

[AJL] L. Alonso, A. Jeremías and J. Lipman, Duality and flat base change on formal schemes, in "Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes," Contemp. Math. **244**, Amer. Math. Soc., 1999, 3-90.

[Be] K. Behrend, Differential Graded Schemes I, preprint.

[Co] B. Conrad, "Grothendieck Duality and Base Change," Lecture Notes in Math. **1750**, Springer, 2000.

[HS] R. Hübl and P. Sastry, Regular differential forms and relative duality, Amer. J. Math. **115** (1993), no. 4, 749-787.

[HK] R. Hübl and E. Kunz, Regular differential forms and duality for projective morphisms, J. Reine Angew. Math. **410** (1990), 84-108.

[Hu] R. Hübl, "Traces of Differential Forms and Hochschild Homology," Lecture Notes in Math. **1368**, Springer, 1989.

[Ke] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63-102.

[Li] J. Lipman, "Residues and Traces of Differential Forms via Hochschild Homology," Contemporary Mathematics **61**, Amer. Math. Soc., Providence, RI, 1987.

[Ne] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, J. Amer. Math. Soc. **9** (1996), no. 1, 205-236.

[RD] R. Hartshorne, "Residues and Duality," Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[VdB] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered ring, J. Algebra **195** (1997), no. 2, 662-679.

[Ye1] A. Yekutieli, "An Explicit Construction of the Grothendieck Residue Complex" (with an appendix by P. Sastry), Astérisque **208** (1992).

[YZ1] A. Yekutieli and J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra **213** (1999), no. 1, 1-51.

[YZ2] A. Yekutieli and J.J. Zhang, Dualizing Complexes and Perverse Sheaves on Noncommutative Ringed Schemes, eprint math.AG/0211309 at http://arXiv.org.

[YZ3] A. Yekutieli and J.J. Zhang, Perverse Sheaves and Rigid Dualizing Complexes on Schemes, in preparation.